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On the dependency in stress-strength models

Saeed Zalzadeh

Department of Statistics, Semnan University, Semnan, Iran

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Abstract

In reliability studies, the probability $R = P(Y < X)$ is called "reliability", and X, Y are typically considered independent. However, in many applications, such an assumption may be unrealistic. In this paper, we consider the stress-strength reliability $R = P(Y < X)$ where the stress Y and strength X are dependent. We use a copula to describe dependence among random variables X and Y. We obtain R for several copula functions with exponential marginal and plot graphs of R versus the dependence parameter to study the effect of (positive and negative) dependency on R.

Keywords: Reliability, Kendall's tau, Copula, Stress-strength models
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1 Introduction

Researchers in various fields, like medicine, psychology, engineering, and quality control, are interested in stress-strength models. In stress-strength models, a component with a random strength X is assumed to be subject to a random stress Y. The reliability of a component in such modes is defined as $R = P(X > Y)$. The reliability of a stress-strength model has been studied in many cases when both stress and strength are independent random variables (see [6, 10, 11]). Jovanović and Rajić [5] presented a stress-strength model where X and Y follow gamma and exponential distributions. Jovanović [4] studied the estimation of the reliability for the independent random variables X and Y. Less attention has been given to the evaluation of R when variables X and Y are dependent.

R has been studied when strength and stress are assumed to have a bivariate distribution (see [7, 2]). Patil et al. [9] investigated the effect of dependency on the estimation of R in the exponential stress-strength models. They considered FGM, AMH, Gumbel's bivariate exponential, and Gumbel-Hougaard copula functions to take into account dependency among the random variables X and Y. In this study, we consider more copula functions describing the positive and negative dependency of variables and investigate the effect of type and strength of dependency on the R. In this work, we denote by F, F_X, and F_Y the joint and marginal distributions of bivariate random variables X and Y, respectively. The joint survival function and the marginal survival functions of (X, Y), that is, $\bar{F}(x, y) = P\{X > x, X > y\}$, $\bar{F}_X(x) = P(X > x)$, and $\bar{F}_Y(y) = P(Y > y)$ will be denoted by $\bar{F}$, $\bar{F}_X$, and $\bar{F}_y$, respectively. During this paper, we assume the distributions F_X and F_Y are continuous.

Email address: szalzadeh@Semnan.ac.ir (Saeed Zalzadeh)

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2 Preliminaries

Let (X, Y) be a random vector with the joint distribution function and marginal distributions F_X and F_Y. The function $C : [0,1]^2 \to [0,1]$ such that, for all $(x, y) \in \mathbb{R}^2$, satisfies

$$F(x, y) = C(F_X(x), F_Y(y)) \tag{2.1}$$

is called *copula* of the vector (X, Y). In this case, it also holds

$$C(u, v) = F(F_X^{-1}(u), F_Y^{-1}(v)),$$

for all $u, v \in [0,1]$. Such a copula is a bivariate distribution function with margins uniformly distributed on $[0,1] \subset \mathbb{R}$, and is unique whenever F_X and F_Y are continuous. For further details on copulas, we refer the reader to the standard references Joe [3] and Nelsen[8].

Definition 2.1. ([8]) If C_1 and C_2 are copulas, we say that C_1 is smaller than C_2, and write $C_1 \prec C_2$ if $C_1(u, v) \leq C_2(u, v)$ for all $u, v \in [0,1]$.

In definition 2.1 C_2 is said to be more concordant than C_1. A parametric family $\{C_\theta\}$ of copulas is said to be positively ordered if $C_\alpha \prec C_\beta$ whenever $\alpha \leq \beta$, and negatively ordered if $C_\alpha \succ C_\beta$ whenever $\alpha \leq \beta$ (see [8, 3]). If X and Y have joint distribution function $F(x, y)$ with continuous margins $F_X(x)$ and $F_Y(y)$, respectively, and copula $C(u, v)$, then X and Y are positively quadrant dependent (PQD) if

$$F(x, y) \geq F_X(x) F_Y(y), \quad \text{for all } (x, y) \in R^2, \tag{2.2}$$

or equivalently if

$$\bar{F}(x, y) \geq \bar{F}_X(x) \bar{F}_Y(y), \quad \text{for all } (x, y) \in R^2, \tag{2.3}$$

where $\bar{F}(x, y) = P(X > x, Y > y)$ is the joint survival function of (X, Y). If X and Y have copula C then (2.3) is equivalent to

$$C(u, v) \geq uv \quad \text{for all } (u, v) \in [0,1]^2. \tag{2.4}$$

Negative quadrant dependence (NQD) is defined analogously by reversing the inequalities in (2.2), (2.3) and (2.4). An important class of copulas known as Archimedean copulas.

Definition 2.2. A copula is said to be Archimedean if it can be written as

$$C(u, v) = \phi(\phi^{-1}(u) + \phi^{-1}(v)), \tag{2.5}$$

where $\phi : \mathbb{R}^+ \longrightarrow [0,1]$ is a continuous, strictly decreasing and convex function such that $\phi(0) = 1$, and $lim_{x\to\infty}\phi(x) = 0$. The ϕ is called generator of the Archimedean copula.

Next, we recall some well-known families of bivariate copulas considered in this paper.

1. The Product (independent) copula

$$C(u, v) = uv \tag{2.6}$$

2. The Clayton copula

The Clayton copula is a PQD Archimedean copula for $\theta > 0$ and describes positive dependence, i.e., it is PQD.

$$C_\theta(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}, \quad \theta > 0. \tag{2.7}$$

3. The Gumbel-Hougaard copula

The Gumbel-Hougaard copula is an Archimedean copula and describes a positive dependence between the variables, i.e., it is PQD.

$$C_\theta(u, v) = \exp\left(-\left[(-\log u)^\theta + (-\log u)^\theta\right]^{\frac{1}{\theta}}\right); \quad \theta \in [1, \infty). \tag{2.8}$$

4. The Frank copula

The Frank copula is an Archimedean copula and is PQD for $\theta > 0$ and NQD for $\theta < 0$. This copula describe negative dependence for $\theta \in (-\infty, 0)$ and a positive dependence for $\theta \in (0, \infty)$.

$$C_\theta(u,v) = -\frac{1}{\theta}\log\left[1+\frac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{e^{-\theta}-1}\right].\theta \in (-\infty,\infty)\setminus\{0\} \tag{2.9}$$

5. The Farlie-Gumbel-Morgenstern (FGM) copula

$$C_\theta(u,v) = uv[1+\theta(1-u)(1-v)]\ ;\theta \in [-1,1]. \tag{2.10}$$

6. The Gumbel-Barnett (GB) copula

The Gumbel-Barnett copula is an Archimedean copula and describes a negative dependence between the variables, i.e., it is NQD.

$$C_\theta(u,v) = uv\exp\{-\theta\log u\log v\}\ ;\theta \in [0,1]. \tag{2.11}$$

The Gumbel-Barnett copula is equal to product copula for $\theta = 0$.

In the next section, we examine the effect of strength of dependence of X and Y on the p by considering different copula functions. The strength of dependence in a copula with dependent parameter θ, can be transformed to Kendall's tau, which is a well-known measure to assess the dependence among two variables. Given the copula model (2.1), Kendall's tau for X and Y is expressed as

$$Kendall's\ \tau = 4\int_0^1\int_0^1 C_\theta(u,v)C_\theta(d_u,d_v)-1.$$

The Kendall's τ does not depend on the marginals, and is solely determined by the copula. Therefore, it is advantageous over the Pearson correlation X and Y. The Kendall's τ of each aforementioned copula is expressed as follows:

1. The Product (independent) copula $\tau_\theta = 0$.
2. The Clayton copula $\tau_\theta = \frac{\theta}{\theta+2}$; $\theta \in [-1,\infty)\setminus\{0\}$.
3. The Gumbel-Hougaard copula $\tau_\theta = \frac{\theta-1}{\theta}$; $\theta \in [1,\infty)$.
4. Frank copula $\tau_\theta = 1-\frac{4}{\theta}[1-D_\theta]$; $\quad D_\theta = \frac{1}{\theta}\int_0^\theta \frac{x}{\exp(x)-1}dx$.
5. The FGM copula $\tau_\theta = \frac{2\theta}{9}$; $\theta \in [-1,1]$.
6. The Gumbel-Barnett copula $\tau_\theta = 1-\frac{4}{\theta}\int_0^1 x(1-\theta\log x)\log(1-\theta\log x)dx$; $\quad \theta \in [0,1]$.

3 Computation of p under different dependency conditions

In this section, we compute R for dependent random variables X and Y. Let $U = F_X(x)$ and $V = F_Y(y)$ be the corresponding marginal distribution functions of X and Y having copula C. Now we define

$$C_\theta^{[1,1]}(u,v) = \frac{\partial^2 C(u,v)}{\partial u\partial v} \tag{3.1}$$

and

$$C_\theta^{[1,0]}(u,v) = P(V<v|U=u) = \frac{\partial C(u,v)}{\partial u}. \tag{3.2}$$

Using Theorem in [1] and (3.2), R can be written in terms of a univariate integral as follows:

$$\begin{aligned} R &= P(X>Y) \\ &= P(F_Y(X)) > F_Y(Y)) = P(F_Y(F_X^{-1}(U)>V)) \\ &= E[P(V<F_Y(F_X^{-1}(U)|U)] \\ &= \int_0^1 P(V<F_Y(F_X^{-1}(u))|U=u)du \\ &= \int_0^1 C^{[1,0]}(u,F_Y(F_X^{-1}(u))du, \end{aligned} \tag{3.3}$$

also we can write

$$
\begin{aligned}
R = P(X > Y) &= \iint_{y<x} f(x, y)dxdy \\
&= \iint_{y<x} C_\theta^{[1,1]}[F_X(x), F_Y(y)]f_X(x)f_Y(y)dxdy.
\end{aligned}
\tag{3.4}
$$

In the following examples, we will see the behavior of R when the strength of dependence changes.

Example 3.1. Let the independent marginals X and Y have the exponential distributions with parameters λ_x and λ_y ($E(X) = 1/\lambda_x$, $E(Y) = 1/\lambda_y$). Then by (3.4) we can write

$$
\begin{aligned}
R = P(X > Y) &= \iint_{y<x} f(x, y)dxdy \\
&= \int_0^\infty \int_0^x \lambda_x \lambda_y e^{-x\lambda_x} e^{-y\lambda_y} dydx = \frac{\lambda_y}{\lambda_x + \lambda_y}.
\end{aligned}
$$

Example 3.2. Let the marginals X and Y have the exponential distributions with parameters λ_x and λ_y and corresponding Clayton copula, i.e.,

$$C_\theta(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}, \quad \theta > 0.$$

Then X and Y have a positive dependence and we obtain

$$C_\theta^{[1,0]}(u, v) = \frac{\partial C(u, v)}{\partial u} = u^{-(\theta+1)}(u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}-1}, \tag{3.5}$$

and also we have

$$F_X(x) = 1 - \exp(-\lambda_x x); \quad F_Y(F_X^{-1}(u)) = 1 - (1 - u)^{\frac{\lambda_x}{\lambda_y}}.$$

Thus we get

$$R = P(X > Y) = \int_0^1 [1 + u^\theta(1 - (1 - u)^{\frac{\lambda_x}{\lambda_y}})^{-\theta} - u^\theta]^{-\frac{1}{\theta}-1} du.$$

In Fig. 1, we have plotted the R in terms of θ where we assume $\lambda_x/\lambda_y = 2, 3$ and 0.5. For the case $\lambda_x/\lambda_y = 2$, Fig. 1 shows (it has been verified numerically with the R package) that the reliability R increases when dependence parameter θ increases from 0 to ∞ (i.e., when positive dependence increases).

Example 3.3. Let the marginals X and Y have the exponential distributions with parameters λ_x and λ_y and corresponding Gumbel-Hougaard copula, i.e.,

$$C_\theta(u, v) = \exp\left(-\left[(-\log u)^\theta + (-\log u)^\theta\right]^{\frac{1}{\theta}}\right); \quad \theta \in [1, \infty).$$

Then

$$
\begin{aligned}
C_\theta^{[1,0]}(u, v) =& \frac{\partial C(u, v)}{\partial u} \\
=& \frac{1}{u}(-\log u)^{\theta-1} \times \left[(-\log u)^\theta + (-\log v)^\theta\right]^{\frac{1}{\theta}-1} \times \exp\left(-\left[(-\log u)^\theta + (-\log u)^\theta\right]^{\frac{1}{\theta}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
C_\theta^{[1,0]}(u, F_Y(F_X^{-1}(u))) =& C_\theta^{[1,0]}(u, 1 - (1 - u)^{\frac{\lambda_x}{\lambda_y}}) \\
=& \frac{1}{u}(-\log u)^{\theta-1} \times \left[(-\log u)^\theta + (-\log[1 - (1 - u)^\lambda])^\theta\right]^{\frac{1}{\theta}-1} \\
& \times \exp\left(-\left[(-\log u)^\theta + (-\log[1 - (1 - u)^\lambda])^\theta\right]^{\frac{1}{\theta}}\right).
\end{aligned}
\tag{3.6}
$$

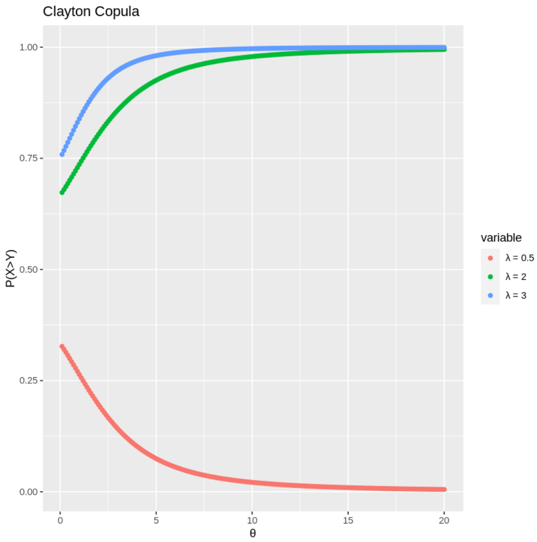

Figure 1: R in term of θ in example 3.2

By replacing relation3.6 in relation 3.3 we can compute the R. In Fig. 2 we have plotted the R in terms of θ where we assume $\lambda_x/\lambda_y = 2, 3$ and 0.5. For the case $\lambda_x/\lambda_y = 0.5$, as shown in Fig. 2, the R increases when positive dependence increases. Noting that for GH copula $\tau_\theta = \frac{\theta-1}{\theta}; \theta \in [1, \infty)$, and the dependence between X and Y is considered positive.

Example 3.4. Let the marginals X and Y have the exponential distributions with parameters λ_x and λ_y and corresponding Gumbel-Barnett copula, i.e.,

$$C_\theta(u, v) = uv \exp\{-\theta \log u \log v\}; \quad \theta \in [0, 1].$$

Then

$$C_\theta^{[1,0]}(u, v) = \frac{\partial C(u, v)}{\partial u} = (1 - \theta)v \exp\{-\theta \log u \log v\},$$

and by letting $\frac{\lambda_x}{\lambda_y} = \lambda$ we get

$$\begin{aligned} C_\theta^{[1,0]}(u, F_Y(F_X^{-1}(u))) &= C_\theta^{[1,0]}(u, 1 - (1 - u)^\lambda) \\ &= (1 - \theta)[1 - (1 - u)^\lambda] \times \exp\{-\theta \log u \log[1 - (1 - u)^\lambda]\}. \end{aligned} \tag{3.7}$$

In Fig. 3, we have plotted the R in terms of θ where we assume $\lambda_x/\lambda_y = 2, 3$ and 0.5. For the cases $\lambda_x/\lambda_y = 2, 3$, as shown in 3, the R decreases when negative dependence increases.

Example 3.5. Let the marginals X and Y have the exponential distributions with parameters λ_x and λ_y and corresponding FGM copula, i.e.,

$$C_\theta(u, v) = uv[1 + \theta(1 - u)(1 - v)]; \quad \theta \in [-1, 1].$$

According to $\tau_\theta = \frac{2\theta}{9}$; $\theta \in [-1, 1]$ this copula describe the positive dependence for $\theta > 0$ and the negative dependence for $\theta < 0$. However, for FGM copula we have $\frac{-2}{9} < \tau < \frac{2}{9}$. Thus, we get

$$c(u, v) = C_\theta^{[1,1]}(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v} = 1 + \theta(1 - 2u)(1 - 2v),$$

and since

$$F_X(x) = 1 - \exp(-\lambda_x x); \quad F_Y(y) = 1 - \exp(-\lambda_y y)$$

then using 3.4 we obtain

$$R = P(X > Y) = \int_0^\infty \int_0^x [1 + \theta(1 - 2F_X(x))(1 - 2F_Y(y))] f_X(x) f_Y(y) dy dx.$$

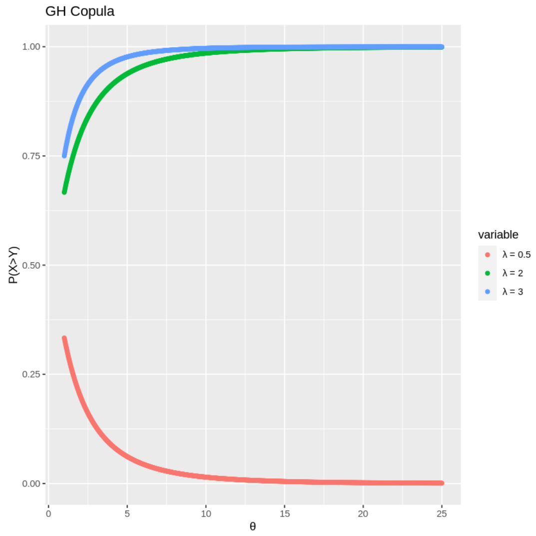

Figure 2: R in term of θ in example 3.3

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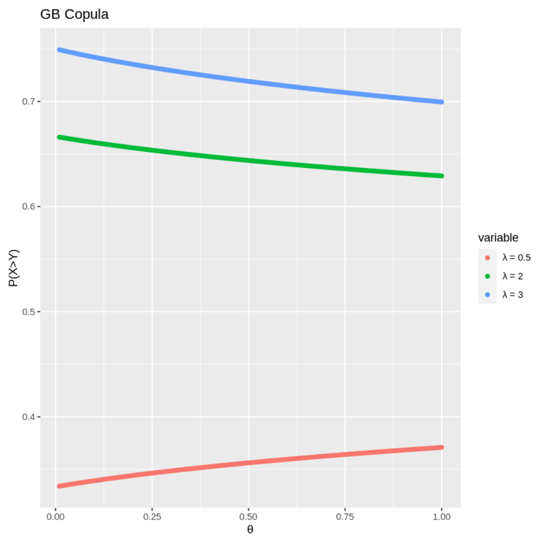

Figure 3: R in term of θ in example 3.4

By replacing marginal density and distribution function in (3.4) and computing the simple integral, we obtain R as

$$R = P(X > Y) = \frac{\lambda_y}{\lambda_x + \lambda_y} - \theta\lambda_x[\frac{2}{\lambda_x + \lambda_y} - \frac{1}{\lambda_x + 2\lambda_y} - \frac{2}{2\lambda_x + \lambda_y}].$$

If assume $\lambda_x = 2,\ \lambda_y = 1$, then $R = \frac{1}{3} - \frac{\theta}{30}$, that is, for $\theta \in [-1, 1]$, R is a linear decreasing function of θ. Since $\tau = \frac{2\theta}{9}$, one can say that R increases when negative dependence increases (i.e., θ decreases in [-1,0]) and R decreases when positive dependence increases (i.e., θ increases in [0,1]). If we suppose $\lambda_x = 1,\ \lambda_y = 2$, this conclusion will be the opposite (see Fig. 4).

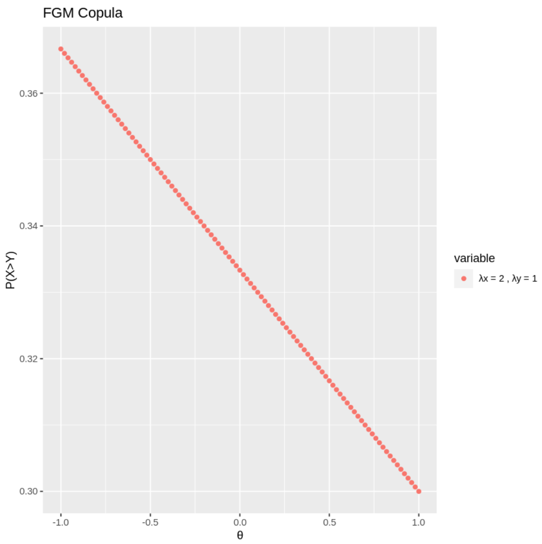

Figure 4: R in term of θ in example 3.5

Example 3.6. Let the marginals X and Y have the exponential distributions with parameter λ_x and λ_y and corresponding Frank copula, i.e.,

$$C_\theta(u, v) = -\frac{1}{\theta}\log\left[1 + \frac{\left(e^{-\theta u} - 1\right)\left(e^{-\theta v} - 1\right)}{e^{-\theta} - 1}\right];\ \theta \in (-\infty, \infty) \setminus \{0\}.$$

Then we obtain

$$C_\theta^{[1,0]}(u, v) = \frac{e^{-\theta u}(e^{-\theta v} - 1)}{e^{-\theta} - 1 + (e^{-\theta u} - 1)(e^{-\theta v} - 1)}.$$

Thus

$$\begin{aligned} C_\theta^{[1,0]}(u, F_Y(F_X^{-1}(u))) =& C_\theta^{[1,0]}(u, 1 - (1 - u)^\lambda) \\ =& \frac{e^{-\theta u}(e^{-\theta(1-(1-u)^\lambda)} - 1)}{e^{-\theta} - 1 + (e^{-\theta u} - 1)(e^{-\theta(1-(1-u)^\lambda)} - 1)}, \end{aligned}$$

where $\lambda = \lambda_x/\lambda_y$. then using (3.3) we have

$$\begin{aligned} R = P(X > Y) =& \int_0^1 \left[C_\theta^{[1,0]}(u, F_Y(F_X^{-1}(u)))\right] du \\ =& \int_0^1 \frac{e^{-\theta u}(e^{-\theta(1-(1-u)^\lambda)} - 1)}{e^{-\theta} - 1 + (e^{-\theta u} - 1)(e^{-\theta(1-(1-u)^\lambda)} - 1)} du. \end{aligned}$$

In Fig.s 5 and 6 , we have plotted the R in terms of θ where we assume $\lambda = 2, 3$ and 0.5. For the case $\lambda = 2, 3$, and $\theta > 0$ as shown in 5, the R increases when positive dependence increases and vice versa for the case $\lambda = 0.5$. There is a similar interpretation but in the opposite direction for the case $\theta < 0$ and Fig.R-Frank-neg.

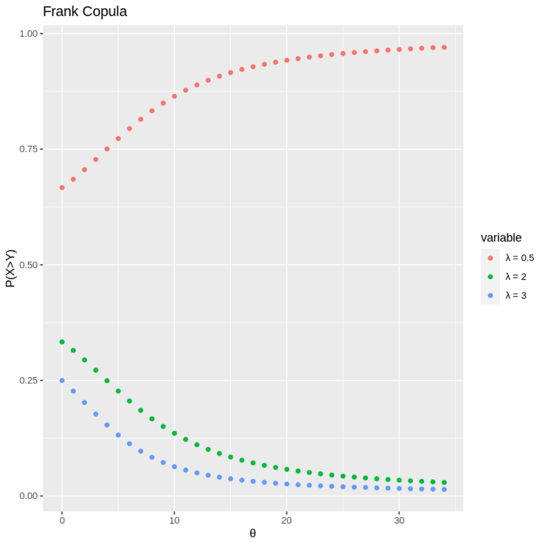

Figure 5: R in term of $\theta > 0$ in example 3.6

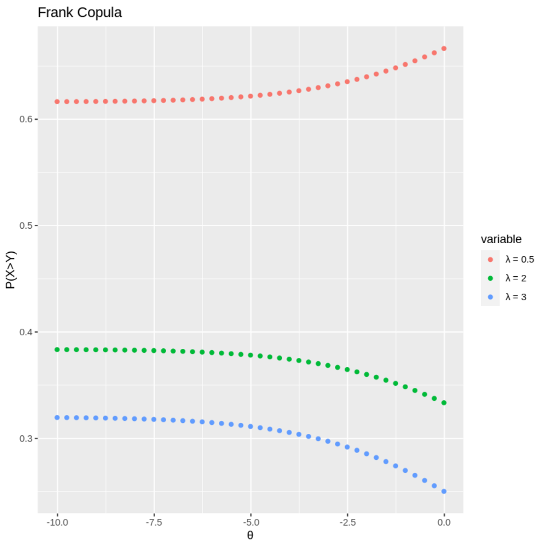

Figure 6: R in term of $\theta < 0$ in example 3.6

4 Conclusion

In this paper, we consider the stress-strength reliability $R = P(X > Y)$, where the stress Y is smaller than the dependent strength variable X. We obtain the reliability function R according to the different types of copulas describing positive and negative dependence with exponential marginal distribution functions. We compute R under a variety of parametric marginal distributions and copulas. The results show that changes in R under a positive dependence for $E(X) < E(Y)$ are increasing in terms of the dependency parameter, and R increases when the positive dependence increases. When we consider the negative dependence between random variables stress and strength, R decreases when the negative dependence increases, i.e., the dependency parameter increases (Examples 3.4, 3.5 and 3.6).

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