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Norm of difference of two general polynomial weighted differentiation composition operators from Cauchy transform space into some analytic function spaces

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Abstract

Let for $0 \leq j \leq n$, $\varphi_j, \psi_j : \mathbb{D} \to \mathbb{D}$ and $u_j, v_j : \mathbb{D} \to \mathbb{C}$. In this paper, we investigate boundedness of operator

$$S = \sum_{j=0}^{n} (D^j_{u_j,\varphi_j} - D^j_{v_j,\psi_j})$$

from Cauchy transform space into some analytic function spaces. Also, we obtain an exact formula for the norm of this operator.

Keywords: boundedness, Cauchy transform space, Dirichlet space, mth weighted type space
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1 Introduction

The space of Cauchy transform functions can be viewed as a connection between analytic function theory and measure theory. If f is analytic in $\overline{\mathbb{D}}$, $\mathbb{D}$ is the open unit disk in the complex plane $\mathbb{C}$, then using the Cauchy formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad z \in \mathbb{D}.$$

The above formula is a special case of the Cauchy transforms: A function f is Cauchy transform or $f \in \mathcal{F}$ if it has a representation as

$$f(z) = \int_{\partial\mathbb{D}} \frac{1}{1 - \overline{\zeta} z} d\nu(\zeta), \qquad z \in \mathbb{D}, \tag{1.1}$$

where $\nu \in \mathcal{M}$, $\mathcal{M}$ be the space of all complex valued Borel measures on $\mathbb{T}$ with the total variation norm. For more information about Cauchy transform space see [4, 5, 6, 13]. The space $\mathcal{F}$ is a Banach space equipped with the norm

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$\|f\|_{\mathcal{F}} = \inf \|\nu\|$ where infimum is taken over $\nu \in \mathcal{M}$ for which (1.1) holds. Here $\|\nu\|$ is the total variation of ν. The function $\|.\|_{\mathcal{F}}$ is well-defined, means that for every $f \in \mathcal{F}$ there exists $\nu \in \mathcal{M}$ such that $\|f\|_{\mathcal{F}} = \|\nu\|$. The Banach space $\mathcal{F}$ is clearly the quotient of the Banach space $\mathcal{M}$ by the subspace of measures with vanishing Cauchy transforms. There are some relations between the space of Cauchy transforms and other Banach spaces of analytic function, see [11]. Moreover for every $f \in \mathcal{F}$

$$|f(z)| \leq \frac{\|f\|_{\mathcal{F}}}{1-|z|}, \quad z \in \mathbb{D}$$

so, $\mathcal{F} \subset \mathcal{A}^{-1}$, where $\mathcal{A}^{-1}$ is the growth space. Let μ be a weight (positive and continuous function on $\mathbb{D}$) and $m \in \mathbb{N}_0 = \{0, 1, \cdots\}$. The mth weighted type space $\mathcal{W}_\mu^m$, consists of all functions f analytic in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{W}_\mu^m} = \sum_{i=0}^{m-1} |f^{(i)}(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f^{(m)}(z)| < \infty.$$

It can be verified that the space $\mathcal{W}_\mu^m$ is a Banach space with the norm $\|.\|_{\mathcal{W}_\mu^m}$. The little mth weighted type space $\mathcal{W}_{\mu,0}^m$ is the closed subspace of $\mathcal{W}_\mu^m$ such that for any $f \in \mathcal{W}_{\mu,0}^m$,

$$\lim_{|z|\to 1} \mu(z)|f^{(m)}(z)| = 0.$$

Many classical and well-known analytic function spaces are included in $\mathcal{W}_\mu^m$ such as weighted Bloch space and weighted Zygmund space. More information about the spaces we use here can be found in [1, 2, 3, 4, 5, 6, 9, 10, 15, 16]. The space of analytic functions on $\mathbb{D}$ is denoted by $H(\mathbb{D})$ and $S(\mathbb{D})$ is the space of analytic self-maps of $\mathbb{D}$. Every $\varphi : \mathbb{D} \to \mathbb{D}$ which is analytic induces an operator using composition called composition operator C_φ, $C_\varphi f = f \circ \varphi$. The main subject in the study of composition operators is to describe operator theoretic properties of C_φ in terms of function theoretic properties of φ. Throughout the recent decades there has been an interest of study of the generalization of composition operators on different spaces of analytic function. Some generalization are weighted composition operator, Stević-Sharma type operator and generalized Stević-Sharma type operator.

Let $\alpha > 0$. The weighted Dirichlet space $\mathcal{D}_\alpha$ consists of all functions $f \in H(\mathbb{D})$ such that

$$(1+\alpha)\int_{\mathbb{D}} (1-|z|^2)^\alpha |f'(z)|^2 dA(z) < \infty,$$

where $dA(z)$ area measure on $\mathbb{D}$ such that $(1+\alpha)\int_{\mathbb{D}}(1-|z|^2)^\alpha dA(z)$. Weighted Dirichlet space is a Banach space with the following norm

$$\|f\|_{\mathcal{D}_\alpha} = |f(0)| + \left((1+\alpha)\int_{\mathbb{D}} (1-|z|^2)^\alpha |f'(z)|^2 dA(z)\right)^{\frac{1}{2}}.$$

Recently Zhu et al. [17] introduced general polynomial differentiation composition operator which includes the previous operators and defined as

$$T_{u,\varphi}^n f(z) = \sum_{j=0}^n u_j(z) f^{(j)}(\varphi_j(z)) = \sum_{j=0}^n (D_{u_j,\varphi_j}^j f)(z), \quad f \in H(\mathbb{D}), \quad n \in \mathbb{N}_0 \tag{1.2}$$

where $u_j \in H(\mathbb{D})$ and $\varphi_j \in S(\mathbb{D})$. Boundedness, compactness and essential norm of the above operator from Besov-type spaces into Bloch-type spaces were characterized in [17]. Beside study of the single operator on some spaces, research on differences or sums of operators are of interest because in this case topological structure of space of operators was regarded. For example differences of (weighted) composition operators on these spaces have been investigated in [7, 12]. Also linear combination (finite sum) of composition operators on Cauchy transform type spaces to Korenblum spaces studied in [8]. Let $i, n \in \mathbb{N}_0$, $u_i, v_i \in H(\mathbb{D})$ and $\varphi_i, \psi_i \in S(\mathbb{D})$. We set

$$S = \sum_{j=0}^n (D_{u_j,\varphi_j}^j - D_{v_j,\psi_j}^j) = T_{u,\varphi}^n - T_{v,\psi}^n \tag{1.3}$$

where, $D^j f = f^{(j)}$.

The aim of this paper is to characterize differences of general polynomial differentiation composition operator S, from Cauchy transform space into mth weighted type space and weighted Dirichlet space.

2 Boundedness of $S : \mathcal{F} \to \mathcal{W}_\mu^m(\mathcal{W}_{\mu,0}^m)$

In this section, we consider boundedness of operator $S : \mathcal{F} \to \mathcal{W}_\mu^m$ and we find exact formula for norm of this operator. We begin with the following lemma.

Lemma 2.1. [1] Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then for any $t \in \mathbb{N}_0$,

$$\left(D_{u,\varphi}^0 f\right)^{(t)}(z) = \sum_{i=0}^{t} f^{(i)}(\varphi(z)) E_{u,\varphi}^{t,i}(z), \quad f \in H(\mathbb{D}),$$

where, $E_{u,\varphi}^{t,i}(z) = \sum_{l=i}^{t} \binom{t}{l} u^{(t-l)}(z) B_{l,i}(\varphi'(z), ..., \varphi^{(l-i+1)}(z))$ and $B_{l,i}(\varphi'(z), ..., \varphi^{(l-i+1)}(z))$ is Bell polynomial.

The interested readers can find additional information about Bell polynomials in [1, 2, 3, 16].

Theorem 2.2. Let $m, n \in \mathbb{N}_0$, μ be a weight. Then the operator $S : \mathcal{F} \to \mathcal{W}_\mu^m$ is bounded if and only if

$$\sup_{\xi \in \partial\mathbb{D}} \left(l_1(\xi) + \sup_{z \in \mathbb{D}} l_2(\xi, z) \right) < \infty, \tag{2.1}$$

where

$$l_1(\xi) = \sum_{k=0}^{m-1} \left| \sum_{j=0}^{n} \sum_{t=0}^{k} ((j+t)! \bar{\xi}^{j+t}) \left(\frac{E_{u_j,\varphi_j}^{k,t}(0)}{(1 - \bar{\xi}\varphi_j(0))^{j+t+1}} - \frac{E_{v_j,\psi_j}^{k,t}(0)}{(1 - \bar{\xi}\psi_j(0))^{j+t+1}} \right) \right|,$$

$$l_2(\xi, z) = \left| \sum_{j=0}^{n} \sum_{t=0}^{m} \mu(z)(j+t)! \bar{\xi}^{j+t} \left(\frac{E_{u_j,\varphi_j}^{m,t}(z)}{(1 - \bar{\xi}\varphi_j(z))^{j+t+1}} - \frac{E_{v_j,\psi_j}^{m,t}(z)}{(1 - \bar{\xi}\psi_j(z))^{j+t+1}} \right) \right|.$$

In this case,

$$\|S\| = \sup_{\xi \in \partial\mathbb{D}} \left(l_1(\xi) + \sup_{z \in \mathbb{D}} l_2(\xi, z) \right).$$

Proof . For any $\xi \in \partial\mathbb{D}$, the function

$$f_\xi(z) = \frac{1}{1 - \bar{\xi}z}, \quad z \in \mathbb{D}$$

belongs to $\mathcal{F}$ with $\|f_\xi\|_{\mathcal{F}} = 1$ and for any $t \in \mathbb{N}$, $f_\xi^{(t)}(z) = \frac{t!\bar{\xi}^t}{(1-\bar{\xi}z)^{t+1}}$ (see[14]). So, by using Lemma 2.1 for any $\xi \in \partial\mathbb{D}$, we obtain

$$\begin{aligned}
\sum_{k=0}^{m-1} \left| (Sf_\xi)^{(k)}(0) \right| &= \sum_{k=0}^{m-1} \left| \left(\sum_{j=0}^{n} (u_j f_\xi^{(j)}(\varphi_j) - v_j f_\xi^{(j)}(\psi_j) \right)^{(k)} (0) \right| \\
&= \sum_{k=0}^{m-1} \left| \sum_{j=0}^{n} \left(u_j (f_\xi^{(j)}(\varphi_j) \right)^{(k)} (0) - \left(v_j f_\xi^{(j)}(\psi_j) \right)^{(k)} (0) \right| \\
&= \sum_{k=0}^{m-1} \left| \sum_{j=0}^{n} \sum_{t=0}^{k} f_\xi^{(j+t)}(\varphi_j)(0) E_{u_j,\varphi_j}^{k,t}(0) - f_\xi^{(j+t)}(\psi_j)(0) E_{v_j,\psi_j}^{k,t}(0) \right| \\
&= \sum_{k=0}^{m-1} \left| \sum_{j=0}^{n} \sum_{t=0}^{k} (j+t)! \bar{\xi}^{j+t} \left(\frac{E_{u_j,\varphi_j}^{k,t}(0)}{(1 - \bar{\xi}\varphi_j(0))^{j+t+1}} - \frac{E_{v_j,\psi_j}^{k,t}(0)}{(1 - \bar{\xi}\psi_j(0))^{j+t+1}} \right) \right| \\
&= l_1(\xi).
\end{aligned} \tag{2.2}$$

Applying Lemma 2.1, for any $\xi \in \partial\mathbb{D}$, we get

$$\begin{aligned}\mu(z)\left|(Sf_\xi)^{(m)}(z)\right| &= \left|\sum_{j=0}^{n}\sum_{t=0}^{m}\mu(z)(j+t)!\bar{\xi}^{j+t}\Big(\frac{E^{m,t}_{u_j,\varphi_j}(z)}{(1-\bar{\xi}\varphi_j(z))^{j+t+1}} - \frac{E^{m,t}_{v_j,\psi_j}(z)}{(1-\bar{\xi}\psi_j(z))^{j+t+1}}\Big)\right| \\ &= l_2(\xi, z). \end{aligned} \tag{2.3}$$

Let $S : \mathcal{F} \to \mathcal{W}^m_\mu$ be bounded. Then

$$\sum_{k=0}^{m-1}\left|(Sf_\xi)^{(k)}(0)\right| + \mu(z)\left|(Sf_\xi)^{(m)}(z)\right| \le \|Sf_\xi\|_{\mathcal{W}^m_\mu} \le \|S\|\|f_\xi\|_{\mathcal{F}} = \|S\|. \tag{2.4}$$

Therefore, by using (2.2), (2.3) and (2.4), we have

$$\sup_{\xi\in\partial\mathbb{D}}\left(l_1(\xi) + \sup_{z\in\mathbb{D}} l_2(\xi, z)\right) \le \|S\|. \tag{2.5}$$

Conversely, we assume that the condition of (2.1) is holds. For any $f \in \mathcal{F}$ there exists $\nu \in \mathcal{M}$ such that

$$f(z) = \int_{\partial\mathbb{D}} \frac{d\nu(\xi)}{1-\bar{\xi}z}, \qquad z \in \mathbb{D},$$

and $\|f\|_{\mathcal{F}} = \|\nu\|$. So for each $k \in \mathbb{N}_0$

$$f^{(k)}(z) = k!\int_{\partial\mathbb{D}} \frac{\bar{\xi}^k d\nu(\xi)}{(1-\bar{\xi}z)^{k+1}}, \qquad z \in \mathbb{D}.$$

Therefore, by using Lemma 2.1 for any $f \in \mathcal{F}$, we have

$$\begin{aligned}\sum_{k=0}^{m-1}\left|(Sf)^{(k)}(0)\right| &= \sum_{k=0}^{m-1}\left|\left(\sum_{j=0}^{n}(u_j f^{(j)}(\varphi_j) - v_j f^{(j)}(\psi_j)\right)^{(k)}(0)\right| \\ &= \sum_{k=0}^{m-1}\left|\sum_{j=0}^{n}\sum_{t=0}^{k} f^{(j+t)}(\varphi_j)(0)E^{k,t}_{u_j,\varphi_j}(0) - f^{(j+t)}(\psi_j)(0)E^{k,t}_{v_j,\psi_j}(0)\right| \\ &= \sum_{k=0}^{m-1}\left|\sum_{j=0}^{n}\sum_{t=0}^{k}(j+t)!\int_{\partial\mathbb{D}}\bar{\xi}^{j+t}\Big(\frac{E^{k,t}_{u_j,\varphi_j}(0)}{(1-\bar{\xi}\varphi_j(0))^{j+t+1}} - \frac{E^{k,t}_{v_j,\psi_j}(0)}{(1-\bar{\xi}\psi_j(0))^{j+t+1}}\Big)d\nu(\xi)\right| \\ &\le \int_{\partial\mathbb{D}}\sum_{k=0}^{m-1}\left|\sum_{j=0}^{n}\sum_{t=0}^{k}(j+t)!\bar{\xi}^{j+t}\Big(\frac{E^{k,t}_{u_j,\varphi_j}(0)}{(1-\bar{\xi}\varphi_j(0))^{j+t+1}} - \frac{E^{k,t}_{v_j,\psi_j}(0)}{(1-\bar{\xi}\psi_j(0))^{j+t+1}}\Big)\right| d|\nu|(\xi) \\ &= \int_{\partial\mathbb{D}} l_1(\xi)d|\nu|(\xi)\end{aligned}$$

and

$$\begin{aligned}\mu(z)\left|(Sf)^{(m)}(z)\right| &= \left|\sum_{j=0}^{n}\sum_{t=0}^{m}\mu(z)(j+t)!\int_{\partial\mathbb{D}}\bar{\xi}^{j+t}\Big(\frac{E^{m,t}_{u_j,\varphi_j}(z)}{(1-\bar{\xi}\varphi_j(z))^{j+t+1}} - \frac{E^{m,t}_{v_j,\psi_j}(z)}{(1-\bar{\xi}\psi_j(z))^{j+t+1}}\Big)d\nu(\xi)\right| \\ &\le \int_{\partial\mathbb{D}}\left|\sum_{j=0}^{n}\sum_{t=0}^{m}\mu(z)(j+t)!\bar{\xi}^{j+t}\Big(\frac{E^{m,t}_{u_j,\varphi_j}(z)}{(1-\bar{\xi}\varphi_j(z))^{j+t+1}} - \frac{E^{m,t}_{v_j,\psi_j}(z)}{(1-\bar{\xi}\psi_j(z))^{j+t+1}}\Big)\right| d|\nu|(\xi) \\ &\le \int_{\partial\mathbb{D}} l_2(\xi, z)|d|\nu|(\xi).\end{aligned} \tag{2.6}$$

By using last inequalities, we obtain

$$\begin{aligned}\sum_{k=0}^{m-1}\left|(Sf)^{(k)}(0)\right| + \mu(z)\left|(Sf)^{(m)}(z)\right| &\leq \int_{\partial\mathbb{D}}(l_1(\xi)+l_2(\xi,z))d|\nu|(\xi)\\ &\leq \sup_{\xi\in\partial\mathbb{D}}\left(l_1(\xi)+\sup_{z\in\mathbb{D}}l_2(\xi,z)\right)\int_{\partial\mathbb{D}}d|\nu|(\xi)\\ &\leq \sup_{\xi\in\partial\mathbb{D}}\left(l_1(\xi)+\sup_{z\in\mathbb{D}}l_2(\xi,z)\right)\|\nu\|\\ &\leq \sup_{\xi\in\partial\mathbb{D}}\left(l_1(\xi)+\sup_{z\in\mathbb{D}}l_2(\xi,z)\right)\|f\|_{\mathcal{F}}.\end{aligned}$$

Hence,

$$\|S\| \leq \sup_{\xi\in\partial\mathbb{D}}\left(l_1(\xi)+\sup_{z\in\mathbb{D}}l_2(\xi,z)\right).$$

Applying (2.5) and the previous inequality, we have

$$\|S\| = \sup_{\xi\in\partial\mathbb{D}}\left(l_1(\xi)+\sup_{z\in\mathbb{D}}l_2(\xi,z)\right).$$

The proof is completed. □

Theorem 2.3. Let $m,n\in\mathbb{N}_0$, μ be a weight. Then the operator $S:\mathcal{F}\to\mathcal{W}^m_{\mu,0}$ is bounded if and only if $S:\mathcal{F}\to\mathcal{W}^m_{\mu}$ be bounded and $\lim_{|z|\to1}l_2(\xi,z)=0$.

Proof . Let $S:\mathcal{F}\to\mathcal{W}^m_{\mu,0}$ be bounded. So $S:\mathcal{F}\to\mathcal{W}^m_{\mu}$ is bounded. Since for each $\xi\in\partial\mathbb{D}$ and $z\in\mathbb{D}$ the function $f_\xi(z)=\frac{1}{1-\xi z}\in\mathcal{F}$, hence $Sf_\xi\in\mathcal{W}^m_{\mu,0}$. Applying (2.3), we get

$$0=\lim_{|z|\to1}\mu(z)|(Sf_\xi)^{(m)}(z)|=\lim_{|z|\to1}l_2(\xi,z).$$

Conversely, let $S:\mathcal{F}\to\mathcal{W}^m_{\mu}$ be bounded, so $l_2(\xi,z)\leq\|S\|$. Also for any $f\in\mathcal{F}$ by using (2.6), we have

$$\mu(z)\left|(Sf)^{(m)}(z)\right|\leq\int_{\partial\mathbb{D}}l_2(\xi,z)d|\nu|(\xi).$$

Applying the Lebesgue dominated convergence theorem for previous inequality, we get

$$\lim_{|z|\to1}\mu(z)\left|(Sf)^{(m)}(z)\right|\leq\int_{\partial\mathbb{D}}\lim_{|z|\to1}l_2(\xi,z)d|\nu|(\xi)=0.$$

The proof is completed. □

Remark 2.4. Let $k<n$. Setting $v_j\equiv0$ for $k<j\leq n$ in Theorems 2.2 and 2.3. So we obtain similar results for operator $T^n_{u,\varphi}-T^k_{v,\psi}:\mathcal{F}\to\mathcal{W}^m_{\mu}$.

Remark 2.5. In Theorems 2.2 and 2.3 putting $u_j\equiv0$ for $0\leq j\leq n-1$, $u_n(z)=u(z),\varphi_n(z)=\varphi(z)$ and for $k\in\{0,1,...,s-1,s+1,...,n\}$, $\nu_k\equiv0$, $\nu_s(z)=v(z),\psi_s(z)=\psi(z)$, we obtain two following corollaries (see [14, Theorems 4.1 and 4.3]).

Corollary 2.6. Let $s,n\in\mathbb{N}_0$ such that $s\leq n$ and μ be a weight. Then the operator $D^n_{\varphi,u}-D^s_{v,\psi}:\mathcal{F}\to\mathcal{W}^m_{\mu}$ is bounded if and only if

$$\sup_{\xi\in\partial\mathbb{D}}\left(q_1(\xi)+\sup_{z\in\mathbb{D}}q_2(\xi,z)\right)<\infty, \tag{2.7}$$

where,

$$q_1(\xi)=\sum_{k=0}^{m-1}\left|\sum_{t=0}^{k}(\frac{(n+t)!\bar{\xi}^{n+t}E^{m,t}_{u,\varphi}(0)}{(1-\bar{\xi}\varphi(0))^{n+t+1}}-\frac{(s+t)!\bar{\xi}^{s+t}E^{m,t}_{v,\psi}(0)}{(1-\bar{\xi}\psi(0))^{s+t+1}})\right|,$$
$$q_2(\xi,z)=\left|\sum_{t=0}^{m}\mu(z)(\frac{(n+t)!\bar{\xi}^{n+t}E^{m,t}_{u,\varphi}(z)}{(1-\bar{\xi}\varphi(z))^{n+t+1}}-\frac{(s+t)!\bar{\xi}^{s+t}E^{m,t}_{v,\psi}(z)}{(1-\bar{\xi}\psi(z))^{s+t+1}})\right|.$$

Moreover

$$\|D^n_{\varphi,u}-D^s_{v,\psi}\|=\sup_{\xi\in\partial\mathbb{D}}\left(q_1(\xi)+\sup_{z\in\mathbb{D}}q_2(\xi,z)\right).$$

Corollary 2.7. Let $s,n\in\mathbb{N}_0$ such that $s\le n$ and μ be a weight. Then the operator $D^n_{\varphi,u}-D^s_{v,\psi}:\mathcal{F}\to\mathcal{W}^m_{\mu,0}$ is bounded if and only if $D^n_{\varphi,u}-D^s_{v,\psi}:\mathcal{F}\to\mathcal{W}^m_{\mu}$ be bounded and $\lim_{|z|\to1}q_2(\xi,z)=0$.

Remark 2.8. When $\varphi(z)=z:=I(z)$ then $E^{t,l}_{u,z}(z)=\binom{t}{l}u^{(t-l)}(z)$. By substitution $\varphi_j(z)=\psi_j(z)=I(z)$ for $0\le j\le n$ in Theorems 2.2 and 2.3, we obtain similar results for the operator $\sum_{j=0}^{n}(D^j_{u_j,I}-D^j_{v_j,I})=\sum_{j=0}^{n}D^j_{u_j-v_j,I}$.

3 Boundedness of $S:\mathcal{F}\to\mathcal{D}_\alpha$

In this section, we consider boundedness of the operator $S:\mathcal{F}\to\mathcal{D}_\alpha$ and we give approximation for the norm of this operator.

Theorem 3.1. Let $\alpha>-1$. Then the operator $S:\mathcal{F}\to\mathcal{D}_\alpha$ is bounded if and only if

$$\sup_{\xi\in\partial\mathbb{D}}\Big(n_1(\xi)+n_2(\xi)\Big)<\infty, \tag{3.1}$$

where

$$n_1(\xi)=\left|\sum_{j=0}^{n}j!\bar{\xi}^j(\frac{u_j(0)}{(1-\bar{\xi}\varphi_j(0))^{j+1}}-\frac{v_j(0)}{(1-\bar{\xi}\psi_j(0))^{j+1}})\right|,$$
$$n_2(\xi)=\left((1+\alpha)\int_{\mathbb{D}}(1-|z|^2)^\alpha\left|\sum_{j=0}^{n}\sum_{t=0}^{1}(j+t)!\bar{\xi}^{j+t}(\frac{E^{m,t}_{u_j,\varphi_j}(z)}{(1-\bar{\xi}\varphi_j(z))^{j+t+1}}-\frac{E^{m,t}_{v_j,\psi_j}(z)}{(1-\bar{\xi}\psi_j(z))^{j+t+1}})\right|^2dA(z)\right)^{\frac12}.$$

Moreover,

$$\sup_{\xi\in\partial\mathbb{D}}\Big(n_1(\xi)+n_2(\xi)\Big)\le\|S\|\le\sup_{\xi\in\partial\mathbb{D}}n_1(\xi)+\sup_{\xi\in\partial\mathbb{D}}n_2(\xi).$$

Proof . Let the operator $S:\mathcal{F}\to\mathcal{D}_\alpha$ be bounded and $f_\xi(z)=\frac{1}{1-\bar{\xi}z}$, $(\xi\in\partial\mathbb{D})$. So

$$\begin{aligned}|(Sf_\xi)(0)|&=\left|\sum_{j=0}^{n}u_j(0)(f^{(j)}_\xi(\varphi_j)(0)-v_j(0)f^{(j)}_\xi(\psi_j)(0)\right|\\&=\left|\sum_{j=0}^{n}j!\bar{\xi}^j(\frac{u_j(0)}{(1-\bar{\xi}\varphi_j(0))^{j+1}}-\frac{v_j(0)}{(1-\bar{\xi}\psi_j(0))^{j+1}})\right|=n_1(\xi)\end{aligned}$$

and

$$\begin{aligned}&(1+\alpha)\int_{\mathbb{D}}(1-|z|^2)^\alpha\left|(Sf_\xi)'(z)\right|^2dA(z)\\=&(1+\alpha)\int_{\mathbb{D}}(1-|z|^2)^\alpha\left|\sum_{j=0}^{n}\sum_{t=0}^{1}(j+t)!\bar{\xi}^{j+t}(\frac{E^{m,t}_{u_j,\varphi_j}(z)}{(1-\bar{\xi}\varphi_j(z))^{j+t+1}}-\frac{E^{m,t}_{v_j,\psi_j}(z)}{(1-\bar{\xi}\psi_j(z))^{j+t+1}})\right|^2dA(z)\\=&n_2(\xi)^2.\end{aligned}$$

Hence,

$$|(Sf_\xi)(0)| + \left((1+\alpha)\int_{\mathbb{D}}(1-|z|^2)^\alpha \left|(Sf_\xi)'(z)\right|^2 dA(z)\right)^{\frac{1}{2}} \le \|S\|\|f_\xi\|_{\mathcal{F}} = \|S\|.$$

Therefore,

$$\sup_{\xi\in\partial\mathbb{D}}(n_1(\xi)+n_2(\xi)) \le \|S\|. \tag{3.2}$$

For any $f\in\mathcal{F}$ there exists $\nu\in M$ such that

$$f(z)=\int_{\partial\mathbb{D}}\frac{d\nu(\xi)}{1-\bar{\xi}z}, \qquad z\in\mathbb{D},$$

and $\|f\|_{\mathcal{F}}=\|\nu\|$. So, for any $f\in\mathcal{F}$, we get

$$\begin{aligned}
|(Sf)(0)| &= \left|\sum_{j=0}^n u_j(0)f^{(j)}(\varphi_j)(0) - v_j(0)f^{(j)}(\psi_j)(0)\right| \\
&= \left|\sum_{j=0}^n j!\int_{\partial\mathbb{D}}\bar{\xi}^j\Big(\frac{u_j(0)}{(1-\bar{\xi}\varphi_j(0))^{j+1}} - \frac{v_j(0)}{(1-\bar{\xi}\psi_j(0))^{j+1}}\Big)d\nu(\xi)\right| \\
&\le \int_{\partial\mathbb{D}}\left|\sum_{j=0}^n j!\bar{\xi}^j\Big(\frac{u_j(0)}{(1-\bar{\xi}\varphi_j(0))^{j+1}} - \frac{v_j(0)}{(1-\bar{\xi}\psi_j(0))^{j+1}}\Big)\right| d|\nu|(\xi) \\
&= \int_{\partial\mathbb{D}} n_1(\xi)d|\nu|(\xi) \le \|\nu\|\sup_{\xi\in\partial\mathbb{D}} n_1(\xi).
\end{aligned}$$

By using Jensen's inequality and Fubini's theorem, we get

$$\begin{aligned}
&(1+\alpha)\int_{\mathbb{D}}(1-|z|^2)^\alpha \left|(Sf)'(z)\right|^2 dA(z) \\
=&(1+\alpha)\int_{\mathbb{D}}(1-|z|^2)^\alpha\left|\sum_{j=0}^n\sum_{t=0}^1 (j+t)!\int_{\partial\mathbb{D}}\bar{\xi}^{j+t}\Big(\frac{E^{m,t}_{u_j,\varphi_j}(z)}{(1-\bar{\xi}\varphi_j(z))^{j+t+1}} - \frac{E^{m,t}_{v_j,\psi_j}(z)}{(1-\bar{\xi}\psi_j(z))^{j+t+1}}\Big)d\nu(\xi)\right|^2 dA(z) \\
\le& \int_{\partial\mathbb{D}}\|\nu\|(1+\alpha)\int_{\mathbb{D}}(1-|z|^2)^\alpha\left|\sum_{j=0}^n\sum_{t=0}^1 (j+t)!\bar{\xi}^{j+t}\Big(\frac{E^{m,t}_{u_j,\varphi_j}(z)}{(1-\bar{\xi}\varphi_j(z))^{j+t+1}} - \frac{E^{m,t}_{v_j,\psi_j}(z)}{(1-\bar{\xi}\psi_j(z))^{j+t+1}}\Big)\right|^2 dA(z)d|\nu|(\xi) \\
\le& \int_{\partial\mathbb{D}}\|\nu\| n_2^2(\xi)d|\nu|(\xi) \le \|\nu\|^2(\sup_{\xi\in\partial\mathbb{D}} n_2(\xi))^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(Sf)(0)| + \left((1+\alpha)\int_{\mathbb{D}}(1-|z|^2)^\alpha \left|(Sf)'(z)\right|^2 dA(z)\right)^{\frac{1}{2}} &\le \left(\sup_{\xi\in\partial\mathbb{D}} n_1(\xi) + \sup_{\xi\in\partial\mathbb{D}} n_2(\xi)\right)\|\nu\| \\
&\le \left(\sup_{\xi\in\partial\mathbb{D}} n_1(\xi) + \sup_{\xi\in\partial\mathbb{D}} n_2(\xi)\right)\|f\|_{\mathcal{F}}.
\end{aligned}$$

Hence,

$$\|S\| \le \sup_{\xi\in\partial\mathbb{D}} n_1(\xi) + \sup_{\xi\in\partial\mathbb{D}} n_2(\xi).$$

The proof is completed. □

With suitable parameters in Theorem 3.1, we can obtain similar results as stated in Remarks 2.4, 2.5, 2.8 and Corollary 2.6.

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