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Existence and uniqueness outcomes for a nonlinear fractional differential equation of high order featuring nonlocal boundary conditions

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Abstract

This study centers on establishing the existence of a unique solution for a class of fractional differential equations that incorporate the Riemann-Liouville fractional derivative. The boundary conditions encompass a nonlocal condition involving integration in a sub-domain near the boundary. Initially, the precise solution is derived for the linear fractional differential equation. Subsequently, the Banach contraction mapping theorem is employed to establish the primary result for the general nonlinearity of the source term. Additionally, the validity and applicability of our primary result are illustrated through a specific example.

Keywords: Fractional differential equations, Integral boundary conditions, Riemann-liouville derivative, Fixed point theorem
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1 Introduction and the problem formulation

Fractional-order differential and integral operators, characterized by their nonlocal nature, find diverse applications in various applied fields. These applications span a range of disciplines, including but not limited to blood flow problems, anomalous diffusion phenomena, the spread of diseases, control processes, and population dynamics. for instance see [13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24].

The utilization of the fixed-point theory, coupled with various other methodologies, is a pivotal approach for examining solutions to boundary value problems. This method plays a significant role in not only establishing the existence of a solution but also in obtaining an approximate solution, see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 18, 25]. There has been a significant surge of interest in investigating nonlocal nonlinear fractional-order phenomena, evident in both single-valued and multi-valued boundary value problems over recent years. Within the research domain focused on establishing existence and uniqueness results for boundary value problems using fixed-point theory, the task of identifying the Green function – a crucial element facilitating the presentation of a unique solution in linear cases – has proven to be particularly challenging, except in instances of simpler cases.

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In this research, we explore the presence of a distinct solution for a category of fractional differential equations with nonlocal boundary conditions, as expressed by:

$$\begin{cases} {}_0D_t^{\alpha} u + f(t, u(t), u'(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \quad u'(\eta) + \int_\eta^1 u(s)ds = 0. \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$, $0 \leq \eta < 1$, ${}_0D_t^{\alpha} u$ is the left Riemann-Liouville fractional derivative and $f : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Definition 1.1. Let $\alpha \in \mathbb{R}^+$. The operator ${}_aD_t^{-\alpha}$, defined on $L^1[a,b]$ by

$${}_aD_t^{-\alpha} u(t) := \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{\alpha-1} u(\tau) d\tau$$

for $t \in [a,b]$, is called the Riemann-Liouville fractional integral of order α.

Definition 1.2. Let $\alpha \in \mathbb{R}^+$ and $m = \lceil \alpha \rceil$.The operator${}_aD_t^{\alpha}$, defined by

$${}_aD_t^{\alpha} u := D^m({}_aD_t^{-(m-\alpha)} u) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d^m}{dt^m}\right) \int_a^t (t-\tau)^{m-1-\alpha} u(\tau) d\tau$$

is called the Riemann-Liouville fractional differential of order α.

2 Solution to the linear equation

First, let we have an observation to

Lemma 2.1. [4, Lemma 5.2] Let $\alpha > 0$, where α is a positive integer, and let m be the ceiling of α. Additionally, assume that all the hypotheses of Theorem 5.1 in [4] are satisfied. Consider the function $u \in C(0,h]$ as a solution to the differential equation:

$${}_0D_t^{\alpha} u(t) = f(t, u(t)),$$

subject to the initial conditions:

$${}_0D_t^{\alpha-k} u(0) = b_k \quad (k = 1, 2, ..., m-1), \quad \lim_{z \to 0^+} {}_0D_t{}^{-(m-\alpha)} u(z) = b_m.$$

This function u is a solution to the Volterra integral equation if and only if it satisfies these conditions.

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (x-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau + \sum_{k=1}^{m} \frac{b_k}{\Gamma(\alpha-k+1)} t^{\alpha-k}. \tag{2.1}$$

Now, we go back to the aimed problem 1.1. Typically, the methodology involves seeking solutions as fixed points of an operator. This operator is defined by utilizing the Green's function associated with the linear version of the problem

$${}_0D_t^{\alpha} u(t) + y(t) = 0, \quad t \in (0,1), \tag{2.2}$$

with respect to the integral boundary conditions.

Theorem 2.2. Let $2 < \alpha \leq 3$. Assume $y \in C[0,1]$, then the problem (2.2) has a unique solution $u \in C^1[0,1]$, given by

$$\begin{aligned} u(t) = \ & \tfrac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + \mu_1(t) \int_0^\eta (\eta-s)^{\alpha-2} y(s)ds + \mu_2(t) \big[\int_0^\eta (\eta-s)^{\alpha} y(s)ds \\ & - \int_0^1 (1-s)^{\alpha} y(s)ds + \tfrac{(1-\eta^\alpha)}{\eta^{\alpha-2}} \int_0^\eta (\eta-s)^{\alpha-2} y(s)ds \big], \end{aligned} \tag{2.3}$$

where

$$\mu_1(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)\eta^{\alpha-2}}, \qquad \text{and} \qquad \mu_2(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)[\alpha(\alpha-1)\eta^{\alpha-2} + \eta^\alpha - 1]}.$$

Proof . By applying Lemma 2.1, we observe that Eq. (2.2) is equivalent to

$$u(t) = {}_0D_t{}^{-\alpha}(-y(t)) = \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \sum_{k=1}^{3}\frac{{}_0D_t^{\alpha-k}u(0)}{\Gamma(\alpha-k+1)}t^{\alpha-k}. \tag{2.4}$$

Set $c_k = \frac{{}_0D_t^{\alpha-k}u(0)}{\Gamma(\alpha-k+1)}$, so

$$u(t) = \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3}. \tag{2.5}$$

Since $2 < \alpha \leq 3$ and $u(0) = 0$ then we should have $c_3 = 0$. Similarly, since $u'(0) = 0$ we have $c_2 = 0$. So the Eq. (2.5) deduces to

$$u(t) = \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1t^{\alpha-1}.$$

Hence

$$u'(t) = \frac{1-\alpha}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}y(s)ds + c_1(\alpha-1)t^{\alpha-2}.$$

Set $t = \eta$ in above equality, by integral BC $u'(\eta) + \int_\eta^1 u(s)ds = 0$, we get

$$-\int_\eta^1 u(s)ds = \frac{1-\alpha}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds + c_1(\alpha-1)\eta^{\alpha-2}.$$

Therefore

$$c_1 = \frac{1}{\Gamma(\alpha)\eta^{\alpha-2}}\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds + \frac{1}{(1-\alpha)\eta^{\alpha-2}}\int_\eta^1 u(s)ds.$$

Which yields:

$$u(t) = \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)\eta^{\alpha-2}}\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds + \frac{t^{\alpha-1}}{(1-\alpha)\eta^{\alpha-2}}\int_\eta^1 u(s)ds. \tag{2.6}$$

Set $A = \int_\eta^1 u(s)ds$. By integrating both sides of the last equality on interval $[\eta, 1]$ with respect to t, we get the following:

$$A = \tfrac{-1}{\Gamma(\alpha)}\int_\eta^1\int_0^t (t-s)^{\alpha-1}y(s)dsdt + \int_\eta^1 \tfrac{t^{\alpha-1}}{\Gamma(\alpha)\eta^{\alpha-2}}\int_0^\eta (\eta-s)^{\alpha-2}y(s)dsdt + \tfrac{A}{(1-\alpha)\eta^{\alpha-2}}\int_\eta^1 t^{\alpha-1}dt. \tag{2.7}$$

or

$$A = \frac{1-\alpha}{\Gamma(\alpha)\big(\alpha(1-\alpha)\eta^{\alpha-2} + \eta^\alpha - 1\big)}\big[\eta^{\alpha-2}\int_0^\eta (\eta-s)^\alpha y(s)ds - \eta^{\alpha-2}\int_0^1 (1-s)^\alpha y(s)ds + (1-\eta^\alpha)\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds\big]. \tag{2.8}$$

Replacing this value to Eq. (2.6), we obtain the following expression for u:

$$\begin{aligned} u(t) ={}& \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \underbrace{\frac{t^{\alpha-1}}{\Gamma(\alpha)\eta^{\alpha-2}}}_{\mu_1(t)}\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds \\ &+ \underbrace{\frac{t^{\alpha-1}}{\Gamma(\alpha)\big(\alpha(1-\alpha)\eta^{\alpha-2} + \eta^\alpha - 1\big)}}_{\mu_2(t)}\big[\int_0^\eta (\eta-s)^\alpha y(s)ds - \int_0^1 (1-s)^\alpha y(s)ds + \frac{(1-\eta^\alpha)}{\eta^{\alpha-2}}\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds\big]. \end{aligned} \tag{2.9}$$

□

3 Fixed point iteration

Consider $C^1[0,1]$, the Banach space comprising all continuously differentiable functions from $[0,1]$ to $\mathbb{R}$. This space is equipped with the usual norm $|u| = |u|\infty + |u'|\infty$. By replacing $y(x)$ with $f(t, u(t), u'(t))$ in Theorem 2.2, we can define an operator $T : C^1[0,1] \to C^1[0,1]$ associated with problem (1.1) as:

$$Tu(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s),u'(s))ds + \mu_1(t) \int_0^\eta (\eta - s)^{\alpha-2} f(s,u(s),u'(s))ds \tag{3.1}$$

$$+ \mu_2(t)\big[\int_0^\eta (\eta - s)^{\alpha} f(s,u(s),u'(s))ds - \int_0^1 (1-s)^{\alpha} f(s,u(s),u'(s))ds \tag{3.2}$$

$$+ \frac{(1-\eta^\alpha)}{\eta^{\alpha-2}} \int_0^\eta (\eta - s)^{\alpha-2} f(s,u(s),u'(s))ds\big]. \tag{3.3}$$

We confirm that, based on Theorem 2.2, the solutions to problem (1.1) precisely correspond to the fixed points of the operator T.

4 Main results

We are prepared to demonstrate the central theorem. To enhance computational efficiency, we place

$$R = \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} + \frac{\eta^{\alpha-1}}{\alpha-1}\|\mu_1\| + \big(\frac{\eta^{\alpha+1}+1}{\alpha+1} + \frac{\eta|1-\eta^\alpha|}{\alpha-1}\big)\|\mu_2\|.$$

Theorem 4.1. Given that the contraction condition described below is satisfied:
(H_1) There exists $L_f > 0$ such that

$$|f(t,u,\overline{u}) - f(t,v,\overline{v})| \le L_f\big(|u-v| + |\overline{u} - \overline{v}|\big), \forall t \in [0,1], u, v, \overline{u}, \overline{v} \in \mathbb{R}. \tag{4.1}$$

If $L_f R < 1$, the boundary value problem (BVP) (1.1) possesses a singular solution within the confined region of the ball $B_r = u \in C^1[0,1] \;:\; |u| \le r$, where $r \ge \frac{N_f R}{1 - L_f R}$. Here, N_f is defined as $\sup_{t\in[0,1]} |f(t,0,0)|$.

Proof . Initially, we establish that $T : B_r \to B_r$, implying that $T(B_r) \subset B_r$. Let's consider $u \in B_r$ and $t \in [0,1]$, indicating that $|u| = |u|\infty + |u'|\infty \le r$. Our objective is to demonstrate that $\|Tu\| = \|Tu\|_\infty + \|(Tu)'\|_\infty \le r$. It is routine to see that

$$\begin{aligned}
|Tu(t)| \le\;& \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} |f(s,u(s),u'(s))|ds + \|\mu_1\|_\infty \int_0^\eta |\eta - s|^{\alpha-2} |f(s,u(s),u'(s))|ds \\
&+ \|\mu_2\|_\infty \big[\int_0^\eta |\eta - s|^{\alpha} |f(s,u(s),u'(s))|ds + \int_0^1 |1-s|^{\alpha} |f(s,u(s),u'(s))|ds \\
&+ \big|\frac{1-\eta^\alpha}{\eta^{\alpha-2}}\big| \int_0^\eta |\eta - s|^{\alpha-2} |f(s,u(s),u'(s)|ds\big] \qquad (4.2) \\
\le\;& \frac{1}{\Gamma(\alpha+1)} t^\alpha \|f\|_\infty + \frac{\eta^{\alpha-1}}{\alpha-1}\|\mu_1\|_\infty \|f\|_\infty + \frac{\eta^{\alpha+1}}{\alpha+1}\|\mu_2\|_\infty \|f\|_\infty \\
&+ \frac{1}{\alpha+1}\|\mu_2\|_\infty \|f\|_\infty + \frac{\eta|1-\eta^\alpha|}{\alpha-1}\|\mu_2\|_\infty \|f\|_\infty. \qquad (4.3)
\end{aligned}$$

But

$$\begin{aligned}
|f(t,u(t),u'(t))| &= |f(t,u(t),u'(t)) + f(t,0,0) - f(t,0,0)| \\
&\le |f(t,u(t),u'(t)) - f(t,0,0)| + |f(t,0,0)| \\
&\le L_f(|u(t)| + |u'(t)|) + |f(t,0,0)|.
\end{aligned}$$

Take supremum of inequality above $\|f\|_\infty \le L_f\|u\| + N_f \le L_f r + N_f$. Therefore

$$\|Tu\|_\infty \le (L_f r + N_f)\big(\frac{1}{\Gamma(\alpha+1)} + \frac{\eta^{\alpha-1}}{\alpha-1}\|\mu_1\|_\infty + \frac{\eta^{\alpha+1}+1}{\alpha+1}\|\mu_2\|_\infty + \frac{\eta|1-\eta^\alpha|}{\alpha-1}\|\mu_2\|_\infty\big). \tag{4.4}$$

Also

$$\begin{aligned}(Tu)'(t) =&\frac{1-\alpha}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}f(s,u(s),u'(s))ds+\mu_1'(t)\int_0^\eta(\eta-s)^{\alpha-2}f(s,u(s),u'(s))ds\\&+\mu_2'(t)\big[\int_0^\eta(\eta-s)^{\alpha}f(s,u(s),u'(s))ds-\int_0^1(1-s)^{\alpha}f(s,u(s),u'(s))ds\\&+\frac{(1-\eta^\alpha)}{\eta^{\alpha-2}}\int_0^\eta(\eta-s)^{\alpha-2}f(s,u(s),u'(s))ds\big].\end{aligned}\tag{4.5}$$

So we have

$$|(Tu)'(t)|\le\frac{t^{\alpha-1}}{\Gamma(\alpha)}\|f\|_\infty+\frac{\eta^{\alpha-1}}{\alpha-1}|\mu_1'\|_\infty\|f\|_\infty+\big(\frac{\eta^{\alpha+1}+1}{\alpha+1}+\frac{\eta|1-\eta^\alpha|}{\alpha-1}\big)\|\mu_2'\|\|f\|_\infty.\tag{4.6}$$

This yields:

$$\|(Tu)'\|_\infty\le(L_fr+N_f)\big[\frac{1}{\Gamma(\alpha)}+\frac{\eta^{\alpha-1}}{\alpha-1}\|\mu_1'\|_\infty+\big(\frac{\eta^{\alpha+1}+1}{\alpha+1}+\frac{\eta|1-\eta^\alpha|}{\alpha-1}\big)\|\mu_2'\|_\infty\big].\tag{4.7}$$

Combining (4.4) and (4.7), we conclude

$$\|Tu\|\le(L_fr+N_f)\big[\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}+\frac{\eta^{\alpha-1}}{\alpha-1}\|\mu_1\|+\big(\frac{\eta^{\alpha+1}+1}{\alpha+1}+\frac{\eta|1-\eta^\alpha|}{\alpha-1}\big)\|\mu_2\|\big]=(L_fr+N_f)R.$$

Selecting r such that $r\ge(L_fr+N_f)R$ is enough to ensure $TB_r\subset B_r$. Moving forward, let's demonstrate that T is a contraction. It's worth noting that, for any arbitrary $u,v\in C^1[0,1]$, we have:

$$\begin{aligned}|(Tu)(t)-(Tv)(t)| \;\le\;& \frac{1}{\Gamma(\alpha)}\int_0^t|t-s|^{\alpha-1}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds\\&+\|\mu_1\|_\infty\int_0^\eta|\eta-s|^{\alpha-2}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds\\&+\|\mu_2\|_\infty\big[\int_0^\eta|\eta-s|^{\alpha}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds\\&+\int_0^1|1-s|^{\alpha}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds\\&+\frac{|1-\eta^\alpha|}{\eta^{\alpha-2}}\int_0^\eta|\eta-s|^{\alpha-2}|f(s,u(s),u'(s)-f(s,v(s),v'(s))|ds\big]\\ \le\;& L_f\|u-v\|\big[\frac{1}{\Gamma(\alpha+1)}+\frac{\eta^{\alpha-1}}{\alpha-1}\|\mu_1\|_\infty+\big(\frac{\eta^{\alpha+1}+1}{\alpha+1}+\frac{\eta|1-\eta^\alpha|}{\alpha-1}\big)\|\mu_2\|_\infty\big].\end{aligned}$$

Similarly

$$|(Tu)'(t)-(Tv)'(t)|\le L_f\|u-v\|\big[\frac{1}{\Gamma(\alpha)}+\frac{\eta^{\alpha-1}}{\alpha-1}\|\mu_1'\|_\infty+\big(\frac{\eta^{\alpha+1}+1}{\alpha+1}+\frac{\eta|1-\eta^\alpha|}{\alpha-1}\big)\|\mu_2'\|_\infty\big].\tag{4.8}$$

Form (4.8) and (4.8) we obtain: $\|Tu-Tv\|\le L_f\|u-v\|R$. Applying the Banach Contraction Mapping Theorem, the equation referenced by (1.1) possesses a sole solution within the interval $[0,1]$. $\square$

5 Illustrative example

Consider the following fractional differential equation

$$\begin{cases}{}_0D_t^{2.5}u+\rho\left(\frac{u(t)}{2+u(t)}+\sin(u'(t))+\cos(t)\right)=0, & t\in(0,1),\ \rho>0\\ u(0)=u'(0)=0, \quad u'(\frac{1}{2})+\int_{\frac{1}{2}}^1u(s)ds=0.\end{cases}\tag{5.1}$$

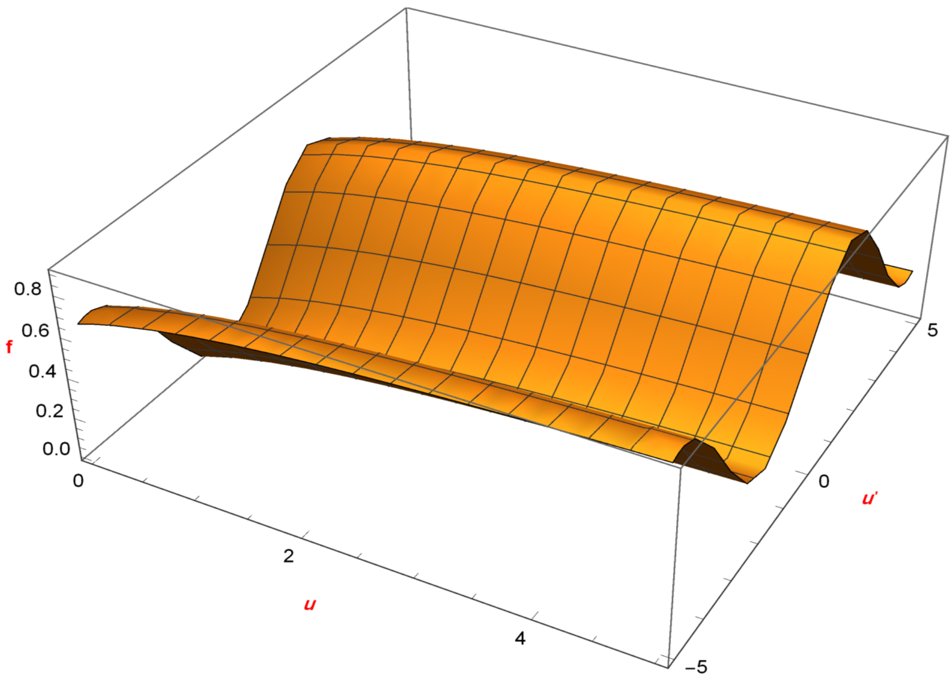

Figure 1: : Diagram of f versus u and u' for the Example.

As it is seen, $\alpha = \frac{5}{2}$, $\eta = \frac{1}{2}$ and $f(t,u(t),u'(t)) = \rho\left(\frac{u(t)}{2+u(t)} + \sin\left(u'(t)\right) + \cos(t)\right)$. It is obviously observed that

$$\begin{aligned}
|f(t,u(t),u'(t)) - f(t,v(t),v'(t))| &= \left|\rho\left(\frac{u(t)}{2+u(t)} + \sin\left(u'(t)\right) + \cos(t)\right) - \rho\left(\frac{v(t)}{2+v(t)} + \sin\left(v'(t)\right) + \cos(t)\right)\right| \\
&= \left|\rho\left(\frac{u(t)}{2+u(t)} - \frac{v(t)}{2+v(t)}\right) - \rho\left(\sin(u'(t) - \sin(v'(t))\right)\right| \\
&\leq \rho\left(\left|\frac{|u(t)|}{2+|u(t)|} - \frac{|v(t)|}{2+|v(t)|}\right|\right) + \rho\left|\sin(u'(t) - \sin(v'(t)\right| \\
&\leq \rho\left|u(t) - v(t)\right| + \rho\left|u'(t) - v'(t)\right| \\
&= \rho\left(|u(t) - v(t)| + |u'(t) - v'(t)|\right). \qquad (5.2)
\end{aligned}$$

On the other hand, routine calculation implies the following results:

$$\mu_1(t) = \frac{4}{3}\sqrt{\frac{2}{\pi}}t^{3/2}, \qquad \mu_2(t) = \frac{4t^{3/2}}{3\left(2\sqrt{2}-1\right)\sqrt{\pi}}, \qquad (5.3)$$

$$\|\mu_1(t)\| = \|\mu_1(t)\|_\infty + \|\mu_1'(t)\|_\infty = 2.6596152026762176, \qquad (5.4)$$

$$\|\mu_2(t)\| = \|\mu_2(t)\|_\infty + \|\mu_2'(t)\|_\infty = 1.0285517643588031, \qquad (5.5)$$

$$N_f = \sup_{t\in[0,1]} |f(t,0,0)| = 1, \qquad (5.6)$$

$$R = 2.2821205857367284. \qquad (5.7)$$

Hence, by selecting a value for ρ such that $\rho R < 1$, we ensure that the conditions outlined in Theorem 4.1 are satisfied. Consequently, we establish the existence of a unique solution within the confines of the ball $B_r = u \in C^1[0,1] \,:\, |u| \leq r$, where $r \geq \frac{R}{1-\rho R}$. The feasibility of the function f for the singular solution at $t = 0$ is illustrated in Fig. 1, with the chosen parameter $\rho = \frac{1}{3}$.

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References

[1] L. AhmadSoltani, *Two-point nonlocal nonlinear fractional boundary value problem with Caputo derivative: Analysis and numerical solution*, Nonlinear Engin. **11** (2022), no. 1, 71–79.

[2] N. Asaithambi and J. Garner, *Pointwise solution bounds for a class of singular diffusion problems in physiology*, Appl. Math. Comput. **30** (1989), no. 3, 215–222.

[3] A.C. Burton, *Rate of growth of solid tumours as a problem of diffusion*, Growth **30** (1966), no. 2, 157–176.

[4] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer Science & Business Media, 2010.

[5] A. Dinmohammadi, A. Razani, and E. Shivanian, *Analytical solution to the nonlinear singular boundary value problem arising in biology*, Bound. Value Prob. **2017** (2017), no. 1, 1–9.

[6] A. Dinmohammadi, E. Shivanian, and A. Razani, *Existence and uniqueness of solutions for a class of singular nonlinear two-point boundary value problems with sign-changing nonlinear terms*, Numer. Funct. Anal. Optim. **38** (2017), no. 3, 344–359.

[7] R. Duggan and A. Goodman, *Pointwise bounds for a nonlinear heat conduction model of the human head*, Bull. Math. Bio. **48** (1986), no. 2, 229–236.

[8] U. Flesch, *The distribution of heat sources in the human head: A theoretical consideration*, J. Theor. Bio. **54** (1975), no. 2, 285–287.

[9] B. Gray, *The distribution of heat sources in the human head—theoretical considerations*, J. Theor. Bio. **82** (1980), no. 3, 473–476.

[10] H. Greenspan, *Models for the growth of a solid tumor by diffusion*, Stud. Appl. Math. **51** (1972), no. 4, 317–340.

[11] D. Haghighi, S. Abbasbandy, and E. Shivanian, *Study of the fragile points method for solving two-dimensional linear and nonlinear wave equations on complex and cracked domains*, Engin. Anal. Bound. Elements **146** (2023), 44–55.

[12] S. Heidari and A. Razani, *Existence and multiplicity of weak solutions for singular fourth-order elliptic systems*, São Paulo J. Math. Sci. **16** (2022), no. 2, 1309–1326.

[13] A. Kilbas, *Theory and Applications of Fractional Differential Equations*, Vol. 204, Elsevier, 2006.

[14] R. Klages, G. Radons, and I.M. Sokolov, *Anomalous Transport*, Wiley Online Library, 2008.

[15] R. Kubo, *The fluctuation-dissipation theorem*, Rep. Prog. Phys. **29** (1966), no. 1, 255.

[16] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II: Nonequilibrium Statistical Mechanics*, vol. 31, Springer Science & Business Media, 2012.

[17] R.L. Magin, *Fractional Calculus in Bioengineering*, vol. 2, Begell House Redding, 2006.

[18] D. McElwain, *A re-examination of oxygen diffusion in a spherical cell with Michaelis-Menten oxygen uptake kinetics*, J. Theor. Bio. **71** (1978), no. 2, 255–263.

[19] S. Rezabeyk, S. Abbasbandy, E. Shivanian, and H. Derili, *A new approach to solve weakly singular fractional-order delay integro-differential equations using operational matrices*, J. Math. Model. **11** (2023), no. 2, 257–275.

[20] E. Shivanian, *Error estimate and stability analysis on the study of a high-order nonlinear fractional differential equation with Caputo-derivative and integral boundary condition*, Comput. Appl. Math. **41** (2022), no. 8, 395.

[21] E. Shivanian, *On the solution of Caputo fractional high-order three-point boundary value problem with applications to optimal control*, J. Nonlinear Math. Phys. **31** (2024), no. 1, 2.

[22] E. Shivanian and A. Dinmohammadi, *On the solution of a nonlinear fractional integro-differential equation with non-local boundary condition*, Int. J. Nonlinear Anal. Appl. **15** (2024), no. 1, 125–136.

[23] E. Shivanian and H. Fatahi, *To study existence of unique solution and numerically solving for a kind of three-point boundary fractional high-order problem subject to Robin condition*, Comput. Meth. Differ. Equ. **11** (2023), no. 2, 332–344.

[24] E. Shivanian, S.J.H. Ghoncheh, and H. Goudarzi, *A unique weak solution for the fractional integro-differential Schrödinger equations*, Math. Sci. **17** (2023), no. 1, 15–19.

[25] E. Shivanian, Z. Hajimohammadi, F. Baharifard, K. Parand, and R. Kazemi, *A novel learning approach for different profile shapes of convecting–radiating fins based on shifted Gegenbauer LSSVM*, New Math. Natural Comput. **19** (2023), no. 1, 195–215.