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Investigation of a nonlocal Stieltjes type coupled boundary value problem of higher order nonlinear ordinary differential equations

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Abstract

We explore the existence criteria for solutions of a coupled system of two higher-order nonlinear ordinary differential equations supplemented with nonlocal and Stieltjes-type coupled boundary conditions. Such problems are useful in view of their occurrence in certain physical phenomena (see Section 1). In our first result, we apply the Leray-Schauder alternative to establish the existence of solutions to the given problem, while the second result deals with the uniqueness of solutions for the problem at hand, and it is based on Banach's fixed-point theorem. Examples are included to illustrate the results obtained. Finally, we indicate some new results arising as special cases of the ones presented in this paper.

Keywords: Ordinary differential equations, coupled system, nonlocal Stieltjes boundary conditions, existence, fixed point
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1 Introduction

During the last few decades, there has been shown a great interest in the study of boundary value problems. It has been mainly due to the occurrence of such problems in diverse disciplines, such as, cellular systems and aging models [1], fluid flow problems [29], conservation laws [8], magneto Maxwell nano-material [17], nano boundary layer fluid flow [7], magnetohydrodynamic flow [16], etc.

It has been observed that much of the work on boundary value problems is concerned with classical boundary conditions. However, the changes happening within the given domain cannot be modeled with such conditions. This led to the concept of nonlocal conditions [19], which can describe the changes happening at some interior points or sub-segments of the given domain. For details and examples of nonlocal boundary conditions, see the articles, [12, 13, 14, 15, 22, 26].

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Integral boundary conditions provide a practical approach to model the flow and drag phenomena in arbitrary shaped blood vessels [25, 28], heat conduction [10, 20, 23], biomedical CFD [11], etc. Some interesting results on boundary value problems with integral boundary conditions can be found in the papers [2, 3, 4, 5, 9, 18, 21, 27].

In a recent article [4] , the authors discussed the existence and uniqueness of solutions for the following problem

$$\begin{cases} u^{(n)}(t) = f(t,u,v), \qquad t \in [0,1], \\ u(0) = \delta u(\xi),\ u'(0) = 0,\ u^{''}(0) = 0,\ \ldots, u^{(n-2)}(0) = 0, \\ \alpha u(1) + \beta u'(1) = \displaystyle\int_0^1 u(s)d\mu(s), \end{cases}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $0 < \xi < 1$, μ is a function of bounded variation and $\alpha, \beta, \delta \in \mathbb{R}$.

In this paper, our objective is to generalize the problem studied in [4] to a coupled system of two higher order nonlinear ordinary differential equations complemented with nonlocal and Stieltjes type coupled boundary conditions. In precise terms, we investigate the problem

$$\begin{cases} u^{(n)}(t) = f(t,u,v), \qquad v^{(m)}(t) = g(t,u,v), \qquad t \in [0,1], \\ u(0) = \zeta_1 v(\eta),\ u'(0) = 0,\ u^{''}(0) = 0, \ldots,\ u^{(n-2)}(0) = 0, \\ v(0) = \zeta_2 u(\eta),\ v'(0) = 0,\ v^{''}(0) = 0, \ldots,\ v^{(m-2)}(0) = 0, \\ \alpha_1 u(1) + \beta_1 u'(1) = \displaystyle\int_0^1 v(s)d\mu(s),\ \alpha_2 v(1) + \beta_2 v'(1) = \displaystyle\int_0^1 u(s)d\mu(s), \end{cases} \tag{1.1}$$

where $\zeta_1, \zeta_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, $0 < \eta < 1$, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and μ is a function of bounded variation.

Here it is imperative to mention that the problem (1.1) is novel in the given setting and the results obtained for this problem specialize to some new ones by fixing the parameters in it (see the Conclusions section).

We organize the remaining paper as follows. In Section 2, an auxiliary lemma dealing with a linear version of the problem (1.1) is proved. The existence and uniqueness results for the given nonlinear problem, based on Leray-Schauder alternative and Banach's fixed point theorem respectively, are derived in Section 3. Examples illustrating the main results are also presented in this section. The paper concludes with some interesting observations.

2 An auxiliary lemma

Lemma 2.1. Let $\zeta_1\zeta_2 \neq 1$ and $y_1, y_2 \in C([0,1], \mathbb{R})$. Then the system of linear higher-order ordinary differential equations

$$u^{(n)}(t) = y_1(t), \qquad v^{(m)}(t) = y_2(t), \qquad t \in [0,1], \tag{2.1}$$

subject to the boundary conditions (1.1) is equivalent to a pair of integral equations

$$\begin{aligned} u(t) = &\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y_1(s)ds + \zeta_1 T_1(t) \int_0^\eta \frac{(\eta-s)^{m-1}}{(m-1)!} y_2(s)ds + \zeta_2 T_2(t) \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} y_1(s)ds \\ &+ T_3(t) \left[\int_0^1 \int_0^s \frac{(s-r)^{m-1}}{(m-1)!} y_2(r)drd\mu(s) - \int_0^1 \frac{(1-s)^{n-2}[\alpha_1(1-s) + \beta_1(n-1)]}{(n-1)!} y_1(s)ds \right] \\ &+ T_4(t) \left[\int_0^1 \int_0^s \frac{(s-r)^{n-1}}{(n-1)!} y_1(r)drd\mu(s) - \int_0^1 \frac{(1-s)^{m-2}[\alpha_2(1-s) + \beta_2(m-1)]}{(m-1)!} y_2(s)ds \right], \end{aligned} \tag{2.2}$$

$$\begin{aligned} v(t) = &\int_0^t \frac{(t-s)^{m-1}}{(m-1)!} y_2(s)ds + \zeta_1 T_5(t) \int_0^\eta \frac{(\eta-s)^{m-1}}{(m-1)!} y_2(s)ds + \zeta_2 T_6(t) \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} y_1(s)ds \\ &+ T_7(t) \left[\int_0^1 \int_0^s \frac{(s-r)^{m-1}}{(m-1)!} y_2(r)drd\mu(s) - \int_0^1 \frac{(1-s)^{n-2}[\alpha_1(1-s) + \beta_1(n-1)]}{(n-1)!} y_1(s)ds \right] \\ &+ T_8(t) \left[\int_0^1 \int_0^s \frac{(s-r)^{n-1}}{(n-1)!} y_1(r)drd\mu(s) - \int_0^1 \frac{(1-s)^{m-2}[\alpha_2(1-s) + \beta_2(m-1)]}{(m-1)!} y_2(s)ds \right], \end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
&T_1(t) = \Omega_1 + \Omega_5 t^{n-1}, \quad T_2(t) = \Omega_2 + \Omega_6 t^{n-1}, \quad T_3(t) = \Omega_3 + \Omega_7 t^{n-1},\\
&T_4(t) = \Omega_4 + \Omega_8 t^{n-1}, \qquad T_5(t) = \Omega_9 + \Omega_{13} t^{m-1}, \quad T_6(t) = \Omega_{10} + \Omega_{14} t^{m-1},\\
&T_7(t) = \Omega_{11} + \Omega_{15} t^{m-1}, \quad T_8(t) = \Omega_{12} + \Omega_{16} t^{m-1},\\
&\Omega_1 = \sigma_1 + \frac{\sigma_4(\zeta_1\eta^{m-1}\mathcal{K}_1 + \zeta_1\zeta_2\eta^{n-1}\mathcal{K}_2) - \sigma_6(\zeta_1\eta^{m-1}\mathcal{J}_1 + \zeta_1\zeta_2\eta^{n-1}\mathcal{J}_2)}{U(1-\zeta_1\zeta_2)},\\
&\Omega_2 = \sigma_2 + \frac{\sigma_5(\zeta_1\eta^{m-1}\mathcal{K}_1 + \zeta_1\zeta_2\eta^{n-1}\mathcal{K}_2) - \sigma_7(\zeta_1\eta^{m-1}\mathcal{J}_1 + \zeta_1\zeta_2\eta^{n-1}\mathcal{J}_2)}{U(1-\zeta_1\zeta_2)},\\
&\Omega_3 = \frac{(\zeta_1\eta^{m-1}\mathcal{K}_1 + \zeta_1\zeta_2\eta^{n-1}\mathcal{K}_2)}{U(1-\zeta_1\zeta_2)}, \quad \Omega_4 = \frac{(\zeta_1\eta^{m-1}\mathcal{J}_1 + \zeta_1\zeta_2\eta^{n-1}\mathcal{J}_2)}{U(1-\zeta_1\zeta_2)},\\
&\Omega_5 = \frac{(\sigma_4\mathcal{K}_2 - \sigma_6\mathcal{J}_2)}{U}, \quad \Omega_6 = \frac{(\sigma_5\mathcal{K}_2 - \sigma_7\mathcal{J}_2)}{U}, \quad \Omega_7 = \frac{\mathcal{K}_2}{U}, \quad \Omega_8 = \frac{\mathcal{J}_2}{U},\\
&\Omega_9 = \sigma_3 + \frac{\sigma_4(\zeta_1\zeta_2\eta^{m-1}\mathcal{K}_1 + \zeta_2\mathcal{K}_2) - \sigma_6(\zeta_1\zeta_2\eta^{m-1}\mathcal{J}_1 + \zeta_2\mathcal{J}_2)}{U(1-\zeta_1\zeta_2)},\\
&\Omega_{10} = \sigma_1 + \frac{\sigma_5(\zeta_1\zeta_2\eta^{m-1}\mathcal{K}_1 + \mathcal{H}_3\mathcal{K}_2) - \sigma_7(\zeta_1\zeta_2\eta^{m-1}\mathcal{J}_1 + \zeta_2\mathcal{J}_2)}{U(1-\zeta_1\zeta_2)},\\
&\Omega_{11} = \frac{\zeta_1\zeta_2\eta^{m-1}\mathcal{K}_1 + \zeta_2\mathcal{K}_2}{U(1-\zeta_1\zeta_2)}, \quad \Omega_{12} = \frac{\zeta_1\zeta_2\eta^{m-1}\mathcal{J}_1 + \zeta_2\mathcal{J}_2}{U(1-\zeta_1\zeta_2)},\\
&\Omega_{13} = \frac{(\sigma_4\mathcal{K}_1 - \sigma_6\mathcal{J}_1)}{U}, \quad \Omega_{14} = \frac{(\sigma_5\mathcal{K}_1 - \sigma_7\mathcal{J}_1)}{U}, \quad \Omega_{15} = \frac{\mathcal{K}_1}{U}, \quad \Omega_{16} = \frac{\mathcal{J}_1}{U},\\
&\sigma_1 = \frac{1}{1-\zeta_1\zeta_2}, \qquad \sigma_2 = \frac{\zeta_1}{1-\zeta_1\zeta_2}, \qquad \sigma_3 = \frac{\zeta_2}{1-\zeta_1\zeta_2},\\
&\sigma_4 = \frac{1}{1-\zeta_1\zeta_2}\left(\int_0^1 \zeta_2 d\mu(s) - \alpha_1\right), \qquad \sigma_5 = \frac{1}{1-\zeta_1\zeta_2}\left(\int_0^1 d\mu(s) - \zeta_1\alpha_1\right),\\
&\sigma_6 = \frac{1}{1-\zeta_1\zeta_2}\left(\zeta_2\alpha_2 - \int_0^1 d\mu(s)\right), \qquad \sigma_7 = \frac{1}{1-\zeta_1\zeta_2}\left(\alpha_2 - \int_0^1 \zeta_1 d\mu(s)\right),\\
&\mathcal{J}_1 = \frac{\zeta_2\eta^{n-1}\left(\zeta_1\alpha_1 - \int_0^1 d\mu(s)\right)}{1-\zeta_1\zeta_2} + \alpha_1 + \beta_1(n-1),\\
&\mathcal{J}_2 = \frac{\zeta_1\eta^{m-1}\left(\int_0^1 \zeta_2 d\mu(s) - \alpha_1\right)}{1-\zeta_1\zeta_2} + \int_0^1 s^{m-1} d\mu(s),\\
&\mathcal{K}_1 = \frac{\zeta_2\eta^{n-1}\left(\zeta_1\int_0^1 d\mu(s) - \alpha_2\right)}{1-\zeta_1\zeta_2} + \int_0^1 s^{n-1} d\mu(s),\\
&\mathcal{K}_2 = \frac{\zeta_1\eta^{m-1}\left(\zeta_2\alpha_2 - \int_0^1 d\mu(s)\right)}{1-\zeta_1\zeta_2} + \alpha_2 + \beta_2(m-1), \; U = \mathcal{J}_1\mathcal{K}_2 - \mathcal{J}_2\mathcal{K}_1.
\end{aligned} \tag{2.4}$$

Proof . Solving the ordinary differential equations in (2.1), we get

$$\begin{cases} u(t) = \displaystyle\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y_1(s) ds + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},\\ v(t) = \displaystyle\int_0^t \frac{(t-s)^{m-1}}{(m-1)!} y_2(s) ds + b_0 + b_1 t + \ldots + b_{m-1} t^{m-1}, \end{cases} \tag{2.5}$$

where $c_0, \ldots, c_{n-1}, b_0, \ldots, b_{m-1} \in \mathbb{R}$ are unknown arbitrary constants. Applying the conditions $u'(0) = 0, u''(0) = 0, \ldots, u^{(n-2)}(0) = 0$ and $v'(0) = 0, v''(0) = 0, \ldots, v^{(m-2)}(0) = 0$ in (2.5), we get $c_1 = c_2 = \ldots, c_{n-2} = 0, b_0 = b_1 = \ldots, b_{m-1} = 0$. Then, (2.5) becomes

$$\begin{cases} u(t) = \displaystyle\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y_1(s) ds + c_0 + c_{n-1} t^{n-1},\\ v(t) = \displaystyle\int_0^t \frac{(t-s)^{m-1}}{(m-1)!} y_2(s) ds + b_0 + b_{m-1} t^{m-1}. \end{cases} \tag{2.6}$$

Using (2.6) in the conditions $u(0)=\zeta_1 v(\eta)$ and $v(0)=\zeta_2 u(\eta)$, we get

$$c_0=\zeta_1\left(\int_0^\eta \frac{(\eta-s)^{m-1}}{(m-1)!}y_2(s)ds+b_0+b_{m-1}\eta^{m-1}\right), \tag{2.7}$$

and

$$b_0=\zeta_2\left(\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}y_1(s)ds+c_0+c_{n-1}\eta^{n-1}\right). \tag{2.8}$$

Now, using (2.6) in the conditions:

$$\alpha_1 u(1)+\beta_1 u'(1)=\int_0^1 v(s)d\mu(s),\quad \alpha_2 v(1)+\beta_2 v'(1)=\int_0^1 u(s)d\mu(s),$$

we obtain

$$\begin{aligned}&\alpha_1\left(\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}y_1(s)ds+c_0+c_{n-1}\right)+\beta_1\left(\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}y_1(s)ds+c_{n-1}(n-1)\right)\\ &=\int_0^1\left(\int_0^s \frac{(s-r)^{m-1}}{(m-1)!}y_2(r)dr+b_0+b_{m-1}s^{m-1}\right)d\mu(s),\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\alpha_2\left(\int_0^1 \frac{(1-s)^{m-1}}{(m-1)!}y_2(s)ds+b_0+b_{m-1}\right)+\beta_2\left(\int_0^1 \frac{(1-s)^{m-2}}{(m-2)!}y_2(s)ds+b_{m-1}(m-1)\right)\\ &=\int_0^1\left(\int_0^s \frac{(s-r)^{n-1}}{(n-1)!}y_1(r)dr+c_0+c_{n-1}s^{n-1}\right)d\mu(s).\end{aligned} \tag{2.10}$$

We can rewrite equations (2.7) - (2.10) as

$$\begin{cases} c_0=\mathcal{A}_1+\mathcal{A}_2 b_0+\mathcal{A}_3 b_{m-1},\\ b_0=\mathcal{B}_1+\mathcal{B}_2 c_0+\mathcal{B}_3 c_{n-1},\\ \mathcal{C}_1 c_0+\mathcal{C}_2 c_{n-1}+\mathcal{C}_3=\mathcal{D}_1 b_0+\mathcal{D}_2 b_{m-1}+\mathcal{D}_3,\\ \mathcal{E}_1 b_0+\mathcal{E}_2 b_{m-1}+\mathcal{E}_3=\mathcal{F}_1 c_0+\mathcal{F}_2 c_{n-1}+\mathcal{F}_3,\end{cases} \tag{2.11}$$

where

$$\begin{aligned}&\mathcal{A}_1=\zeta_1\Big[\int_0^\eta \frac{(\eta-s)^{m-1}}{(m-1)!}y_2(s)ds\Big],\ \mathcal{A}_2=\zeta_1,\ \mathcal{A}_3=\zeta_1\eta^{m-1}, \mathcal{B}_1=\zeta_2\Big[\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}y_1(s)ds\Big],\\ &\mathcal{B}_2=\zeta_2,\ \mathcal{B}_3=\zeta_2\eta^{n-1},\ \mathcal{C}_1=\alpha_1,\ \mathcal{C}_2=\alpha_1+\beta_1(n-1),\\ &\mathcal{C}_3=\int_0^1 \frac{(1-s)^{n-2}[\alpha_1(1-s)+\beta_1(n-1)]}{(n-1)!}y_1(s)ds,\\ &\mathcal{D}_1=\int_0^1 d\mu(s),\ \mathcal{D}_2=\int_0^1 s^{m-1}d\mu(s),\ \mathcal{D}_3=\int_0^1\int_0^s \frac{(s-r)^{m-1}}{(m-1)!}y_2(r)drd\mu(s),\ \mathcal{E}_1=\alpha_2,\\ &\mathcal{E}_2=\alpha_2+\beta_2(m-1),\ \mathcal{E}_3=\int_0^1 \frac{(1-s)^{m-2}[\alpha_2(1-s)+\beta_2(m-1)]}{(m-1)!}y_2(s)ds,\\ &\mathcal{F}_1=\int_0^1 d\mu(s),\ \mathcal{F}_2=\int_0^1 s^{n-1}d\mu(s),\ \mathcal{F}_3=\int_0^1\int_0^s \frac{(s-r)^{n-1}}{(n-1)!}y_1(r)drd\mu(s).\end{aligned} \tag{2.12}$$

Solving the first two equations in (2.11) for c_0 and b_0 in term of c_{n-1} and $b_{m-1}$and using the notation in (2.12), we obtain

$$c_0=\mathcal{G}_1+\mathcal{G}_2 b_{m-1}+\mathcal{G}_3 c_{n-1},\ \ b_0=\mathcal{H}_1+\mathcal{H}_2 b_{m-1}+\mathcal{H}_3 c_{n-1}, \tag{2.13}$$

where

$$\mathcal{G}_1=\frac{\mathcal{A}_1+\mathcal{A}_2\mathcal{B}_1}{m_1},\ \mathcal{G}_2=\frac{\mathcal{A}_3}{m_1},\ \mathcal{G}_3=\frac{\mathcal{A}_2\mathcal{B}_3}{m_1},\ \mathcal{H}_1=\frac{\mathcal{A}_1\mathcal{B}_2+\mathcal{B}_1}{m_1},\ \mathcal{H}_2=\frac{\mathcal{A}_3\mathcal{B}_2}{m_1},\ \mathcal{H}_3=\frac{\mathcal{B}_3}{m_1}, \tag{2.14}$$

and $m_1=1-\mathcal{A}_2\mathcal{B}_2\neq 0$. Substituting the values of c_0 and b_0 from (2.13) in the last two equations of (2.11), we get

$$c_{n-1}\mathcal{J}_1=b_{m-1}\mathcal{J}_2+\mathcal{J}_3,\ \ c_{n-1}\mathcal{K}_1=b_{m-1}\mathcal{K}_2+\mathcal{K}_3, \tag{2.15}$$

where $\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}_1, \mathcal{K}_2$ are given in (2.4) and

$$\begin{aligned} \mathcal{J}_3 &= \frac{\mathcal{A}_1[\mathcal{B}_2\mathcal{D}_1 - \mathcal{C}_1] + \mathcal{B}_1[\mathcal{D}_1 - \mathcal{A}_2\mathcal{C}_1]}{m_1} + \mathcal{D}_3 - \mathcal{C}_3, \\ \mathcal{K}_3 &= \frac{\mathcal{A}_1[\mathcal{B}_2\mathcal{E}_1 - \mathcal{F}_1] + \mathcal{B}_1[\mathcal{E}_1 - \mathcal{A}_2\mathcal{F}_1]}{m_1} + \mathcal{E}_3 - \mathcal{F}_3. \end{aligned} \tag{2.16}$$

Solving the system (2.15) for b_{m-1} and c_{n-1}, we find that

$$c_{n-1} = \frac{1}{U}(\mathcal{J}_3\mathcal{K}_2 - \mathcal{J}_2\mathcal{K}_3),\ b_{m-1} = \frac{1}{U}(\mathcal{J}_3\mathcal{K}_1 - \mathcal{J}_1\mathcal{K}_3), \tag{2.17}$$

where U is given in (2.4). Inserting (2.17) in (2.13), we obtain

$$\begin{cases} c_0 = \mathcal{G}_1 + \dfrac{1}{U}(\mathcal{G}_2(\mathcal{J}_3\mathcal{K}_1 - \mathcal{J}_1\mathcal{K}_3) + \mathcal{G}_3(\mathcal{J}_3\mathcal{K}_2 - \mathcal{J}_2\mathcal{K}_3)), \\ b_0 = \mathcal{H}_1 + \dfrac{1}{U}(\mathcal{H}_2(\mathcal{J}_3\mathcal{K}_1 - \mathcal{J}_1\mathcal{K}_3) + \mathcal{H}_3(\mathcal{J}_3\mathcal{K}_2 - \mathcal{J}_2\mathcal{K}_3)). \end{cases} \tag{2.18}$$

Substituting the values of c_{n-1}, b_{m-1}, c_0 and b_0 into (2.11) together with the notation (2.4), we obtain the solution (2.2) and (2.3). One can obtain the converse of the lemma by direct computation. □ In the sequel, we set

$$\begin{aligned} &\overline{T}_i = \max_{t\in[0,1]} |T_i(t)|,\ i = 1, 2, \ldots, 8,\ \omega_1 = \zeta_1 \frac{\eta^m}{m!},\ \omega_2 = \zeta_2 \frac{\eta^n}{n!},\ \omega_3 = \frac{1}{n!}(\alpha_1 + \beta_1 n), \\ &\omega_4 = \frac{1}{m!}(\alpha_2 + \beta_2 m),\ \omega_5 = \int_0^1 \frac{s^m}{m!} d\mu(s),\ \omega_6 = \int_0^1 \frac{s^n}{n!} d\mu(s), \end{aligned} \tag{2.19}$$

where $T_i(t)$ are given in (2.4)

3 Main results

Let $\mathcal{Q} = \{u(t) \mid u(t) \in C([0, 1])\}$ be the space equipped with norm $\|u\| = \sup\{|u(t)|, t \in [0, 1]\}$. Then, $(\mathcal{Q}, \|\cdot\|)$ is a Banach space and consequently, the product space $(\mathcal{Q} \times \mathcal{Q}, \|u, v\|)$ is also a Banach space endowed with the norm $\|(u, v)\| = \|u\| + \|v\|$ for $(u, v) \in \mathcal{Q} \times \mathcal{Q}$.

By Lemma 1, we define an operator $\mathcal{P} : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q} \times \mathcal{Q}$ associated with the problem (1.1) as

$$\mathcal{P}(u, v)(t) := (\mathcal{P}_1(u, v)(t), \mathcal{P}_2(u, v)(t)), \tag{3.1}$$

where

$$\begin{aligned} \mathcal{P}_1(u, v)(t) = &\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, u, v) ds + \zeta_1 T_1(t) \int_0^\eta \frac{(\eta - s)^{m-1}}{(m-1)!} g(s, u, v) ds \\ &+ \zeta_2 T_2(t) \int_0^\eta \frac{(\eta - s)^{n-1}}{(n-1)!} f(s, u, v) ds + T_3(t) \Big[\int_0^1 \int_0^s \frac{(s-r)^{m-1}}{(m-1)!} g(r, u, v) dr d\mu(s) \\ &- \int_0^1 \frac{(1-s)^{n-2}[\alpha_1(1-s) + \beta_1(n-1)]}{(n-1)!} f(s, u, v) ds \Big] + T_4(t) \Big[\int_0^1 \int_0^s \frac{(s-r)^{n-1}}{(n-1)!} f(r, u, v) dr d\mu(s) \\ &- \int_0^1 \frac{(1-s)^{m-2}[\alpha_2(1-s) + \beta_2(m-1)]}{(m-1)!} g(s, u, v) ds \Big], \end{aligned} \tag{3.2}$$

$$\begin{aligned} \mathcal{P}_2(u, v)(t) = &\int_0^t \frac{(t-s)^{m-1}}{(m-1)!} g(s, u, v) ds \\ &+ T_5(t) \left[\zeta_1 \int_0^\eta \frac{(\eta - s)^{m-1}}{(m-1)!} g(s, u, v) ds \right] + T_6(t) \left[\zeta_2 \int_0^\eta \frac{(\eta - s)^{n-1}}{(n-1)!} f(s, u, v) ds \right] \\ &+ T_7(t) \left[\int_0^1 \int_0^s \frac{(s-r)^{m-1}}{(m-1)!} g(r, u, v) dr d\mu(s) - \int_0^1 \frac{(1-s)^{n-2}[\alpha_1(1-s) + \beta_1(n-1)]}{(n-1)!} f(s, u, v) ds \right] \\ &+ T_8(t) \left[\int_0^1 \int_0^s \frac{(s-r)^{n-1}}{(n-1)!} f(r, u, v) dr d\mu(s) - \int_0^1 \frac{(1-s)^{m-2}[\alpha_2(1-s) + \beta_2(m-1)]}{(m-1)!} g(s, u, v) ds \right]. \end{aligned} \tag{3.3}$$

To establish our main results, we need the following assumptions.

(M_1) There exist real constants $m_i, n_i \geq 0, (i = 1, 2)$ and $m_0 \geq, n_0 \geq 0$, such that, $\forall u, v \in \mathbb{R}$,

$$|f(t, u, v)| \leq m_0 + m_1|u| + m_2|v|,$$

$$|g(t, u, v)| \leq n_0 + n_1|u| + n_2|v|.$$

(M_2) There exist positive constants ℓ_1 and ℓ_2, such that, $\forall t \in [a, b]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$,

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \ell_1(|u_1 - u_2| + |v_1 - v_2|),$$

$$|g(t, u_1, v_1) - g(t, u_2, v_2)| \leq \ell_2(|u_1 - u_2| + |v_1 - v_2|).$$

For the sake of convenience in the forthcoming analysis, we set

$$\begin{aligned} &R_0 = \min\{1 - (R_1 m_1 + R_2 n_1), 1 - (R_1 m_2 + R_2 n_2)\}, \quad m_i, n_i \geq 0, \ i = 1, 2, \\ &R_1 = r_1 + r_2, \ R_2 = \bar{r}_1 + \bar{r}_2, \\ &r_1 = \frac{1}{n!} + \overline{T}_2\omega_2 + \overline{T}_3\omega_3 + \overline{T}_4\omega_6, \ \bar{r}_1 = \overline{T}_1\omega_1 + \overline{T}_3\omega_5 + \overline{T}_4\omega_4, \\ &r_2 = \overline{T}_6\omega_2 + \overline{T}_7\omega_3 + \overline{T}_8\omega_6, \ \bar{r}_2 = \frac{1}{m!} + \overline{T}_5\omega_1 + \overline{T}_7\omega_5 + \overline{T}_8\omega_4. \end{aligned} \tag{3.4}$$

3.1 Existence of solutions

In this subsection, we discuss the existence of solutions for the problem (1.1) by using Leray-Schauder's alternative, which is stated below.

Lemma 3.1. Let $T : \mathcal{Q} \to \mathcal{Q}$ to be a completely continuous operator (that is, a map that restricted to any bounded set in $\mathcal{Q}$ is compact). Let $\theta(T) = \{x \in \mathcal{Q} : x = \lambda T(x) \text{ for some } 0 < \lambda < 1\}$. Then, either the set $\theta(T)$ is unbounded or T has at least one fixed point.

Theorem 3.2. Assume that condition (M_1) holds and

$$R_1 m_1 + R_2 n_1 < 1, \ R_1 m_2 + R_2 n_2 < 1, \tag{3.5}$$

where R_1 and R_2 are given in (3.4). Then, there exists at least one solution for the problem (1.1) on $[0, 1]$.

Proof . First of all, we show that the operator $\mathcal{P} : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q} \times \mathcal{Q}$ defined in (3.1) is completely continuous. Notice that the operator $\mathcal{P}$ is continuous as the functions f and g are continuous. Let $\Theta \subset \mathcal{Q} \times \mathcal{Q}$ be bounded. Then, there exist positive constants ι_f and ι_g, such that, $|f(t, u(t), v(t))| \leq \iota_f, |g(t, u(t), v(t))| \leq \iota_g, \forall (u, v) \in \Theta$. Then, for any $(u, v) \in \Theta$, we can obtain

$$\begin{aligned} |\mathcal{P}_1(u, v)(t)| \leq & \sup_{t \in [0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s, u, v)|ds + \zeta_1|T_1(t)| \int_0^\eta \frac{(\eta - s)^{m-1}}{(m-1)!}|g(s, u, v)|ds \\ & + \zeta_2|T_2(t)| \int_0^\eta \frac{(\eta - s)^{n-1}}{(n-1)!}|f(s, u, v)|ds + |T_3(t)| \Big[\int_0^1 \int_0^s \frac{(s-r)^{m-1}}{(m-1)!}|g(r, u, v)|drd\mu(s) \\ & + \int_0^1 \frac{(1-s)^{n-2}[|\alpha_1|(1-s) + |\beta_1|(n-1)]}{(n-1)!}|f(s, u, v)|ds \Big] + |T_4(t)| \Big[\int_0^1 \int_0^s \frac{(s-r)^{n-1}}{(n-1)!}|f(r, u, v)|drd\mu(s) \\ & + \int_0^1 \frac{(1-s)^{m-2}[|\alpha_2|(1-s) + |\beta_2|(m-1)]}{(m-1)!}|g(s, u, v)|ds \Big] \Bigg\} \\ \leq & \iota_f \Big[\frac{1}{n!} + \overline{T}_2\omega_2 + \overline{T}_3\omega_3 + \overline{T}_4\omega_6 \Big] + \iota_g \Big[\overline{T}_1\omega_1 + \overline{T}_3\omega_5 + \overline{T}_4\omega_4 \Big] \leq \iota_f r_1 + \iota_g \bar{r}_1, \end{aligned}$$

which implies that

$$\|\mathcal{P}_1(u,v)\| \le \iota_f r_1 + \iota_g \bar{r}_1.$$

Similarly, one can obtain that

$$\|\mathcal{P}_2(u,v)\| \le \iota_f r_2 + \iota_g \bar{r}_2.$$

From the forgoing inequalities, we get $\|\mathcal{P}(u,v)(t)\| \le \iota_f R_1 + \iota_g R_2$, where R_1 and R_2 are given in (3.4), which shows that the operator $\mathcal{P}$ is uniformly bounded. Next, we establish that $\mathcal{P}$ is equicontinuous. For $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&|\mathcal{P}_1(u_1,v_1)(t_2) - \mathcal{P}_1(u_2,v_2)(t_1)| \\
\le & \iota_f \Bigg\{ \left| \int_0^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} f(s,u,v)\,ds - \int_0^{t_1} \frac{(t_1-s)^{n-1}}{(n-1)!} f(s,u,v)\,ds \right| + |T_1(t_2) - T_1(t_1)|\,\zeta_1 \int_0^{\eta} \frac{(\eta-s)^{m-1}}{(m-1)!} |g(s,u,v)|\,ds \\
& + |T_2(t_2) - T_2(t_1)|\,\zeta_2 \int_0^{\eta} \frac{(\eta-s)^{n-1}}{(n-1)!} |f(s,u,v)|\,ds + |T_3(t_2) - T_3(t_1)| \left[\int_0^1 \int_0^s \frac{(s-r)^{m-1}}{(m-1)!} |g(r,u,v)|\,dr\,d\mu(s) \right. \\
& \left. + \int_0^1 \frac{(1-s)^{n-2}[\alpha_1(1-s) + \beta_1(n-1)]}{(n-1)!} |f(s,u,v)|\,ds \right] + |T_4(t_2) - T_4(t_1)| \left[\int_0^1 \int_0^s \frac{(s-r)^{n-1}}{(n-1)!} |f(r,u,v)|\,dr\,d\mu(s) \right. \\
& \left. + \int_0^1 \frac{(1-s)^{m-2}[\alpha_2(1-s) + \beta_2(m-1)]}{(m-1)!} |g(s,u,v)|\,ds \right] \Bigg\} \\
\le & \frac{\iota_f}{n!} (2(t_2-t_1)^n + |t_2^n - t_1^n|) + |T_1(t_2) - T_1(t_1)|\iota_g \omega_1 + |T_2(t_2) - T_2(t_1)|\iota_f \omega_2 + |T_3(t_2) - T_3(t_1)|(\iota_f \omega_3 + \iota_g \omega_5) \\
& + |T_4(t_2) - T_4(t_1)|(\iota_f \omega_6 + \iota_g \omega_4),
\end{aligned} \tag{3.6}$$

which tends to zero independent of $(u,v) \in \Theta$ as $(t_2 - t_1) \to 0$. Similarly, one can find that

$$|\mathcal{P}_2(u_1,v_1)(t_2) - \mathcal{P}_2(u_2,v_2)(t_1)| \le \frac{\iota_g}{n!} (2(t_2-t_1)^n + |t_2^n - t_1^n|) \to 0,$$

independent of $(u,v) \in \Theta$ as $(t_2 - t_1) \to 0$. Thus, the operator $\mathcal{P}$ is equicontinuous. Finally, we verify that the set $\Phi = \{(u,v) \in \mathcal{Q} \times \mathcal{Q} | (u,v) = \lambda \mathcal{P}(u,v), 0 < \lambda < 1\}$ is bounded. Let $(u,v) \in \Phi$. Then, $(u,v) = \lambda \mathcal{P}(u,v)$ and so $u(t) = \lambda \mathcal{P}_1(u,v)(t),\ v(t) = \lambda \mathcal{P}_2(u,v)(t)$, $t \in [0,1]$. Thus, it is easy to find that

$$|u(t)| = r_1 m_0 + \bar{r}_1 n_0 + (r_1 m_1 + \bar{r}_1 n_1)\|u\| + (r_1 m_2 + \bar{r}_1 n_2)\|v\|, \tag{3.7}$$

and

$$|v(t)| = r_2 m_0 + \bar{r}_2 n_0 + (r_2 m_1 + \bar{r}_2 n_1)\|u\| + (r_2 m_2 + \bar{r}_2 n_2)\|v\|, \tag{3.8}$$

where $r_1, r_2, \bar{r}_1$, and $\bar{r}_2$ are given in (3.4). Hence, we have

$$\|u\| + \|v\| \le (r_1 + r_2)m_0 + (\bar{r}_1 + \bar{r}_2)n_0 + [(r_1 + r_2)m_1 + (\bar{r}_1 + \bar{r}_2)n_1]\|u\| + [(r_1 + r_2)m_2 + (\bar{r}_1 + \bar{r}_2)n_2]\|v\|,$$

which, in view of (3.4), can be written as

$$\|(u,v)\| \le \frac{R_1 m_0 + R_2 n_0}{R_0}.$$

Therefore, the set Φ is bounded. Hence, by Lemma 3.1, the operator $\mathcal{P}$ has at least one fixed point. Therefore, the problem (1.1) has at least one solution on [0,1]. □

3.2 Uniqueness of solutions

Here, we establish the uniqueness of solutions for the problem (1.1) by means of the Banach's contractions mapping principle.

Theorem 3.3. Assume that (M_2) and the following condition hold

$$R_1 \ell_1 + R_2 \ell_2 < 1, \tag{3.9}$$

where R_1 and R_2 are given in (3.4). Then, the problem (1.1) has a unique solution on $[0,1]$.

Proof . Define a closed ball $B_r = \{(u,v) \in \mathcal{Q} \times \mathcal{Q} : \|(u,v)\| \leq r\}$ with

$$r \geq \frac{R_1\mu_1 + R_2\mu_2}{1-(R_1\ell_1 + R_2\ell_2)}, \tag{3.10}$$

where

$$\sup_{t\in[0,1]} |f(t,0,0)| = \mu_1, \ \sup_{t\in[0,1]} |g(t,0,0)| = \mu_2.$$

Then we show that $\mathcal{P}B_r \subset B_r$, where $\mathcal{P}$ is defined in (3.1). For $(u,v) \in B_r$, it follows by assumption (M_2) that

$$\begin{aligned} |f(s,u(s),v(s))| &= |f(s,u(s),v(s)) - f(s,0,0) + f(s,0,0)| \\ &\leq |f(s,u(s),v(s)) - f(s,0,0)| + |f(s,0,0)| \\ &\leq \ell_1(\|u\| + \|v\|) + \mu_1 \leq \ell_1\|(u,v)\| + \mu_1 \leq \ell_1 r + \mu_1. \end{aligned}$$

Similarly, we have

$$|g(s,u(s),v(s))| \leq \ell_2 r + \mu_2.$$

Then, for $(u,v) \in B_r$, it follows by using the arguments used in the previous theorem that

$$\begin{aligned} \|\mathcal{P}_1(u,v)\| &\leq [\ell_1 r + \mu_1]\Big[\frac{1}{n!} + \overline{T}_2\omega_2 + \overline{T}_3\omega_3 + \overline{T}_4\omega_6\Big] + [\ell_2 r + \mu_2]\Big[\overline{T}_1\omega_1 + \overline{T}_3\omega_5 + \overline{T}_4\omega_4\Big] \\ &\leq r_1(\ell_1 r + \mu_1) + \bar{r}_1(\ell_2 r + \mu_2). \end{aligned}$$

Similarly, we get

$$\|\mathcal{P}_2(u,v))\| \leq r_2(\ell_1 r + \mu_1) + \bar{r}_2(\ell_2 r + \mu_2).$$

From the above estimates together with (3.10), it follows that $\|\mathcal{P}(u,v)\| \leq r$. Therefore, $\mathcal{P}B_r \subset B_r$ as $(u,v) \in B_r$ is an arbitrary element. Next, we show that the operator $\mathcal{P}$ is a contraction. For $(u_1,v_1),(u_2,v_2) \in \mathcal{Q}\times\mathcal{Q}$, we have

$$\begin{aligned} &|\mathcal{P}_1(u_1,v_1)(t) - \mathcal{P}_1(u_2,v_2)(t)| \\ &\leq \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |f(s,u_1,v_1) - f(s,u_2,v_2)|\, ds + |T_1(t)| \left[\zeta_1 \int_0^\eta \frac{(\eta-s)^{m-1}}{(m-1)!} |g(s,u_1,v_1) - g(s,u_2,v_2)|\, ds\right] \\ &+ |T_2(t)| \left[\zeta_2 \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} |f(s,u_1,v_1) - f(s,u_2,v_2)|\, ds\right] \\ &+ |T_3(t)| \Bigg[\int_0^1 \int_0^s \frac{(s-r)^{m-1}}{(m-1)!} |g(r,u_1,v_1) - g(r,u_2,v_2)|\, dr d\mu(s) \\ &+ \int_0^1 \frac{(1-s)^{n-2}|\alpha_1(1-s) + \beta_1(n-1)|}{(n-1)!} |f(s,u_1,v_1) - f(s,u_2,v_2)|\, ds \Bigg] \\ &+ |T_4(t)| \Bigg[\int_0^1 \int_0^s \frac{(s-r)^{n-1}}{(n-1)!} |f(r,u_1,v_1) - f(r,u_2,v_2)|\, dr d\mu(s) \\ &+ \int_0^1 \frac{(1-s)^{m-2}|\alpha_2(1-s) + \beta_2(m-1)|}{(m-1)!} |g(s,u_1,v_1) - g(s,u_2,v_2)|\, ds \Bigg] \Bigg\} \\ &\leq \ell_1 \Big[\frac{1}{n!} + \overline{T}_2\omega_2 + \overline{T}_3\omega_3 + \overline{T}_4\omega_6\Big](||u_1 - u_2|| + ||v_1 - v_2||) + \ell_2 \Big[\overline{T}_1\omega_1 + \overline{T}_3\omega_5 + \overline{T}_4\omega_4\Big](||u_1 - u_2|| + ||v_1 - v_2||) \\ &\leq (\ell_1 r_1 + \ell_2 \bar{r}_1)(|u_1 - u_2| + |v_1 - v_2|), \end{aligned}$$

which, on taking the norm for $t \in [0,1]$, yields

$$\|\mathcal{P}_1(u_1,v_1) - \mathcal{P}_1(u_2,v_2)\| \leq (\ell_1 r_1 + \ell_2 \bar{r}_1)(||u_1 - u_2|| + ||v_1 - v_2||). \tag{3.11}$$

In a similar manner, we get

$$\|\mathcal{P}_2(u_1,v_1) - \mathcal{P}_2(u_2,v_2)\| \leq (\ell_1 r_2 + \ell_2 \bar{r}_2)(||u_1 - u_2|| + ||v_1 - v_2||). \tag{3.12}$$

From (3.11) and (3.12), we deduce that

$$\|\mathcal{P}(u_1,v_1)-\mathcal{P}(u_2,v_2)\| \le (R_1\ell_1+R_2\ell_2)(\|u_1-u_2\|+\|v_1-v_2\|), \tag{3.13}$$

where R_1 and R_2 are given in (3.4). By the assumption(3.9), it follows from 3.13 that the operator $\mathcal{P}$ is a contraction. Thus, by the Banach's contractions mapping principle, the operator $\mathcal{P}$ has a unique fixed point, which corresponds to a unique solution to the problem (1.1) on $[0,1]$. □

Example 3.4. Consider the following boundary value problem:

$$\begin{cases} u^{(4)}(t) = \dfrac{e^{-t}}{7} + \dfrac{u\sin(v)}{\sqrt{t^2+16}} + \dfrac{1}{\sqrt{t^2+4}}\dfrac{|v^2|}{(1+|v|)}, & t\in[0,1],\\ v^{(3)}(t) = \dfrac{1}{t^4+5} + \dfrac{e^{-t}}{t^2+4}\sin(u) + \dfrac{v}{(t^2+3)}\dfrac{|u^2|}{(1+|u|)}, & t\in[0,1],\\ u(0)=\zeta_1\displaystyle\int_0^{\eta} v(s)\,ds, \quad u'(0)=0, \quad u''(0)=0, \\ v(0)=\zeta_2\displaystyle\int_0^{\eta} u(s)\,ds, \quad v'(0)=0, \quad v''(0)=0, \\ \alpha_1 u(1)+\beta_1 u'(1) = \displaystyle\int_0^1 v(s)d\mu s, \ \ \alpha_2 v(1)+\beta_2 v'(1) = \int_0^1 u(s)d\mu s, \end{cases} \tag{3.14}$$

where $\zeta_1=0.1,\ \zeta_2=0.2,\ \alpha_1=2.3,\ \alpha_2=1.6,\ \beta_1=0.9,\ \beta_2=1.2,\ \eta=0.5,\ \mu(s)=s^2.$

Using the given data in (2.4), (2.19) and (3.4), we find that $\overline{T}_1\approx 0.604769,\ \overline{T}_2\approx 0.217167,\ \overline{T}_3\approx 0.203422,\ \overline{T}_4\approx 0.029208,\ \overline{T}_5\approx 0.330367,\ \overline{T}_6\approx 0.648649,\ \overline{T}_7\approx 0.023645,\ \overline{T}_8\approx 0.255020,\ \omega_1\approx 0.002083,\ \omega_2\approx 0.000521,\ \omega_3\approx 0.245833,\ \omega_4\approx 0.866667,\ \omega_5\approx 0.066667,\ \omega_6\approx 0.013889,\ r_1\approx 0.092193,\ r_2\approx 0.009692,\ \bar{r}_1\approx 0.040135,\ \bar{r}_2\approx 0.389949,\ R_1\approx 0.101886,\ R_2\approx 0.430084$. Also, it is easy to find that

$$|f(t,u,v)| \le \frac{1}{7}+\frac{1}{4}|u|+\frac{1}{2}|v|, |g(t,u,v)| \le \frac{1}{5}+\frac{1}{4}|u|+\frac{1}{3}|v|,$$

$R_1m_1+R_2n_1\approx 0.132992<1$ and $R_1m_2+R_2n_2\approx 0.194304<1$. Clearly, all the assumptions of Theorem 3.2 are satisfied. Therefore, there exists at least one solution to the problem (3.14).

Example 3.5. Consider the problem (3.14) with $f(t,u,v)$ and $g(t,u,v)$ given by

$$\begin{cases} f(t,u,v) = \dfrac{1}{\sqrt{t^2+100}}\tan^{-1}(u) + \dfrac{1}{(t^2+10)}\dfrac{|v|}{(1+|v|)} + \dfrac{e^{-t}}{4}, \\ g(t,u,v) = \dfrac{e^{-t}}{t^2+2}\sin(u) + \dfrac{1}{\sqrt{t^2+4}}\cos(v) + \dfrac{t^2+4}{\sqrt{t^3+4}}. \end{cases} \tag{3.15}$$

Note that $\ell_1=1/10$ and $\ell_2=1/2$ as

$$|f(t,u_1,v_1)-f(t,u_2,v_2)| \le \frac{1}{10}(|u_1-u_2|+|v_1-v_2|),$$

$$|g(t,u_1,v_1)-g(t,u_2,v_2)| \le \frac{1}{2}(|u_1-u_2|+|v_1-v_2|).$$

Moreover, $R_1\ell_1+R_2\ell_2\approx 0.225230<1$. Clearly the conditions of Theorem 3.3 are satisfied. Hence, the problem (3.14) with f and g given in (3.15) has a unique solution on $[0,1]$.

4 Conclusions

In this paper, we have derived the existence and uniqueness results for a coupled system of two higher order nonlinear ordinary differential equations supplemented with nonlocal and Stieltjes type coupled boundary conditions. Our results are new in the given configuration and enrich the literature on nonlocal coupled boundary value problems of systems of higher order nonlinear ordinary differential equations. Furthermore, some new results arise from the present ones as special cases by fixing the parameters in the boundary conditions. For instance, by taking $\zeta_1 = 0 = \zeta_2$, our results correspond to the ones associated with the boundary conditions

$$\begin{cases} u(0) = 0, u'(0) = 0, u^{''}(0) = 0, \dots, u^{(n-2)}(0) = 0, \\ v(0) = 0, v'(0) = 0, v^{''}(0) = 0, \dots, v^{(m-2)}(0) = 0, \\ \alpha_1 u(1) + \beta_1 u'(1) = \displaystyle\int_0^1 v(s)d\mu(s),\ \alpha_2 v(1) + \beta_2 v'(1) = \int_0^1 u(s)d\mu(s). \end{cases}$$

In case we take $\beta_1 = 0 = \beta_2$ and $\alpha_1 = 1 = \alpha_2$ in the present results, we get the ones for the boundary conditions of the form

$$\begin{cases} u(0) = \zeta_1 v(\eta), u'(0) = 0, u^{''}(0) = 0, \dots, u^{(n-2)}(0) = 0, \\ v(0) = \zeta_2 u(\eta), v'(0) = 0, v^{''}(0) = 0, \dots, v^{(m-2)}(0) = 0, \\ u(1) = \displaystyle\int_0^1 v(s)d\mu(s),\ v(1) = \int_0^1 u(s)d\mu(s). \end{cases}$$

Letting $\alpha_1 = 0 = \alpha_2$ and $\beta_1 = 1 = \beta_2$, our results correspond to the ones with the boundary conditions

$$\begin{cases} u(0) = \zeta_1 v(\eta), u'(0) = 0, u^{''}(0) = 0, \dots, u^{(n-2)}(0) = 0, \\ v(0) = \zeta_2 u(\eta), v'(0) = 0, v^{''}(0) = 0, \dots, v^{(m-2)}(0) = 0, \\ u'(1) = \displaystyle\int_0^1 v(s)d\mu(s),\ v'(1) = \int_0^1 u(s)d\mu(s). \end{cases}$$

In future, we plan to extend our work by considering boundary conditions involving multipoint, multi-strip and flux-type integral terms. We will also study the multivalued variant of the problem (1.1).

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