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Notes on matrix inequalities and positive multilinear mappings

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Abstract

In this paper, our aim is to prove some matrix inequalities involving arbitrary matrix means and positive multilinear mappings. For example, it is shown that for Hermitian matrices A_i, B_i such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$,

$$\Phi^2(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k) \leq \left(K(u^k)\right)^2 \Phi^2(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k),$$

where σ_1 and σ_2 are two arbitrary matrix means between the arithmetic and harmonic means, $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$ is a positive unital multilinear mapping, $u = \frac{M}{m}$ and $K(u) = \frac{(1+u)^2}{4u}$. We also give the obtained results for the adjoint and the dual of an arbitrary matrix mean.

Keywords: Matrix mean, Positive multilinear mapping, Positive matrix
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1 Introduction

Let $\mathscr{M}_n(\mathbb{C})$ denote the C^*-algebra of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$ with the identity I. For a Hermitian matrix $A \in \mathscr{M}_n(\mathbb{C})$, we write $A \geq 0$ if all its eigenvalues are non-negative. We also write $A > 0$, if it be invertible, moreover the before condition. For two Hermitian matrices $A, B \in \mathscr{M}_n(\mathbb{C})$, we use the nonation $A \leq B$(or $A > B$) to mean that $B - A \geq 0$(or $B - A > 0$). A mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) = \mathscr{M}_n(\mathbb{C}) \times \cdots \times \mathscr{M}_n(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$ is said to be multilinear whenever it is linear in each of its variable and also is called positive if $A_i \geq 0$ for $i = 1, \cdots, k$ implies that $\Phi(A_1, \cdots, A_k) \geq 0$. Moreover, Φ is called unital if $\Phi(I, \cdots, I) = I$. If $A \in \mathscr{M}_n(\mathbb{C})$ is a Hermitian matrix, the Gelfand map $f(t) \mapsto f(A)$ is an isometrically $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of the Hermitian matrix A and the C^*-algebra generated by I and A. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$. For $J \subset \mathbb{R}$, a real valued continuous function $f : J \to \mathbb{R}$ is called matrix monotone if $A \leq B$ implies that $f(A) \leq f(B)$ for all Hermitian matrices A and B whose eigenvalues are in J. Kubo and Ando [1] defined the matrix mean σ for pairs of positive definite matrices A and B and the nonnegative matrix monotone function $f : (0, \infty) \to (0, \infty)$ with $f(1) = 1$

$$A\sigma B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

where f is the representing function for σ.

For two positive definite matrices $A, B \in \mathscr{M}_n(\mathbb{C})$, the Löwner–Heinz inequality states that, if $A \leq B$, then

$$A^p \leq B^p, \qquad (0 \leq p \leq 1). \tag{1.1}$$

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In general (1.1) is not true for $p > 1$. The authors [8] proved that if A and B are two positive definite matrices such that $0 < m \leq A, B \leq M$, then

$$\Phi^2(A\sigma_1 B) \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 \Phi^2(A\sigma_2 B), \tag{1.2}$$

where Φ is a positive unital linear mapping and σ_1, σ_2 are two arbitrary matrix means between the harmonic and arithmetic matrix means. The authors in [6] generalized the inequality (1.2) for $p > 0$ as follows:

$$\Phi^p(A\sigma_1 B) \leq \left(\frac{(M+m)^2}{4Mm}\right)^p \Phi^p(A\sigma_2 B), \tag{1.3}$$

where $0 < m \leq A, B \leq M$, Φ is a positive unital linear mapping, σ_1, σ_2 are two arbitrary matrix means between the harmonic and arithmetic matrix means. For more information on the above inequalities see [5, 10, 12, 13, 14, 15, 7] and references therein. In this paper, we intend to extend the inequality (1.3) for multilinear positive mappings. Then, we shall generalize the derived results for the adjoint and the dual of an arbitrary matrix mean.

2 Main results

2.1 Some inequalities for arbitrary matrix means

This section is started by recalling several well known lemmas.

Lemma 2.1. [4] Suppose that $A_i \in \mathscr{M}_n(\mathbb{C})$ $(i = 1, \cdots, k)$ be positive definite. Then for every unital multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$

$$\Phi(A_1, \cdots, A_k)^{-1} \leq \Phi(A_1^{-1}, \cdots, A_k^{-1}).$$

Lemma 2.2. [1, 2, 3] Let $A, B \in \mathscr{M}_n(\mathbb{C})$ be positive definite and $\alpha > 0$. Then
(i) $||\mathrm{AB}|| \leq \frac{1}{4}||\mathrm{A}+\mathrm{B}||^2$.
(ii) $||\mathrm{A}^\alpha + \mathrm{B}^\alpha|| \leq ||(\mathrm{A}+\mathrm{B})^\alpha||$ for $\alpha \geq 1$.
(iii) $\mathrm{A} \leq \alpha \mathrm{B}$ if and only if $||A^{\frac{1}{2}}B^{-\frac{1}{2}}|| \leq \alpha^{\frac{1}{2}}$.

Lemma 2.3. [9] Let $X \in \mathscr{M}_n(\mathbb{C})$. Then $\|X\| \leq t$ if and only if

$$\begin{pmatrix} tI & X \\ X^* & tI \end{pmatrix} \geq 0.$$

Lemma 2.4. [13] Let f be a strictly positive convex twice differential function on the interval $[m, M]$ with $0 < m < M$ and let $C_i \in \mathscr{M}_n(\mathbb{C})$ such that $\sum_{i=1}^k C_i^* C_i = I$. If $A_i \in M_n(\mathbb{C})$ such that $0 < m \leq A_i \leq M$ $(i = 1, \cdots, k)$, then

$$\sum_{i=1}^k C_i^* f(A_i) C_i \leq a_f \sum_{i=1}^k C_i^* A_i C_i + b_f I \leq \alpha f\left(\sum_{i=1}^k C_i^* A_i C_i\right),$$

where $a_f = \frac{f(M)-f(m)}{M-m}$, $b_f = \frac{Mf(m)-mf(M)}{M-m}$ and $\alpha = \max_{m \leq t \leq M}\left\{\frac{a_f t + b_f}{f(t)}\right\}$.

First, we prove following efficient lemma.

Lemma 2.5. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ be such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$, for some scalars $0 < m < M$. Then, for every positive unital multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$,

$$\Phi(A_1, \cdots, A_k) + M^k m^k \Phi(A_1^{-1}, \cdots, A_k^{-1}) \leq M^k + m^k.$$

Proof . It follows the spectral decomposition of $A_i \in \mathscr{M}_n(\mathbb{C})$ that $A_i = \sum_{j=1}^n \lambda_{ij} Q_{ij}$ $(i = 1, \cdots, k)$ such that $\sum_{j=1}^n Q_{ij} = I$. Putting

$$C(j_1, \cdots, j_k) = (\Phi(Q_{1j_1}, \cdots, Q_{kj_k}))^{\frac{1}{2}}$$

such that
$$\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_k=1}^{n} C^*(j_1,\cdots,j_k)C(j_1,\cdots,j_k)=I.$$

One can get

$$\begin{aligned}
&\Phi(A_1^{-1},\cdots,A_k^{-1})\\
&\le \Phi\left(\sum_{j=1}^{n}\lambda_{1j}^{-1}Q_{1j},\cdots,\sum_{j=1}^{n}\lambda_{kj}^{-1}Q_{kj}\right) \quad \text{(by the convexity of } f(t)=t^{-1})\\
&=\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_k=1}^{n}\lambda_{1j_1}^{-1}\lambda_{2j_2}^{-1}\cdots\lambda_{kj_k}^{-1}\Phi(Q_{1j_1},\cdots,Q_{kj_k}) \quad \text{(by the multilinearity of }\Phi)\\
&\le a_1\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_k=1}^{n} C(j_1,\cdots,j_k)\lambda_{1j_1}\lambda_{2j_2}\cdots\lambda_{kj_k}C(j_1,\cdots,j_k)+b_1 I\\
&\qquad\qquad\qquad\qquad \text{(by Lemma 2.4)}\\
&=a_1\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_k=1}^{n}\lambda_{1j_1}\lambda_{2j_2}\cdots\lambda_{kj_k}\Phi(Q_{1j_1},\cdots,Q_{kj_k})+b_1 I\\
&=a_1\Phi(A_1,A_2,\cdots,A_k)+b_1 I,
\end{aligned}$$

where $a_1=\frac{-1}{M^k m^k}$ and $b_1=\frac{M^k+m^k}{M^k m^k}$. Therefore,
$$\Phi(A_1,\cdots,A_k)+M^k m^k\Phi(A_1^{-1},\cdots,A_k^{-1})\le M^k+m^k.$$

This proves the desired inequality. □

By applying Lemma 2.5, we have the following result:

Lemma 2.6. Let $A_i,B_i\in\mathscr{M}_n(\mathbb{C})$ be such that $0<m\le A_i,B_i\le M$ $(i=1,\cdots,k)$, for some scalars $0<m<M$ and $\Phi:\mathscr{M}_n^k(\mathbb{C})\to\mathscr{M}_l(\mathbb{C})$ be a positive unital multilinear mapping, then
$$\Phi(A_1\nabla B_1,\cdots,A_k\nabla B_k)+M^k m^k\Phi(A_1^{-1}\nabla B_1^{-1},\cdots,A_k^{-1}\nabla B_k^{-1})\le(M^k+m^k). \tag{2.1}$$

Proof . Since Φ is multilinear and by Lemma 2.5, it follows that
$$\begin{aligned}
&\Phi(A_1\nabla B_1,\cdots,A_k\nabla B_k)+M^k m^k\Phi(A_1^{-1}\nabla B_1^{-1},\cdots,A_k^{-1}\nabla B_k^{-1})\\
&=\left(\frac{1}{2}\right)^k\left[\Phi(A_1,\cdots,A_k)+M^k m^k\Phi(A_1^{-1},\cdots,A_k^{-1})+\Phi(B_1,\cdots,B_k)+M^k m^k\Phi(B_1^{-1},\cdots,B_k^{-1})\right]\\
&\le(M^k+m^k).
\end{aligned}$$

□

Remark 2.7. If we choose $k=1$, in the inequality (2.1), then we get
$$\Phi(A\nabla B)+Mm\Phi(A^{-1}\nabla B^{-1})\le M+m.$$

Lemma 2.8. Let $A_i,B_i\in\mathscr{M}_n(\mathbb{C})$ be such that $0<A_i\le B_i$ $(i=1,\cdots,k)$ and $\Phi:\mathscr{M}_n^k(\mathbb{C})\to\mathscr{M}_l(\mathbb{C})$ be any positive unital multilinear mapping. Then
$$\Phi(A_1,\cdots,A_k)\le\Phi(B_1,\cdots,B_k). \tag{2.2}$$

Proof . Since $0<A_i\le B_i$ $(i=1,\cdots,k)$, and in result $B_i-A_i\ge 0$. Thus, we can write
$$\begin{aligned}
\Phi(A_1,A_2,\cdots,A_k)&\le\Phi(B_1,A_2\cdots,A_k)\\
&\le\Phi(B_1,B_2,\cdots,A_k)\\
&\le\Phi(B_1,B_2,\cdots,B_k).
\end{aligned}$$

□

Theorem 2.9. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ be such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$, for some scalars $0 < m < M$ and let σ_1 and σ_2 be two arbitrary matrix means between the arithmetic and harmonic means. Then, for every positive unital multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$,

$$\Phi^2(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) \leq \left(K(u^k)\right)^2 \Phi^2(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k), \tag{2.3}$$

where $u = \frac{M}{m}$ and $K(u) = \frac{(1+u)^2}{4u}$.

Proof . By our assumption and using Lemma 2.6, we have

$$\begin{aligned} &\Phi(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + M^km^k\Phi((A_1\sigma_2B_1)^{-1}, \cdots, (A_k\sigma_2B_k)^{-1}) \\ &\leq \Phi(A_1\nabla B_1, \cdots, A_k\nabla B_k) + M^km^k\Phi((A_1!B_1)^{-1}, \cdots, (A_k!B_k)^{-1}) \\ &= \Phi(A_1\nabla B_1, \cdots, A_k\nabla B_k) + M^km^k\Phi(A_1^{-1}\nabla B_1^{-1}, \cdots, A_k^{-1}\nabla B_k^{-1}) \\ &\leq (M^k + m^k). \end{aligned} \tag{2.4}$$

Applying Lemma 2.2(i), Lemma 2.1, respectively, and utilizing (2.4), we have

$$\begin{aligned} &\left\|\Phi(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k)M^km^k\Phi^{-1}(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k)\right\| \\ &\leq \frac{1}{4}\left\|\Phi(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + M^km^k\Phi^{-1}(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k)\right\|^2 \\ &\leq \frac{1}{4}\left\|\Phi(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + M^km^k\Phi((A_1\sigma_2B_1)^{-1}, \cdots, (A_k\sigma_2B_k)^{-1})\right\|^2 \\ &\leq \frac{(M^k + m^k)^2}{4}, \end{aligned}$$

which completes the proof. □

Remark 2.10. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ be such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$, let σ_1 and σ_2 be two arbitrary matrix means between the arithmetic and harmonic means and $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$ be a positive unital multilinear mapping. Then,

$$\Phi^p(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) \leq \left(K(u^k)\right)^p \Phi^p(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k), \qquad 0 \leq p \leq 2 \tag{2.5}$$

where $K(u) = \frac{(1+u)^2}{4u}$ and $u = \frac{M}{m}$.

Note that for $0 \leq p \leq 2$, it follows $0 \leq \frac{p}{2} \leq 1$. So, by the (2.3) and the Lowner-Heinz inequality, we have (2.5).

The following theorem is an extension of inequality (2.5) for $p \geq 2$.

Theorem 2.11. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$ and let σ_1 and σ_2 be two arbitrary matrix means between the arithmetic and harmonic means. Then for every positive unital multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$ and $p \geq 2$,

$$\Phi^p(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) \leq \left(K(u^k)\right)^p \Phi^p(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k), \tag{2.6}$$

where $u = \frac{M}{m}$ and $K(u) = \frac{(1+u)^2}{4u}$.

Proof . By Lemma 2.2(i) and (ii), Lemma 2.1 and (2.4), respectivly, one can get

$$\begin{aligned} &\left\|\Phi^{\frac{p}{2}}(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k)M^{\frac{pk}{2}}m^{\frac{pk}{2}}\Phi^{-\frac{p}{2}}(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k)\right\| \\ &\leq \frac{1}{4}\left\|\Phi^{\frac{p}{2}}(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + M^{\frac{pk}{2}}m^{\frac{pk}{2}}\Phi^{-\frac{p}{2}}(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k)\right\|^2 \\ &\leq \frac{1}{4}\left\|(\Phi(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + M^km^k\Phi((A_1\sigma_2B_1)^{-1}, \cdots, (A_k\sigma_2B_k)^{-1}))^{\frac{p}{2}}\right\|^2 \\ &= \frac{1}{4}\left\|\Phi(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + M^km^k\Phi((A_1\sigma_2B_1)^{-1}, \cdots, (A_k\sigma_2B_k)^{-1})\right\|^p \\ &\leq \frac{(M^k + m^k)^p}{4}. \end{aligned}$$

This shows (2.6). □

Remark 2.12. If we take $k = 1$, the inequality (2.6) reduces the inequality (1.3). So, (2.6) is an extension of (1.3).

Now, we prove the folllowing useful result to present the another extension of inequality (2.5) for $p \geq 4$.

Lemma 2.13. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ be such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$, for some scalars $0 < m < M$ and let σ_1 and σ_2 be two arbitrary matrix means between the arithmetic and harmonic means. Then, for every positive unital multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$, we have

$$\Phi^2(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + M^{2k}m^{2k}\Phi^{-2}(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) \leq (M^{2k} + m^{2k}). \tag{2.7}$$

Proof . From $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$ and (2.2), it is clear that

$$\Phi(mI, \cdots, mI) = m^k \leq \Phi(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) \leq \Phi(MI, \cdots, MI) = M^k. \tag{2.8}$$

We know that for every matrix T such that $0 < m \leq T \leq M$,

$$M^2m^2T^{-2} + T^2 \leq M^2 + m^2.$$

If we apply the same property for (2.8), (2.7) deduces. □

In below, we obtain the another extension of (2.5) for $p \geq 4$.

Theorem 2.14. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$ and let σ_1 and σ_2 be two arbitrary matrix means between the arithmetic and harmonic means. Then for every positive unital multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$ and $p \geq 4$,

$$\Phi^p(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) \leq \left(\frac{K(u^k)(M^{2k} + m^{2k})}{4^{\frac{2}{p}}M^km^k}\right)^p \Phi^p(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k), \tag{2.9}$$

where $u = \frac{M}{m}$ and $K(u) = \frac{(1+u)^2}{4u}$.

Proof . By(2.3) and (2.7), we get

$$\begin{aligned}
&K(u^k)\Phi^2(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + \frac{M^{2k}m^{2k}}{K(u^k)}\Phi^{-2}(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k) \\
&\leq K(u^k)\Phi^2(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + M^{2k}m^{2k}K(u^k)\Phi^{-2}(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) \\
&\leq K(u^k)(M^{2k} + m^{2k}).
\end{aligned} \tag{2.10}$$

Applying Lemma 2.2(i) and (ii), together with (2.10), we derive

$$\begin{aligned}
&\left\|\Phi^{\frac{p}{2}}(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k)M^{\frac{pk}{2}}m^{\frac{pk}{2}}\Phi^{-\frac{p}{2}}(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k)\right\| \\
&\leq \frac{1}{4}\left\|K^{\frac{p}{4}}(u^k)\Phi^{\frac{p}{2}}(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + \left(\frac{M^{2k}m^{2k}}{K(u^k)}\right)^{\frac{p}{4}}\Phi^{-\frac{p}{2}}(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k)\right\|^2 \\
&\leq \frac{1}{4}\left\|(K(u^k)\Phi^2(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + \frac{M^{2k}m^{2k}}{K(u^k)}\Phi^{-2}(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k))^{\frac{p}{4}}\right\|^2 \\
&= \frac{1}{4}\left\|K(u^k)\Phi^2(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) + \frac{M^{2k}m^{2k}}{K(u^k)}\Phi^{-2}(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k)\right\|^{\frac{p}{2}} \\
&\leq \frac{(K(u^k)(M^{2k} + m^{2k}))^{\frac{p}{2}}}{4},
\end{aligned}$$

which leads to the desired result. □

Theorem 2.15. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ be such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$ with $0 < m < M$ and let σ_1 and σ_2 be two arbitrary matrix means between the arithmetic and harmonic means. Then, for every positive unital multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$ and $1 \leq \alpha \leq 2$,

$$\Phi^p(A_1\sigma_1B_1, \cdots, A_k\sigma_1B_k) \leq \frac{\left(K^{\frac{\alpha}{2}}(u^k)(M^{k\alpha} + m^{k\alpha})\right)^{\frac{2p}{\alpha}}}{16M^{kp}m^{kp}}\Phi^p(A_1\sigma_2B_1, \cdots, A_k\sigma_2B_k), \quad p \geq 2\alpha, \tag{2.11}$$

where $u = \frac{M}{m}$ and $K(u) = \frac{(1+u)^2}{4u}$ is the Kantorovich constant.

Proof . For $1 \leq \alpha \leq 2$, the (2.3) and the Löwner–Heinz inequality ensures us that

$$\Phi^{\alpha}(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k) \leq \left(K(u^k)\right)^{\alpha} \Phi^{\alpha}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k),$$

which is equivalent to

$$\Phi^{-\alpha}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k) \leq \left(K(u^k)\right)^{\alpha} \Phi^{-\alpha}(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k). \tag{2.12}$$

Making use of our assumptions, one can easily prove that

$$\Phi^{\alpha}(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k) + M^{k\alpha}m^{k\alpha}\Phi^{-\alpha}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k) \leq M^{k\alpha} + m^{k\alpha}. \tag{2.13}$$

To prove (2.11), from Lemma 2.2(i) and (ii) and inequalities (2.12) and (2.13), we have

$$\begin{aligned}
&\left\| M^{\frac{kp}{2}} m^{\frac{kp}{2}} \Phi^{\frac{p}{2}}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k)\Phi^{-\frac{p}{2}}(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k)\right\| \\
&\leq \frac{1}{4}\left\| K^{-\frac{p}{4}}(u^k) M^{\frac{kp}{2}} m^{\frac{kp}{2}} \Phi^{-\frac{p}{2}}(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k) + K^{\frac{p}{4}}(u^k)\Phi^{\frac{p}{2}}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k)\right\|^2 \\
&\leq \frac{1}{4}\left\| \left(K^{-\frac{\alpha}{2}}(u^k) M^{k\alpha} m^{k\alpha} \Phi^{-\alpha}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k) + K^{\frac{\alpha}{2}}(u^k)\Phi^{\alpha}(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k)\right)^{\frac{p}{2\alpha}}\right\|^2 \\
&= \frac{1}{4}\left\| K^{-\frac{\alpha}{2}}(u^k) M^{k\alpha} m^{k\alpha} \Phi^{-\alpha}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k) + K^{\frac{\alpha}{2}}(u^k)\Phi^{\alpha}(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k)\right\|^{\frac{p}{\alpha}} \\
&\leq \frac{K^{\frac{p}{2}}(u^k)(M^{k\alpha} + m^{k\alpha})^{\frac{p}{\alpha}}}{4},
\end{aligned}$$

This is equivalent to

$$\Phi^{p}(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k) \leq \frac{\left(K^{\frac{\alpha}{2}}(u^k)(M^{k\alpha} + m^{k\alpha})\right)^{\frac{2p}{\alpha}}}{16M^{kp}m^{kp}} \Phi^{p}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k)$$

This shows the desired inequality. □

Remark 2.16. Choosing $\alpha = 1$ and $\alpha = 2$, in the inequality (2.11), we get to the inequalities (2.6) and (2.9), respectively.

Theorem 2.17. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ be such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$, when $0 < m < M$ and let σ_1 and σ_2 be two arbitrary matrix means between the arithmetic and harmonic means. Then, for every positive unital multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$,

$$\|\gamma_2 + \gamma_1\| \leq 2K^p(u^k), \qquad p > 0,$$

where

$$\begin{aligned}
\gamma_1 &= \Phi^{\frac{p}{2}}(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k)\Phi^{\frac{-p}{2}}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k), \\
\gamma_2 &= \Phi^{\frac{-p}{2}}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k)\Phi^{\frac{p}{2}}(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k)
\end{aligned}$$

and $K(u) = \frac{(1+u)^2}{4u}$.

Proof . From inequality (2.6), it follows that

$$\|\Phi^{p}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k)\Phi^{-p}(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k)\| \leq \left(K(u^k)\right)^{p}.$$

Lemma 2.3 leads to

$$\begin{pmatrix} K^p(u^k)I & \gamma_1 \\ \gamma_2 & K^p(u^k)I \end{pmatrix} \geq 0$$

and

$$\begin{pmatrix} K^p(u^k)I & \gamma_2 \\ \gamma_1 & K^p(u^k)I \end{pmatrix} \geq 0,$$

where

$$\gamma_1 = \Phi^p(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k)\Phi^{-p}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k)$$

and

$$\gamma_2 = \Phi^{-p}(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k)\Phi^p(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k).$$

By summing up the two above inequalities, we obtain the following inequality:

$$\begin{pmatrix} 2(K^p(u^k))I & \gamma_1+\gamma_2 \\ \gamma_2+\gamma_1 & 2(K^p(u^k))I \end{pmatrix} \geq 0.$$

Again using Lemma 2.3, we obtain result. □

Now, utilizing the continuous functional calculus, we show the following result.

Theorem 2.18. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ be such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$, σ_1 and σ_2 be two arbitrary matrix means between the arithmetic and harmonic means and $f, g : [m, M] \to [0, \infty)$ be two continuous functions such that g is non-zero, increasing and concave. Then, for every positive unital multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$

$$f(\Phi(A_1\sigma_1 B_1, \cdots, A_n\sigma_1 B_n)) \leq \gamma g(\Phi(A_1\sigma_2 B_1, \cdots, A_n\sigma_2 B_n))$$

where $\alpha_g = \frac{g(M)-g(m)}{M-m}$, $\beta_g = \frac{Mg(m)-mg(M)}{M-m}$, $u = \frac{M}{m}$, $K(u) = \frac{(1+u)^2}{4u}$ and

$$\gamma = \max\left\{\frac{f(t)}{\alpha_g K^{-1}(u^k)t + \beta_g} : t \in [m, M]\right\}.$$

Proof . It follows from concavity of g that for $t \in [m, M]$

$$g(t) \geq \alpha_g t + \beta_g.$$

On the other hand, by our assumptions $m \leq \Phi(A_1\sigma_1 B_1, \cdots, A_n\sigma_1 B_n) \leq M$. Applying monotonicity principle for operator functions and using lowner Heinz inequality for(2.3), we get

$$\begin{aligned} g(\Phi(A_1\sigma_2 B_1, \cdots, A_n\sigma_2 B_n)) &\geq \alpha_g \Phi(A_1\sigma_2 B_1, \cdots, A_n\sigma_2 B_n) + \beta_g \\ &\geq \alpha_g K^{-1}(u^k)\Phi(A_1\sigma_1 B_1, \cdots, A_n\sigma_1 B_n) + \beta_g \end{aligned} \tag{2.14}$$

where $K(u^k) = \frac{(1+u^k)^2}{4u^k}$. From $\alpha_g K^{-1}(u^k)t + \beta_g \neq 0$ for $t \in [m, M]$, it concludes that the function $\frac{f(t)}{\alpha_g K^{-1}(u^k)t+\beta_g}$ is continuous on the interval $[m, M]$. Now, we can put

$$\gamma = \max\left\{\frac{f(t)}{\alpha_g K^{-1}(u^k)t + \beta_g} : t \in [m, M]\right\}.$$

Therefore, $\gamma(\alpha_g K^{-1}(u^k)t + \beta_g) \geq f(t)$. Again with the aid of monotonicity principle for operator functions, we get

$$\gamma(\alpha_g K^{-1}(u^k)\Phi(A_1\sigma_1 B_1, \cdots, A_n\sigma_1 B_n) + \beta_g) \geq f(\Phi(A_1\sigma_1 B_1, \cdots, A_n\sigma_1 B_n)). \tag{2.15}$$

Combining inequalities (2.14) and (2.15), we derive the desired result. □

3 Some inequalities for dual and adjoint of matrix means

The authors in [11] introduced the adjoint and the dual of a matrix mean as follows: Let σ be an operator mean with representing function f. A matrix mean with representing function $f(t^{-1})^{-1}$ and $tf(t)^{-1}$ are called the adjoint and the dual of σ, respectively, and are denoted by σ^* and $\sigma^{\perp}$, respectively. Thus by the definition, we have

$$A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1} \quad \text{and} \quad A\sigma^{\perp} B = (B^{-1}\sigma A^{-1})^{-1}.$$

The next Theorem is the key result of this section.

Theorem 3.1. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$ with $0 < m < M$ and σ be an arbitrary matrix mean. Then

$$\Phi^2(A_1\sigma B_1, \cdots, A_k\sigma B_k) \leq K^2(u^k)\Phi^2(A_1\sigma^* B_1 \cdots A_k\sigma^* B_k),$$

where σ^* is adjoint σ, $K(u) = \frac{(1+u)^2}{4u}$ and $u = \frac{M}{m}$.

Proof . Applying [4, Proposition 3.6],

$$\begin{aligned}&\Phi(A_1\sigma B_1, \cdots, A_k\sigma B_k) + M^k m^k \Phi(A_1^{-1}\sigma B_1^{-1}, \cdots, A_k^{-1}\sigma B_k^{-1})\\ &\leq \Phi(A_1, \cdots, A_k)\sigma\Phi(B_1, \cdots, B_k) + M^k m^k(\Phi(A_1^{-1}, \cdots, A_k^{-1})\sigma\Phi(B_1^{-1}, \cdots, B_k^{-1})).\end{aligned}$$

By Lemma 2.5 and the subadditivity and the monotonicity property of the matrix mean σ, respectively, we derive

$$\Phi(A_1\sigma B_1, \cdots, A_k\sigma B_k) + M^k m^k \Phi(A_1^{-1}\sigma B_1^{-1}, \cdots, A_k^{-1}\sigma B_k^{-1}) \leq M^k + m^k. \tag{3.1}$$

Using Lemma 2.2(i), Lemma 2.1 and inequality (3.1) and applying a method similar to (2.3), the rest of proof is trivial. □

As a consequence, we have the following result:

Remark 3.2. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$, let σ be an arbitrary matrix mean and $p > 0$. Then,

$$\Phi^p(A_1\sigma B_1, \cdots, A_k\sigma B_k) \leq \left(\frac{(1+u^k)^2}{4u^k}\right)^p \Phi^p(A_1\sigma^* B_1 \cdots A_k\sigma^* B_k)$$

for every positive unital multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_n(\mathbb{C})$ and $u = \frac{M}{m}$. For $0 \leq p \leq 2$, by Theorem 3.1 and the Lowner-Heinz inequality, the result is clear. If $p > 2$, then using a similar method in Theorem 3.2 and using of Lemma 2.2(ii), we get the desired result.

Using the same proof as in Theorem 2.11, we have the following interesting Corollary.

Corollary 3.3. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ be such that $0 < m \leq A_i, B_i \leq M$ $(1 \leq i \leq k)$ with $0 < m < M$, let σ_1 and σ_2 be two arbitrary matrix means between arbitrary matrix mean σ and its adjoint σ^*. Then, for every multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$,

$$\Phi^p(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k) \leq K^p(u^k)\Phi^p(A_1\sigma_2 B_1, \cdots, A_k\sigma_2 B_k), \quad p > 0 \tag{3.2}$$

where $u = \frac{M}{m}$ and $K(u) = \frac{(1+u)^2}{4u}$.

Remark 3.4. Taking $\sigma = \nabla$, then $\sigma^* =!$ and the inequality (3.2) becomes the inequality (2.5). Thus, this result generalize the obtained results in the first section.

Now, we present inequality (1.2) for the dual of a matrix mean.

Theorem 3.5. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$ with $0 < m < M$ and σ be an arbitrary matrix mean. Then, for every multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$

$$\Phi^2(A_1\sigma B_1, \cdots, A_k\sigma B_k) \leq K^2(u^k)\Phi^2(B_1\sigma^\perp A_1 \cdots B_k\sigma^\perp A_k),$$

where $\sigma^\perp$ is the dual of σ, $u = \frac{M}{m}$ and $K(u) = \frac{(1+u)^2}{4u}$.

Proof . Notice that, by Lemma 2.2(i) and the inequality (3.1), we get

$$\left\|\Phi(A\sigma B, \cdots, A\sigma B)M^k m^k \Phi^{-1}(B_1\sigma^\perp A_1, \cdots, B_k\sigma^\perp A_k)\right\| \leq (M^k + m^k).$$

This shows the assertion as desired. □

Using the same idea of the adjoint of a matrix mean, we can state:

Corollary 3.6. Let $A_i, B_i \in \mathscr{M}_n(\mathbb{C})$ be such that $0 < m \leq A_i, B_i \leq M$ $(i = 1, \cdots, k)$ with $0 < m < M$, let σ_1 and σ_2 be two arbitrary matrix means between arbitrary matrix mean σ and its dual $\sigma^\perp$. Then, for every multilinear mapping $\Phi : \mathscr{M}_n^k(\mathbb{C}) \to \mathscr{M}_l(\mathbb{C})$,

$$\Phi^p(A_1\sigma_1 B_1, \cdots, A_k\sigma_1 B_k) \leq K^p(u^k)\Phi^p(B_1\sigma_2 A_1, \cdots B_k\sigma_2 A_k), \quad p > 0,$$

where $u = \frac{M}{m}$ and $K(u) = \frac{(1+u)^2}{4u}$.

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