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Determination of an unknown time dependent coefficient for a semilinear time-fractional parabolic equation

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Abstract

In this paper, an inverse problem of determining the time-dependent coefficient of a semilinear parabolic equation involving the Caputo fractional derivative in time, with nonlocal boundary and integral overdetermination conditions, is considered. Existence, uniqueness, and stability results of a classical solution are established using the Fourier method, the iteration method, and Gronwall's Lemma. Moreover, we provide an example to illustrate the obtained results.

Keywords: Inverse problem, semilinear parabolic equation, fractional derivative, nonlocal boundary condition, iteration method, Gronwall's Lemma
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1 Introduction

In recent years, partial differential equations in particular of the parabolic type, involving partial derivative of non-integer order known as fractional order have attracted many researchers because of their applications in many areas of science and engineering, as shown in [5, 14, 20] and references therein. In many practical situations, various parameters are unknown such as fractional order, diffusion coefficient, source or reaction term in parabolic or fractional parabolic problems, which were studied in [1, 3, 4, 8, 9, 10, 11, 13, 17, 19, 21] according to the data. Let us mention that in [10, 11], the authors treated the inverse problem of identifying the unknown time-dependent coefficient in the quasilinear parabolic equation with the nonlocal boundary and integral overdetermination conditions. In [4], the inverse problem of finding the time-dependent source coefficient for the a semilinear fractional reaction diffusion equation with the nonlocal boundary condition

$$u(0,t) = u(1,t);\ u_x(1,t) = 0, \qquad t \in [0,T].$$

has been considered.

In this article, we are concerned with the inverse problem of determining the pair $\{u(x,t), a(t)\}$ for the following time fractional equation

$$^{C}\partial^{\alpha}_{0+,t}u(x,t) = u_{xx}(x,t) - a(t)u(x,t) + F(x,t,u(x,t)),\ (x,t) \in D_T \tag{1.1}$$

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with the initial condition

$$u(x,0)=\varphi(x), \qquad x\in[0,1] \tag{1.2}$$

the nonlocal boundary conditions

$$u_x(0,t)=u_x(1,t);\; u(1,t)=0, \qquad t\in[0,T] \tag{1.3}$$

and the integral overdetermination data

$$\int_0^1 u(x,t)dx=E(t), \qquad t\in[0,T] \tag{1.4}$$

where $D_T=(0,1)\times(0,T]$, ${}^C\partial^\alpha_{0^+,t}$ stands for the Caputo fractional derivative of order $0<\alpha<1$ with respect to the time variable, $\varphi(x)$ and $F(x,t,u(x,t))$ are given functions on $[0,1]$ and $\bar{D}_T\times\mathbb{R}$ respectively, meanwhile, the unknown potential coefficient $a(t)$ which can be regarded as a control parameter, see [3] has to be determined. The integral data (1.4) is needed for the unique solvability of our inverse problem. The nonlocal boundary conditions (1.3) are commonly referred to as the boundary conditions describing the relationship between the desired solution values on multiple points, which can be more useful than the standard classical boundary conditions for describing some applications of heat conduction or thermo-elasticity, see [3, 2].

The keys factor in the solvability of our inverse problem is based on the expansion of the solution by using a bi-orthogonal system of functions obtained from the nonlocal boundary conditions, the construction of convergent iterations and the use of some known inequalities, mainly those of Gronwall, Cauchy and Hölder.

The rest of the paper is organized as follows: In Section 2, we recall some definitions and basic results for the convenience of the readers. In Section 3, our main results are discussed about the existence, uniqueness of the solution $\{u(x,t),a(t)\}$ of the inverse problem (1.1)-(1.4) and its continuous dependence upon the data $(\phi,E(t))$.

2 Preliminaries

In this section, we introduce notations, definitions from fractional analysis (see [16, 12, 18]) and preliminary facts which are used throughout this paper.

Definition 2.1. For an integrable function $f:\mathbb{R}^+\to\mathbb{R}$, the left sided Riemann-Liouville fractional integral of order $0<\alpha<1$ is defined by

$$I^\alpha_{0^+}f(t):=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \quad t>0,$$

where $\Gamma(.)$ denotes the Gamma function which satisfies $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$.

Definition 2.2. The left sided Caputo fractional derivative for a function f of order $0<\alpha<1$, is defined by

$${}^CD^\alpha_{0^+}f(t):=I^{1-\alpha}_{0^+}\frac{d}{dt}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f'(s)ds, \quad t>0.$$

Notice that the Caputo's fractional derivative of $f(t)$ exists and is continuous for $t>0$ when $f\in AC[0,T]$. Furthermore, we have

$${}^CD^\alpha_{0^+}f(t)=D^\alpha_{0^+}\left(f(t)-f(0)\right), \tag{2.1}$$

where $D^\alpha_{0^+}$ is left sided Riemann-Liouville fractional derivative defined by

$$D^\alpha_{0^+}f(t):=\frac{d}{dt}I^{1-\alpha}_{0^+}f(t), \quad \text{for } 0<\alpha<1.$$

Proposition 2.1. For $0<\alpha<1$ and $f\in AC[0,T]$, we have

$$I^{\alpha}_{0^+}{}^CD^\alpha_{0^+}f(t)=f(t)-f(0).$$

Let us give some results concerning the Mittag-Leffler function and start by its most used definition.

Definition 2.3. The Mittag-Leffler function of two-parameter is defined by

$$E_{\alpha,\beta}(z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(\alpha n + \beta)}; \quad \alpha > 0, \beta > 0, \text{ with } z \in \mathbb{C}.$$

In particular, for $\beta = 1$ we recover the classical Mittag-Leffler function $E_{\alpha,1}(z) = E_\alpha(z)$.

Corollary 2.1. The following properties hold

(i) for $0 < \alpha \leq \beta \leq 1, \lambda \in \mathbb{R}^+$, $E_{\alpha,\beta}(-\lambda t^\alpha)$ is a bounded positive monotonic decreasing function on the positive real axis.

(ii) The Mittag-Leffer function $E_\alpha(\lambda t^\alpha)$ is invariant with respect to the left Caputo fractional derivative, that is

$$^C D^\alpha_{0+} E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha), \quad \text{for } \alpha > 0, \lambda \in \mathbb{R}. \tag{2.2}$$

The following lemma is a well known property of Mittag-Leffler function and can be found in Podlubny's book [16].

Lemma 2.1. If $0 < \alpha < 2$, β is an arbitrary real number, μ satisfies that $\pi\alpha/2 < \mu < \min(\pi, \alpha\pi)$ and C^* is a real constant, then

$$|E_{\alpha,\beta}(z)| \leq \frac{C^*}{1 + |z|}, \quad \mu \leq |\arg(z)| \leq \pi, |z| \geq 0.$$

From this result, we conclude that for $0 < \alpha < 1$, $\lambda > 0$ and $t \in [0, T]$ that

$$\lambda E_{\alpha,\alpha}(-\lambda t^\alpha) \leq \frac{\lambda C^*}{1 + \lambda t^\alpha} < \infty. \tag{2.3}$$

Now, because of the relation (2.1) and linearity of fractional differential operators, we can extend Lemma 15.2 of Samko et al [18] to the following Lemma.

Lemma 2.2. Let $(f_i(t))_{i\geq 0}$ be a sequence of functions defined on $(0, T]$. Suppose the following conditions are fulfilled:

(i) For a given $\alpha > 0$, the α-derivatives $^C D^\alpha_{0+} f_i(t), i \geq 0$ for $t \in (0, T]$ exists.

(ii) $\sum_{i=1}^\infty f_i(t)$ and $\sum_{i=1}^\infty {}^C D^\alpha_{0+} f_i(t)$ are uniformly convergent series on the interval $[\varepsilon, T]$ for any $\varepsilon > 0$.

Then, the function defined by the series $\sum_{i=1}^\infty f_i(t)$ is α-differentiable and satisfies

$$^C D^\alpha_{0+} \sum_{i=1}^\infty f_i(t) = \sum_{i=1}^\infty {}^C D^\alpha_{0+} f_i(t).$$

Theorem 2.1. Let $\alpha \in (0, 1), f \in L^1[0, T], c \in \mathbb{R}$. Then the solution of the problem

$$\begin{cases} ^C D^\alpha_{0+} u(t) + \lambda u(t) &= f(t), \quad t > 0 \\ u(0) &= c \end{cases}$$

is given in $AC[0, T]$ and satisfies

$$u(t) = cE_\alpha(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s) ds.$$

Next, we consider the following system of functions on $[0, 1]$:

$$\Phi = \{\phi_0(x), \phi_{1,n}(x), \phi_{2,n}(x)\}_{n=1}^\infty, \quad \Psi = \{\psi_0(x), \psi_{1,n}(x), \psi_{2,n}(x)\}_{n=1}^\infty, \tag{2.4}$$

where

$$\phi_0(x) = 2(1-x), \quad \phi_{1,n}(x) = 4(1-x)\cos(\sqrt{\lambda_n}x), \quad \phi_{2,n}(x) = 4\sin(\sqrt{\lambda_n}x), \tag{2.5}$$

$$\psi_0 = 1, \quad \psi_{1,n}(x) = \cos(\sqrt{\lambda_n}x), \quad \psi_{2,n} = x\sin(\sqrt{\lambda_n}x), \tag{2.6}$$

with $\lambda_n = (2n\pi)^2, n \geq 1$. Since the nonlocal boundary value spectral problem obtained from (1.1) and (1.3) is nonself-adjoint and the set of eigenvectors with the eigenvalues $\lambda_n, n \geq 1$ of this spectral problem is not complete in the space $L^2(0,1)$, another complete set of eigenvectors (2.5) and associated set of eigenvectors (2.6) of the adjoint problem is obtained. The system (2.4) is bi-orthogonal and forms a Riez basis in $L^2(0,1)$. For more details, see [7, 6, 15]. We denote the inner product in $L^2(0,1)$ by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx, \quad \forall f, g \in L^2(0,1).$$

To define what we mean by a classical solution, let us introduce the following classes of functions

$$C^{1,0}\left(\overline{D}_T\right) = \left\{u(.,t) \in C^1[0,1], \quad t \in [0,T] \text{ and } u(x,.) \in C[0,T], x \in [0,1]\right\};$$
$$C^{2,\alpha}(D_T) = \left\{u(.,t) \in C^2(0,1), \quad t \in (0,T] \text{ and } {}^C D^{\alpha}_{0+,t} u(x,.) \in C(0,T], x \in (0,1)\right\}.$$

Definition 2.4. The pair $\{u(x,t), a(t)\}$ is said to be a classical solution of inverse problem (1.1)-(1.4) if $\{u(x,t), a(t)\} \in \left[C^{2,\alpha}(D_T) \cap C^{1,0}\left(\bar{D}_T\right)\right] \times C[0,T]$ and $a(t)$ is positive on $[0,T]$ such that equation (1.1) and conditions (1.2)-(1.4) are satisfied.

Finally, we recall the Gronwall's lemma.

Lemma 2.3. Let $u(t)$ and $v(t)$ be nonnegative continuous functions on some interval $[0,T]$. Also, let the function $f(t)$ be positive, continuous, and monotonically nondecreasing on $[0,T]$ and satisfies the inequality

$$u(t) \leq f(t) + \int_0^t v(s)\, u(s)\, ds,$$

then

$$u(t) \leq f(t) \exp\left(\int_0^t v(s)\, ds\right), \quad \text{for } t \in [0,T].$$

3 Main results

In this section, we will give the results about existence, uniqueness of the solution and the continuous dependence upon the data of our inverse problem (1.1)-(1.4).

At this stage, for arbitrary $a \in C[0,T]$ and by applying the Fourier method, the solution $u(x,t)$ of the direct problem (1.1)-(1.3) can be expanded to uniformly convergent series form using the bi-orthogonal system (2.4) in $L^2(0,1)$ as follows

$$u(x,t) = u_0(t)\phi_0(x) + \sum_{n\geq 1} u_{1,n}(t)\phi_{1,n}(x) + \sum_{n\geq 1} u_{2,n}(t)\phi_{2,n}(x); \tag{3.1}$$

$$\text{where } u_0(t) = \langle u(.,t), \psi_0 \rangle, \quad u_{k,n}(t) = \langle u(.,t), \psi_{k,n} \rangle, \quad n \geq 1, k = 1,2$$

are to be determined. We denote, the coefficients of the series expansion of $\varphi(x)$ and $F(x,t,u(x,t))$ in the basis (2.6) by

$$\varphi_0 = \langle \varphi, \psi_0 \rangle, \quad \varphi_{k,n} = \langle \varphi, \psi_{k,n} \rangle; \quad k = 1,2, n \geq 1$$

$$F_0(t,u) = \langle F(.,t,u(.,t)), \psi_0 \rangle, \quad F_{k,n}(t,u) = \langle F(.,t,u(.,t)), \psi_{k,n} \rangle; \quad k = 1,2, n \geq 1,$$

respectively. By properties of bi-orthogonal system (2.4), we obtain from (1.1)-(1.3), the following Cauchy problems on $[0,T]$ and $n \geq 1$

$$\begin{cases} {}^C D^{\alpha}_{0+} u_0(t) = -a(t)u_0(t) + F_0(t,u), \\ u_0(0) = \varphi_0, \end{cases} \tag{3.2}$$

$$\begin{cases} {}^C D_{0+}^{\alpha} u_{1,n}(t) = -\lambda_n u_{1,n}(t) - a(t) u_{1,n}(t) + F_{1,n}(t,u), \\ u_{1,n}(0) = \varphi_{1,n}, \end{cases} \tag{3.3}$$

$$\begin{cases} {}^C D_{0+}^{\alpha} u_{2,n}(t) = -\lambda_n u_{2,n}(t) + 2\sqrt{\lambda_n} u_{1,n}(t) - a(t) u_{2,n}(t) + F_{2,n}(t,u), \\ u_{2,n}(0) = \varphi_{2,n}. \end{cases} \tag{3.4}$$

Thus, by applying the integral operator I_{0+}^{α} to the fractional differential equation of the Cauchy problem (3.2) and from Proposition 2.1, we get

$$u_0(t) = \varphi_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[a(s) u_0(s) - F_0(s,u) \right] ds, \tag{3.5}$$

While, according to Theorem 2.1, the solutions of problems (3.3),(3.4) satisfy

$$u_{1,n}(t) = \varphi_{1,n} E_{\alpha,1}(-\lambda_n t^{\alpha}) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) \left[a(s) u_{1,n}(s) - F_{1,n}(s,u) \right] ds, \tag{3.6}$$

$$\begin{aligned} u_{2,n}(t) =& \varphi_{2,n} E_{\alpha,1}(-\lambda_n t^{\alpha}) + 2\sqrt{\lambda_n} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) u_{1,n}(s) ds \\ &- \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) \left[a(s) u_{2,n}(s) - F_{2,n}(s,u) \right] ds, \end{aligned} \tag{3.7}$$

for $n \geq 1$, respectively. On other wise, from (1.4) we observe that

$$u_0(t) = E(t). \tag{3.8}$$

Now, applying ${}^C D_{0+}^{\alpha}$ to the additional condition (1.4) we obtain

$$\begin{aligned} {}^C D_{0+}^{\alpha} E(t) &= \int_0^1 {}^C \partial_{0+,t}^{\alpha} u(x,t) dx \\ &= \int_0^1 \left[u_{xx}(x,t) - a(t) u(x,t) + F(x,t,u(x,t)) \right] dx \\ &= u_x(1,t) - u_x(0,t) - a(t) E(t) + \int_0^1 F(x,t,u(x,t)) dx, \end{aligned}$$

which yields

$$a(t) = \frac{F_0(t,u) - {}^C D_{0+}^{\alpha} E(t)}{E(t)}. \tag{3.9}$$

Throughout this paper, we assume that the following conditions hold

(A_1) $\varphi \in C^4[0,1]$, $\quad \varphi(1) = 0, \varphi'(0) = \varphi'(1), \varphi''(1) = 0, \varphi^{(3)}(0) = \varphi^{(3)}(1)$.

(A_2) $F(x,t,u)$ is continuous on $\bar{D}_T \times \mathbb{R}$ and satisfies the following conditions:

- (1) $F(.,t,u) \in C^4[0,1]$, for $t \in [0,T]$, $F(x,t,u)|_{x=1} = 0$,
  $F_x(x,t,u)|_{x=0} = F_x(x,t,u)|_{x=1} = 0$, $F_{xx}(x,t,u)|_{x=1} = 0$,
  $F_{xxx}(x,t,u)|_{x=0} = F_{xxx}(x,t,u)|_{x=1}$;
- (2) there exists a nonnegative function $b \in L^2(D_T)$ such that for each $u, \tilde{u} \in \mathbb{R}$

  $$\left| \frac{\partial^n}{\partial x^n} F(x,t,u) - \frac{\partial^n}{\partial x^n} F(x,t,\tilde{u}) \right| \leq b(x,t) |u - \tilde{u}|, \quad n = 0,1,2$$

  with $B = \|b\|_{L^2(D_T)}$ and $M_b = \max\limits_{0 \leq t \leq T} \|b\,(.,t)\,\|_{L^2(0,1)} < \infty$;
- (3) $M = \max\{\|\frac{\partial^n}{\partial x^n} F(.,.,0)\|_{L^2(D_T)}; \quad n = 0,1,2\}$,
  $M_F = \max\limits_{0 \leq t \leq T} \|F(.,t,0)\|_{L^2(0,1)} < \infty$;

(4) $F_0(t,u) > 0$ for $t \in [0,T]$.

(A_3) $E \in AC\,[0,T]\,,\quad \min\limits_{0\leq t\leq T} E\,(t) = E_m > 0, \quad {}^C D^{\alpha}_{0^+} E(t) \leq 0$ for $t \in [0,T]$ and $E\,(0) = \int_0^1 \varphi(x)dx$.

Definition 3.1. We denote $\mathcal{B}$ the set of continuous functions on $[0,T]$

$$\{u(t)\} = \{u_0(t), u_{1,n}(t), u_{2,n}(t); \quad n \geq 1\}$$

satisfying

$$|u_0(t)| + \sum_{n\geq 1} \left(|u_{1,n}(t)| + |u_{2,n}(t)|\right) < \infty \text{ for } t \in [0,T].$$

It can be shown that $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a Banach space with

$$\|u\|_{\mathcal{B}} = \max_{0\leq t\leq T} |u_0(t)| + \sum_{n\geq 1} \left(\max_{0\leq t\leq T} |u_{1,n}(t)| + \max_{0\leq t\leq T} |u_{2,n}(t)| \right).$$

Lemma 3.1. If (A_1) is satisfied, then the series $\sum\limits_{n\geq 1} |\varphi_{1,n}|, \sum\limits_{n\geq 1} |\varphi_{2,n}|$ are uniformly convergent. Moreover for some positive constant C, we have

$$|\varphi_0| + \sum_{n\geq 1} \left(|\varphi_{1,n}| + |\varphi_{2,n}|\right) \leq C\|\varphi\|_{C^4[0,1]}.$$

Proof . First, we have by Cauchy's inequality

$$|\varphi_0| = |\int_0^1 \varphi(x)dx| \leq \|\varphi\|_{L^2[0,1]}.$$

By integration by parts four times, we get for $n \geq 1$

$$\begin{aligned}
\varphi_{1,n} &= \int_0^1 \varphi(x)\cos(2n\pi x)dx = \frac{1}{(2n\pi)^4}\varphi^{(4)}_{1,n};\\
\varphi_{2,n} &= \int_0^1 \varphi(x) x \sin(2n\pi x)dx = \frac{1}{(2n\pi)^4}\varphi^{(4)}_{2,n} + \frac{4}{(2n\pi)^5}\varphi^{(4)}_{1,n}.
\end{aligned}$$

Taking the sum and using Hölder's inequality, we obtain

$$\begin{aligned}
\sum_{n\geq 1} |\varphi_{1,n}| &\leq \left(\sum_{n\geq 1} \frac{1}{(2n\pi)^8}\right)^{1/2} \left(\sum_{n\geq 1} |\varphi^{(4)}_{1,n}|^2\right)^{1/2} \leq C_1\|\varphi^{(4)}\|_{L^2(0,1)};\\
\sum_{n\geq 1} |\varphi_{2,n}| &\leq \sum_{n\geq 1} \frac{1}{(2n\pi)^4}|\varphi^{(4)}_{2,n}| + \sum_{n\geq 1} \frac{4}{(2n\pi)^5}|\varphi^{(4)}_{1,n}| \leq C_2\|\varphi^{(4)}\|_{L^2[0,1]}.
\end{aligned}$$

Thus, we conclude the uniform convergence of the above series. Moreover

$$\begin{aligned}
|\varphi_0| + \sum_{n\geq 1} |\varphi_{1,n}| + \sum_{n\geq 1} |\varphi_{2,n}| &\leq \|\varphi\|_{L^2[0,1]} + (C_1 + C_2)\,\|\varphi^{(4)}\|_{L^2[0,1]}\\
&\leq C\|\varphi\|_{C^4[0,1]}.
\end{aligned}$$

□

Remark 3.1. In the same way, from (A_1)-(A_2), we can deduce the uniform convergence of the following series

$$\sum_{n\geq 1} \lambda_n|\varphi_{2,n}|, \quad \sum_{n\geq 1} \lambda_n|F_{2,n}(t,u)|, \quad \sum_{n\geq 1} (\sqrt{\lambda_n})^k|\varphi_{1,n}|, \quad \sum_{n\geq 1} (\sqrt{\lambda_n})^k|F_{1,n}(t,u)| \quad \text{for } k = 1,2,3.$$

Now, from the decrease of the Mittag-Leffler type functions and the estimation (2.3), we have for $\lambda_n = (2\pi n)^2\,, n \geq 1$

$$\sup_{0<t<T} E_{\alpha}(-\lambda_n t^{\alpha}) = 1, \sup_{0<t<s<T} E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) = \frac{1}{\Gamma(\alpha)} < 1 \quad \text{and} \quad \lambda_n E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) \leq N_{\lambda}, \quad 0 < t < s < T. \tag{3.10}$$

3.1 Existence and uniqueness of the solution

Theorem 3.1. Let $(A_1)-(A_3)$ be satisfied. If $1/2 < \alpha < 1$, then the inverse problem (1.1)-(1.4) has a unique classical solution $\{u(x,t), a(t)\}$ on $[0,T]$ for a large T.

Proof . Before going further, we will explain two techniques noted **Step K** and **Step G**, that we will use in most of all the estimates in the other steps.

Step K: Adding and subtracting $\int_0^t \int_0^1 (t-s)^{\alpha-1} F(x,s,0) W_n(x) dxds$ to the first memeber of the following inequality, we obtain

$$\left|\int_0^t \int_0^1 (t-s)^{\alpha-1} F(x,s,u(x,s)) W_n(x) dxds\right| \leq \int_0^t (t-s)^{\alpha-1} \left|\int_0^1 [F(x,s,u(x,s)) - F(x,s,0)] W_n(x) dx\right| ds + \int_0^t (t-s)^{\alpha-1} \left|\int_0^1 F(x,s,0) W_n(x) dx\right| ds,$$

using the Cauchy-Schwartz inequality, we find

$$\begin{aligned}
&\left|\int_0^t \int_0^1 (t-s)^{\alpha-1} F(x,s,u(x,s)) W_n(x) dxds\right| \\
&\leq \left(\int_0^t (t-s)^{2\alpha-2} ds\right)^{1/2} \left(\int_0^t \left(\int_0^1 [F(x,s,u(x,s)) - F(x,s,0)] W_n(x) dx\right)^2 ds\right)^{1/2} \\
&\quad + \left(\int_0^t (t-s)^{2\alpha-2} ds\right)^{1/2} \left(\int_0^t \left(\int_0^1 F(x,s,0) W_n(x) dx\right)^2 ds\right)^{1/2} \\
&\leq \frac{T^{\alpha-1/2}}{\sqrt{2\alpha-1}} \left(\int_0^t \left(\int_0^1 [F(x,s,u(x,s)) - F(x,s,0)] W_n(x) dx\right)^2 ds\right)^{1/2} + \frac{T^{\alpha-1/2}}{\sqrt{2\alpha-1}} \left(\int_0^t \left(\int_0^1 F(x,s,0) W_n(x) dx\right)^2 ds\right)^{1/2}.
\end{aligned}$$

Step G: When we get the following form of inequality on $[0,T]$ with continuous functions $v(t)$ and $P_1(t)$

$$v(t) \leq P_1(t) + P_2 \left(\int_0^t (v(s))^2 ds\right)^{1/2},$$

we take the square to have

$$(v(t))^2 \leq 2P_1^2(t) + 2P_2^2 \int_0^t (v(s))^2 ds. \tag{3.11}$$

Appling the Gronwall's Lemma on (3.11), we get $(v(t))^2 \leq 2P_1^2(t) \exp(2P_2^2 t)$, therefore, we obtain $v(t) \leq \sqrt{2} P_1(t) \exp(P_2^2 t)$.

Step 1: Let us define an iteration for the Fourier coefficient of (3.1) for $N \geq 0$ as follows:

$$u_0^{(N+1)}(t) = E(t); \tag{3.12}$$

$$\begin{aligned}
u_{1,n}^{(N+1)}(t) =& u_{1,n}^{(0)}(t) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) a^{(N)}(s) u_{1,n}^{(N+1)}(s) ds \\
&+ \int_0^t \int_0^1 (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) F(x,s,u^{(N)}(x,s)) \cos(\sqrt{\lambda_n} x) dxds,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
u_{2,n}^{(N+1)}(t) =& u_{2,n}^{(0)}(t) + 2\sqrt{\lambda_n} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) u_{1,n}^{(N+1)}(s) ds \\
&- \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) a^{(N)}(s) u_{2,n}^{(N+1)}(s) ds \\
&+ \int_0^t \int_0^1 (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) F(x,s,u^{(N)}(x,s)) x \sin(\sqrt{\lambda_n} x) dxds,
\end{aligned} \tag{3.14}$$

with

$$u_0^{(0)}(t)=\varphi_0,\quad u_{1,n}^{(0)}(t)=\varphi_{1,n}E_{\alpha,1}(-\lambda_n t^\alpha),\quad u_{2,n}^{(0)}(t)=\varphi_{2,n}E_{\alpha,1}(-\lambda_n t^\alpha). \tag{3.15}$$

Again, we define an iteration for (3.9) on $t\in[0,T]$ for $N\geq 0$

$$a^{(N)}(t)=\frac{1}{E(t)}\left[-^C D_{0^+}^\alpha E(t)+\int_0^1 F(x,t,u^{(N)}(x,t))dx\right]. \tag{3.16}$$

First, by definition 3.1, the estimate of Mittag-Leffer type function in (3.10) and Lemma 3.1, we get

$$\|u^{(0)}\|_{\mathcal{B}}\leq|\varphi_0|+\sum_{n\geq 1}(|\varphi_{1,n}|+|\varphi_{2,n}|)\leq C\|\varphi\|_{C^4[0,1]},$$

thus $u^{(0)}\in\mathcal{B}$. For $N=0$, we have on $[0,T]$

$$a^{(0)}(t)=\frac{1}{E(t)}\left[-^C D_{0^+}^\alpha E(t)+\int_0^1 F(x,t,u^{(0)}(x,t))dx\right]>0.$$

Adding and subtracting $\frac{1}{E(t)}\int_0^1 F(x,t,0)dx$ to the above equation, then using the Lipschitz condition, we get

$$\begin{aligned}a^{(0)}(t)&\leq\frac{|^C D_{0^+}^\alpha E(t)|}{E(t)}+\frac{1}{E(t)}\left|\int_0^1\left[F(x,t,u^{(0)}(x,t))-F(x,t,0)\right]dx\right|+\frac{1}{E(t)}\int_0^1 F(x,t,0)dx\\&\leq\frac{|^C D_{0^+}^\alpha E(t)|}{E(t)}+\frac{1}{E(t)}\int_0^1 b(x,t)\left|u^{(0)}(x,t)\right|dx+\frac{1}{E(t)}\int_0^1 F(x,t,0)dx.\end{aligned}$$

From Cauchy's inequality, (A_2) and (A_3), we find

$$a^{(0)}(t)\leq E_m^{-1}\left(\|^C D_{0^+}^\alpha E\|_{C[0,T]}+\|b\,(.,t)\,\|_{L^2(0,1)}\|u^{(0)}\|_{\mathcal{B}}+\|F(.,t,0)\|_{L^2(0,1)}\right).$$

Taking into consideration that

$$\|^C D_{0^+}^\alpha E\|_{C[0,T]}\leq\frac{T^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\|E'\|_{L^\infty[0,T]},\ \text{for } 0<\alpha<1,$$

then for some positive constant D, we get

$$\|a^{(0)}\|_{C[0,T]}\leq E_m^{-1}\left(D\|E\|_{AC[0,T]}+M_b\|u^{(0)}\|_{\mathcal{B}}+M_F\right).$$

Since $u^{(0)}\in\mathcal{B}$, we deduce that $a^{(0)}\in C[0,T]$. It's clear that, from (3.12) we have for $N\geq 0$

$$\max_{0\leq t\leq T}|u_0^{(N+1)}(t)|\leq\|E\|_{C[0,T]}. \tag{3.17}$$

Next, we consider $u_{1,n}^{(1)}(t)$ and $u_{2,n}^{(1)}(t)$ are given by (3.13) and (3.14) for $N=0$ respectively. Using (3.10) and $a^{(0)}\in C[0,T]$ we obtain

$$|u_{1,n}^{(1)}(t)|\leq|\varphi_{1,n}|+\|a^{(0)}\|_{C[0,T]}\int_0^t(t-s)^{\alpha-1}\left|u_{1,n}^{(1)}(s)\right|ds+\left|\int_0^t\int_0^1(t-s)^{\alpha-1}F(x,s,u^{(0)}(x,s))\cos(2\pi nx)dxds\right|.$$

Applying Cauchy-Schwartz's inequality and **Step K**, then we integrate twice by parts the integrals depending on F with respect to x on $[0,1]$, we get

$$\begin{aligned}|u_{1,n}^{(1)}(t)|\leq&|\varphi_{1,n}|+\|a^{(0)}\|_{C[0,T]}L_\alpha\left(\int_0^t|u_{1,n}^{(1)}(t)|^2ds\right)^{1/2}\\&+\frac{L_\alpha}{4\pi^2}\left(\int_0^T\left(\int_0^1[F_{xx}(x,s,u^{(0)}(x,s))-F_{xx}(x,s,0)]\frac{\cos(2\pi nx)}{n^2}dx\right)^2ds\right)^{1/2}\\&+\frac{L_\alpha}{4\pi^2}\left(\int_0^T\left(\int_0^1 F_{xx}(x,s,0)\frac{\cos(2\pi nx)}{n^2}dx\right)^2ds\right)^{1/2},\end{aligned}$$

where

$$L_\alpha = \frac{T^{\alpha-1/2}}{\sqrt{2\alpha-1}}, \text{ for } \frac{1}{2} < \alpha < 1. \tag{3.18}$$

Applying **Step G** and after that taking the maximum, we get

$$\begin{aligned}
&\max_{0\le t\le T} |u_{1,n}^{(1)}(t)| \\
&\le \sqrt{2}\exp\left(\|a^{(0)}\|_{C[0,T]}^2 L_\alpha^2 T\right)\Bigg[|\varphi_{1,n}| + \frac{L_\alpha}{4\pi^2}\left(\int_0^T\left(\int_0^1 [F_{xx}(x,s,u^{(0)}(x,s)) - F_{xx}(x,s,0)]\frac{\cos(2\pi nx)}{n^2}dx\right)^2 ds\right)^{1/2} \\
&\quad + \frac{L_\alpha}{4\pi^2}\left(\int_0^T\left(\int_0^1 F_{xx}(x,s,0)\frac{\cos(2\pi nx)}{n^2}dx\right)^2 ds\right)^{1/2}\Bigg].
\end{aligned}$$

Taking the sum of both sides of the last inequality, we find

$$\begin{aligned}
\sum_{n\ge1}\max_{0\le t\le T}|u_{1,n}^{(1)}(t)| \le & A\left(a^{(0)}\right)\sum_{n\ge1}|\varphi_{1,n}| \\
&+ A\left(a^{(0)}\right)\frac{L_\alpha}{4\pi^2}\sum_{n\ge1}\frac{1}{n^2}\left(\int_0^T\left(\int_0^1 [F_{xx}(x,s,u^{(0)}(x,s)) - F_{xx}(x,s,0)]\cos(2\pi nx)dx\right)^2 ds\right)^{1/2} \\
&+ A\left(a^{(0)}\right)\frac{L_\alpha}{4\pi^2}\sum_{n\ge1}\frac{1}{n^2}\left(\int_0^T\left(\int_0^1 F_{xx}(x,s,0)\cos(2\pi nx)dx\right)^2 ds\right)^{1/2},
\end{aligned}$$

where

$$A\left(a^{(0)}\right) = \sqrt{2}\exp\left(\|a^{(0)}\|_{C[0,T]}^2 L_\alpha^2 T\right). \tag{3.19}$$

Then, Hölder's inequality gives

$$\begin{aligned}
\sum_{n\ge1}\max_{0\le t\le T}|u_{1,n}^{(1)}(t)| \le & A\left(a^{(0)}\right)\sum_{n\ge1}|\varphi_{1,n}| \\
&+ A\left(a^{(0)}\right)\frac{L_\alpha}{4\pi^2}\left(\sum_{n\ge1}\frac{1}{n^4}\right)^{1/2}\left(\sum_{n\ge1}\int_0^T\left(\int_0^1 [F_{xx}(x,s,u^{(0)}(x,s)) - F_{xx}(x,s,0)]\cos(2\pi nx)dx\right)^2 ds\right)^{1/2} \\
&+ A\left(a^{(0)}\right)\frac{L_\alpha}{4\pi^2}\left(\sum_{n\ge1}\frac{1}{n^4}\right)^{1/2}\left(\sum_{n\ge1}\int_0^T\left(\int_0^1 F_{xx}(x,s,0)\cos(2\pi nx)dx\right)^2 ds\right)^{1/2}.
\end{aligned}$$

Using the fact that $\sum_{n\ge1}\frac{1}{n^4} = \frac{\pi^4}{90}$, then from Bessel's inequality, the Lipschitz condition and by (A_2), we get

$$\sum_{n\ge1}\max_{0\le t\le T}|u_{1,n}^{(1)}(t)| \le A\left(a^{(0)}\right)\left[\sum_{n\ge1}|\varphi_{1,n}| + \frac{L_\alpha}{12\sqrt{10}}\left[B\|u^{(0)}\|_{\mathcal{B}} + M\right]\right]. \tag{3.20}$$

In the same way, according to (3.10) and $a^{(0)} \in C[0,T]$, we have

$$\begin{aligned}
|u_{2,n}^{(1)}(t)| \le & |\varphi_{2,n}| + \frac{N_\lambda}{\pi n}\max_{0\le t\le T}|u_{1,n}^{(1)}(t)|\int_0^t (t-s)^{\alpha-1}ds + \|a^{(0)}\|_{C[0,T]}\int_0^t (t-s)^{\alpha-1}\left|u_{2,n}^{(1)}(s)\right|ds \\
&+ \left|\int_0^t\int_0^1 (t-s)^{\alpha-1}F(x,s,u^{(0)}(x,s))x\sin(2\pi nx)dxds\right|.
\end{aligned}$$

Applying Cauchy-Schwartz's inequality and **Step K**, after that we integrate by parts the integrals depending on F with respect to x on $[0,1]$, we find

$$\begin{aligned}|u_{2,n}^{(1)}(t)| \leq & |\varphi_{2,n}| + \frac{N_\lambda T^\alpha}{\pi n \alpha} \max_{0\leq t\leq T} |u_{1,n}^{(1)}(t)| + \|a^{(0)}\|_{C[0,T]} \frac{T^{\alpha-1/2}}{\sqrt{2\alpha-1}} \left(\int_0^t |u_{2,n}^{(1)}(s)|^2 ds\right)^{1/2} \\ & + \frac{T^{\alpha-1/2}}{\sqrt{2\alpha-1}} \left(\int_0^t \left(\int_0^1 [F_x(x,s,u^{(0)}(x,s)) - F_x(x,s,0)] \frac{\cos(2\pi n x)}{n} dx\right)^2 ds\right)^{1/2} \\ & + \frac{T^{\alpha-1/2}}{\sqrt{2\alpha-1}} \left(\int_0^t \left(\int_0^1 F_x(x,s,0) \frac{\cos(2\pi n x)}{n} dx\right)^2 ds\right)^{1/2}.\end{aligned}$$

Applying **Step G** to the last inequality, we get

$$\begin{aligned}|u_{2,n}^{(1)}(t)| \leq & \sqrt{2} \exp\left(\|a^{(0)}\|_{C[0,T]}^2 L_\alpha^2 T\right) \Bigg[|\varphi_{2,n}| + \frac{N_\lambda T^\alpha}{\alpha\pi} \frac{1}{n} \max_{0\leq t\leq T} |u_{1,n}^{(1)}(t)| \\ & + \frac{L_\alpha}{2\pi} \left(\int_0^T \left(\int_0^1 [F_x(x,s,u^{(0)}(x,s)) - F_x(x,s,0)] \frac{\cos(2\pi n x)}{n} dx\right)^2 ds\right)^{1/2} \\ & + \frac{L_\alpha}{2\pi} \left(\int_0^T \left(\int_0^1 F_x(x,s,0) \frac{\cos(2\pi n x)}{n} dx\right)^2 ds\right)^{1/2} \Bigg],\end{aligned}$$

where L_α is given by(3.18). Taking the maximum and the sum of both sides of the last inequality, by Hölder's inequality, we get

$$\begin{aligned}\sum_{n\geq1} \max_{0\leq t\leq T} |u_{2,n}^{(1)}(t)| \leq & A\left(a^{(0)}\right) \sum_{n\geq1} |\varphi_{2,n}| + A\left(a^{(0)}\right) \frac{N_\lambda T^\alpha}{\alpha\pi} \left(\sum_{n\geq1} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n\geq1} \left(\max_{0\leq t\leq T} |u_{1,n}^{(1)}(t)|\right)^2\right)^{1/2} \\ & + A\left(a^{(0)}\right) \frac{L_\alpha}{2\pi} \left(\sum_{n\geq1} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n\geq1} \int_0^T \left(\int_0^1 [F_x(x,s,u^{(0)}(x,s)) - F_x(x,s,0)] \cos(2\pi n x) dx\right)^2 ds\right)^{1/2} \\ & + A\left(a^{(0)}\right) \frac{L_\alpha}{2\pi} \left(\sum_{k\geq1} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n\geq1} \int_0^T \left(\int_0^1 F_x(x,s,0) \cos(2\pi n x) dx\right)^2 ds\right)^{1/2},\end{aligned}$$

where $A\left(a^{(0)}\right)$ is defined by (3.19). By the fact that

$$\left(\sum_{n\geq1} |y_n|^2\right)^{1/2} \leq \sum_{n\geq1} |y_n|, \quad \sum_{n\geq1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and Bessel's inequality, we get

$$\begin{aligned}\sum_{n\geq1} \max_{0\leq t\leq T} |u_{2,n}^{(1)}(t)| \leq & A\left(a^{(0)}\right) \sum_{n\geq1} |\varphi_{2,n}| + A\left(a^{(0)}\right) \frac{L_\lambda}{\sqrt{6}} \sum_{n\geq1} \max_{0\leq t\leq T} |u_{1,n}^{(1)}(t)| \\ & + A\left(a^{(0)}\right) \frac{L_\alpha}{2\sqrt{6}} \left(\int_0^T \int_0^1 \left|F_x(x,s,u^{(0)}(x,s)) - F_x(x,s,0)\right|^2 dx ds\right)^{1/2} \\ & + A\left(a^{(0)}\right) \frac{L_\alpha}{2\sqrt{6}} \|F_x(.,.,0)\|_{L^2(D_T)},\end{aligned}$$

where $L_\lambda = \dfrac{N_\lambda T^\alpha}{\alpha}$. The Lipschitz condition, $(A_2) - 3$ and estimate (3.20) give

$$\sum_{n\geq 1} \max_{0\leq t\leq T} |u_{2,n}^{(1)}(t)| \leq A\left(a^{(0)}\right) \sum_{n\geq 1} |\varphi_{2,n}| + \left(A\left(a^{(0)}\right)\right)^2 \frac{L_\lambda}{\sqrt{6}} \sum_{n\geq 1} |\varphi_{1,n}| + A\left(a^{(0)}\right) L_\alpha \left[A\left(a^{(0)}\right) \frac{L_\lambda}{24\sqrt{15}} + \frac{1}{2\sqrt{6}}\right] \left[B\|u^{(0)}\|_{\mathcal{B}} + M\right]. \tag{3.21}$$

According the estimates (3.17),(3.20), (3.21) and the fact that

$$\|u^{(1)}\|_{\mathcal{B}} = \max_{0\leq t\leq T} |u_0^{(1)}(t)| + \sum_{n\geq 1} \left(\max_{0\leq t\leq T} |u_{1,n}^{(1)}(t)| + \max_{0\leq t\leq T} |u_{2,n}^{(1)}(t)|\right),$$

we get

$$\|u^{(1)}\|_{\mathcal{B}} \leq \|E\|_{C[0,T]} + Q_1(a^0)\|\varphi\|_{C^4[0,1]} + Q_2(a^0)\left[B\|u^{(0)}\|_{\mathcal{B}} + M\right],$$

where

$$Q_1(a^0) = \max\left(A\left(a^{(0)}\right); \frac{A^2\left(a^{(0)}\right) L_\lambda}{\sqrt{6}}\right), \quad Q_2(a^0) = A\left(a^{(0)}\right) L_\alpha \left[\frac{1}{12\sqrt{10}} + \frac{A\left(a^{(0)}\right) L_\lambda}{24\sqrt{15}} + \frac{1}{2\sqrt{6}}\right].$$

As a result $u^{(1)} \in \mathcal{B}$. In the same way, by induction we get for general value of $N \geq 1$

$$\|u^{(N)}\|_{\mathcal{B}} \leq \|E\|_{C[0,T]} + Q_1(a^{(N-1)})\|\varphi\|_{C^4[0,1]} + Q_2(a^{(N-1)})\left[B\|u^{(N-1)}\|_{\mathcal{B}} + M\right].$$

By induction, we have $u^{(N-1)}(t) \in \mathcal{B}$ and $a^{(N-1)}(t) \in C[0,T]$, then $u^{(N)}(t) \in \mathcal{B}$. Also, $a^{(N)}(t) > 0$ on $[0,T]$ for general value of N, we have

$$\|a^{(N)}\|_{C[0,T]} \leq E_m^{-1}\left(D\|E\|_{AC[0,T]} + M_b\|u^{(N)}\|_{\mathcal{B}} + M_F\right).$$

Finally, since $u^{(N)} \in \mathcal{B}$, we deduce that $a^{(N)} \in C[0,T]$.

Step 2: In the sequel, we will study the convergence of the bounded iterations $\left(u^{(N+1)}(t)\right)_{N\geq 0}$ and $\left(a^{(N)}(t)\right)_{N\geq 0}$ in $\mathcal{B}$ and $C[0,T]$ respectively. Let us denote

$$M_a = \max\{\|a^{(N)}\|_{C[0,T]}; N \geq 0\}; \;\; M_u = \max\{\|u^{(N)}\|_{\mathcal{B}}; N \geq 0\}. \tag{3.22}$$

First, from (3.16), we have for $N \geq 0$

$$a^{(N+1)}(t) - a^{(N)}(t) = \frac{1}{E(t)}\left[\int_0^1 F(x,t,u^{(N+1)}(x,t))dx - \int_0^1 F(x,t,u^{(N)}(x,t))dx\right],$$

by using the Lipschitz condition and (A_3), we find

$$|a^{(N+1)}(t) - a^{(N)}(t)| \leq E_m^{-1} \int_0^1 b(x,t)|u^{(N+1)}(x,t) - u^{(N)}(x,t)|dx. \tag{3.23}$$

Then, Cauchy's inequality and $(A_2) - 2$ give

$$\|a^{(N+1)} - a^{(N)}\|_{C[0,T]} \leq E_m^{-1} M_b \|u^{(N+1)} - u^{(N)}\|_{\mathcal{B}}. \tag{3.24}$$

Next, to estimate $\|u^{(N+1)} - u^{(N)}\|_{\mathcal{B}}$ for $N = 0$ we start with the first term

$$u_0^{(1)}(t) - u_0^{(0)}(t) = E(t) - \varphi_0,$$

then

$$\max_{0\leq t\leq T} |u_0^{(1)}(t) - u_0^{(0)}(t)| \leq \|E\|_{C[0,T]} + \|\varphi\|_{C^4[0,1]}. \tag{3.25}$$

Then, applying the same estimation used in **Step 1** on

$$u_{1,n}^{(1)}(t)-u_{1,n}^{(0)}(t)=-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)a^{(0)}(s)u_{1,n}^{(1)}(s)ds + \int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)F(x,s,u^{(0)}(x,s))\cos(2\pi nx)dxds,$$

we obtain

$$\sum_{n\geq 1}\max_{0\leq t\leq T}|u_{1,n}^{(1)}(t)-u_{1,n}^{(0)}(t)|\leq \frac{T^\alpha}{\alpha}M_a\|u^{(1)}\|_{\mathcal{B}}+\frac{L_\alpha}{12\sqrt{10}}\left[B\|u^{(0)}\|_{\mathcal{B}}+M\right]. \tag{3.26}$$

In the same manner from

$$\begin{aligned} u_{2,n}{}^{(1)}(t)-u_{2,n}^{(0)}(t)&=2\sqrt{\lambda_n}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)u_{1,n}^{(1)}(s)ds\\ &-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)a^{(0)}(s)u_{2,n}^{(1)}(s)ds\\ &+\int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)F(x,s,u^{(0)}(x,s))x\sin(2\pi nx)dxds, \end{aligned}$$

we get

$$\sum_{n\geq 1}\max_{0\leq t\leq T}|u_{2,n}^{(1)}(t)-u_{2,n}^{(0)}(t)|\leq \left[\frac{L_\lambda}{\sqrt{6}}+\frac{T^\alpha M_a}{\alpha}\right]\|u^{(1)}\|_{\mathcal{B}}+\frac{L_\alpha}{2\sqrt{6}}\left[B\|u^{(0)}\|_{\mathcal{B}}+M\right]. \tag{3.27}$$

Consequently, from (3.25),(3.26) and (3.27) we find

$$\|u^{(1)}-u^{(0)}\|_{\mathcal{B}}\leq \|E\|_{C[0,T]}+\|\varphi\|_{C^4[0,1]}+\left(\frac{2M_aT^\alpha}{\alpha}+\frac{L_\lambda}{\sqrt{6}}\right)M_u+\left(\frac{L_\alpha}{12\sqrt{10}}+\frac{L_\alpha}{2\sqrt{6}}\right)\left[B\|u^{(0)}\|_{\mathcal{B}}+M\right]:=\mathcal{E}.$$

For $N=1$, to estimate $\|u^{(2)}-u^{(1)}\|_{\mathcal{B}}$, we start with the fact

$$|u_0^{(N+1)}(t)-u_0^{(N)}(t)|=0,\quad N\geq 1. \tag{3.28}$$

Then, to estimate the following expression

$$\begin{aligned} u_{1,n}^{(2)}(t)-u_{1,n}^{(1)}(t)=&-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)a^{(1)}(s)\left[u_{1,n}^{(2)}(s)-u_{1,n}^{(1)}(s)\right]ds\\ &-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)u_{1,n}^{(1)}(s)\left[a^{(1)}(s)-a^{(0)}(s)\right]ds\\ &+\int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\left[F(x,s,u^1(x,s))-F(x,s,u^{(0)}(x,s))\right]\cos(2\pi nx)dxds, \end{aligned}$$

we use (3.10), (3.22), Cauchy's inequality and estimate (3.23), we get

$$\begin{aligned} |u_{1,n}^{(2)}(t)-u_{1,n}^{(1)}(t)|\leq& M_aL_\alpha\left(\int_0^t\left|u_{1,n}^{(2)}(s)-u_{1,n}^{(1)}(s)\right|^2ds\right)^{1/2}\\ &+\max_{0\leq t\leq T}\left|u_{1,n}^{(1)}(t)\right|E_m^{-1}L_\alpha\left(\int_0^t\left(\int_0^1 b(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2ds\right)^{1/2}\\ &+L_\alpha\left(\int_0^t\left(\int_0^1\left[F(x,s,u^1(x,s))-F(x,s,u^{(0)}(x,s))\right]\cos(2\pi nx)dx\right)^2ds\right)^{1/2}. \end{aligned}$$

Applying **Step G**, we obtain

$$\begin{aligned} |u_{1,n}^{(2)}(t)-u_{1,n}^{(1)}(t)|\leq&\sqrt{2}\exp\left((L_\alpha M_a)^2T\right)\max_{0\leq t\leq T}\left|u_{1,n}^{(1)}(t)\right|E_m^{-1}L_\alpha\left(\int_0^t\left(\int_0^1 b(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2ds\right)^{1/2}\\ &+\sqrt{2}\exp\left((L_\alpha M_a)^2T\right)L_\alpha\left(\int_0^t\left(\int_0^1\left[F(x,s,u^{(1)}(x,s))-F(x,s,u^{(0)}(x,s))\right]\cos(2\pi nx)dx\right)^2ds\right)^{1/2}. \end{aligned} \tag{3.29}$$

Taking the sum of both sides of the obtained inequality, we integrate twice by parts the terms depending on F with respect to x, by Hölder's inequality, Bessel's inequality and the Lipschitz condition, we find

$$\sum_{n\geq 1}|u^{(2)}_{1,n}(t)-u^{(1)}_{1,n}(t)|\leq H_1\left(\int_0^t\int_0^1 b^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2} \tag{3.30}$$

with

$$H_1=\sqrt{2}\exp\left(\left(L_\alpha M_a\right)^2T\right)\left(M_uL_\alpha E_m^{-1}+\frac{L_\alpha}{12\sqrt{10}}\right).$$

In the same way, using previous approximation techniques on

$$\begin{aligned}u^{(2)}_{2,n}(t)-u^{(1)}_{2,n}(t)=&2\sqrt{\lambda_n}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\left[u^{(2)}_{1,n}(s)-u^{(1)}_{1,n}(s)\right]ds\\&-\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)a^{(1)}(s)\left[u^{(2)}_{2,n}(s)-u^{(1)}_{2,n}(s)\right]ds\\&-\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)u^{(1)}_{2,n}(s)\left[a^{(1)}(s)-a^{(0)}(s)\right]ds\\&+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\int_0^1\left[F(x,s,u^{(1)}(x,s))-F(x,s,u^{(0)}(x,s))\right]x\sin(2n\pi x)dxds,\end{aligned}$$

from (3.10), Cauchy's inequality and estimate (3.23), we get

$$\begin{aligned}|u^{(2)}_{2,n}(t)-u^{(1)}_{2,n}(t)|\leq&N_\lambda\int_0^t(t-s)^{\alpha-1}\frac{1}{n\pi}|u^{(2)}_{1,n}(s)-u^{(1)}_{1,n}(s)|ds+L_\alpha M_a\left(\int_0^t|u^{(2)}_{2,n}(s)-u^{(1)}_{2,n}(s)|^2ds\right)^{1/2}\\&+E_m^{-1}L_\alpha\max_{0\leq t\leq T}|u^{(1)}_{2,n}(t)|\left(\int_0^t\left(\int_0^1 b(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2ds\right)^{1/2}\\&+L_\alpha\left(\int_0^t\left(\int_0^1\left[F_x(x,s,u^{(1)}(x,s))-F_x(x,s,u^{(0)}(x,s))\right]\frac{\cos(2n\pi x)}{2\pi n}dx\right)^2ds\right)^{1/2}.\end{aligned}$$

Applying **step G**, we get

$$\begin{aligned}&|u^{(2)}_{2,n}(t)-u^{(1)}_{2,n}(t)|\\\leq&\sqrt{2}\exp\left(\left(L_\alpha M_a\right)^2T\right)N_\lambda\int_0^t(t-s)^{\alpha-1}\frac{1}{\pi n}|u^{(2)}_{1,n}(s)-u^{(1)}_{1,n}(s)|ds\\&+\sqrt{2}\exp\left(\left(L_\alpha M_a\right)^2T\right)E_m^{-1}L_\alpha\max_{0\leq t\leq T}|u^{(1)}_{2,n}(t)|\left(\int_0^t\left(\int_0^1 b(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2ds\right)^{1/2}\\&+\sqrt{2}\exp\left(\left(L_\alpha M_a\right)^2T\right)L_\alpha\left(\int_0^t\left(\int_0^1\left[F_x(x,s,u^{(1)}(x,s))-F_x(x,s,u^{(0)}(x,s))\right]\frac{\cos(2n\pi x)}{2\pi n}dx\right)^2ds\right)^{1/2}.\end{aligned} \tag{3.31}$$

Taking into account that, by the estimate (3.29) we get concerning the following term

$$\begin{aligned}&\sum_{n\geq 1}\frac{1}{n\pi}|u^{(2)}_{1,n}(s)-u^{(1)}_{1,n}(s)|\\\leq&\sqrt{2}\exp\left(\left(L_\alpha M_a\right)^2T\right)E_m^{-1}L_\alpha\sum_{n\geq 1}\frac{1}{n\pi}\max_{0\leq t\leq T}\left|u^{(1)}_{1,n}(t)\right|\left(\int_0^t\left(\int_0^1 b(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2ds\right)^{1/2}\\&+\sqrt{2}\exp\left(\left(L_\alpha M_a\right)^2T\right)L_\alpha\sum_{n\geq 1}\frac{1}{n\pi}\times\left(\int_0^t\left(\int_0^1\left[F(x,s,u^{(1)}(x,s))-F(x,s,u^{(0)}(x,s))\right]\cos(2\pi nx)dx\right)^2ds\right)^{1/2},\end{aligned}$$

by Hölder's inequality, the Lipschitz condition and Bessel's inequality, we get

$$\sum_{n\geq 1}\frac{1}{n\pi}|u_{1,n}^{(2)}(s)-u_{1,n}^{(1)}(s)|\leq\sqrt{\frac{2}{6}}\exp\left((L_\alpha M_a)^2T\right)L_\alpha\left[E_m^{-1}M_u+1\right]\left(\int_0^t\left(\int_0^1 b(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2ds\right)^{1/2}.$$

Hence, we find

$$\begin{aligned}\int_0^t(t-s)^{\alpha-1}\sum_{n\geq 1}\frac{1}{n\pi}|u_{1,n}^{(2)}(s)-u_{1,n}^{(1)}(s)|ds\leq&\sqrt{\frac{2}{6}}\exp\left((L_\alpha M_a)^2T\right)L_\alpha\left[E_m^{-1}M_u+1\right]\\&\times\left(\int_0^t\left(\int_0^1 b(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2ds\right)^{1/2}\int_0^t(t-s)^{\alpha-1}ds.\end{aligned}\tag{3.32}$$

Taking the sum of both sides of the inequality (3.31), according to (3.32), Hölder's inequality, Bessel's inequality and the Lipschitz condition, we obtain

$$\sum_{n\geq 1}|u_{2,n}^{(2)}(t)-u_{2,n}^{(1)}(t)|\leq H_2\left(\int_0^t\int_0^1 b^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2},\tag{3.33}$$

with

$$H_2=\frac{2T^\alpha}{\alpha\sqrt{6}}\left(\exp\left((L_\alpha M_a)^2T\right)\right)^2L_\alpha L_\lambda\left(E_m^{-1}M_u+1\right)+\sqrt{\frac{2}{6}}\exp\left((L_\alpha M_a)^2T\right)L_\alpha\left(E_m^{-1}M_u+\frac{1}{2}\right).$$

Thus, from formula (3.28), (3.30) and (3.33) we deduce

$$|u^{(2)}(t)-u^{(1)}(t)|\leq H\left(\int_0^t\int_0^1 b^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2},\tag{3.34}$$

where $H=[H_1+H_2]$. As a result

$$\|u^{(2)}-u^{(1)}\|_{\mathcal{B}}\leq HB\|u^{(1)}-u^{(0)}\|_{\mathcal{B}}.$$

Applying the same estimation techniques and from (3.34), we obtain for $N=2$

$$\begin{aligned}|u^{(3)}(t)-u^{(2)}(t)|&\leq H\left(\int_0^t\int_0^1 b^2(x,s)|u^{(2)}(x,s)-u^{(1)}(x,s)|^2dxds\right)^{1/2}\\&\leq H\max_{0\leq t\leq T}\|b(.,t)\|_{L^2(0,1)}\left(\int_0^t\left|u^{(2)}(s)-u^{(1)}(s)\right|^2ds\right)^{1/2}\\&\leq H^2\left(\max_{0\leq t\leq T}\|b(.,t)\|_{L^2(0,1)}\right)^2\left(\int_0^t\int_0^s|u^{(1)}(r)-u^{(0)}(r)|^2drds\right)^{1/2}\\&\leq H^2M_b^2\mathcal{E}\left(\int_0^t\int_0^s drds\right)^{1/2}\leq H^2M_b^2\frac{t}{\sqrt{2}}\mathcal{E}.\end{aligned}$$

Similarly, for a general value of N, we get

$$\|u^{(N+1)}-u^{(N)}\|_{\mathcal{B}}\leq H^NM_b^N\frac{T^{\frac{N}{2}}}{\sqrt{N!}}\mathcal{E}.\tag{3.35}$$

Thus clearly, we have $\|u^{(N+1)}-u^{(N)}\|_{\mathcal{B}}\to 0$ when $N\to\infty$ and from (3.24) we deduce that $\|a^{(N+1)}-a^{(N)}\|_{C[0,T]}\to 0$ when $N\to\infty$. We conclude that the iterations $\left(u^{(N+1)}(t)\right)_{N\geq 0}$ and $\left(a^{(N)}(t)\right)_{N\geq 0}$ are of Cauchy's type in the Banach spaces $\mathcal{B}$ and $C[0,T]$ respectively. Following this, these sequences are uniformly convergent to elements of $\mathcal{B}$ and

$C[0,T]$ respectively. Then, we will show that $\lim_{N\to\infty} u^{(N+1)}(t) = u(t)$, $\lim_{N\to\infty} a^{(N+1)}(t) = a(t)$. Similarly to the previous step, we obtain

$$\begin{aligned}|u(t) - u^{(N+1)}(t)| &\le H\left(\int_0^t\int_0^1 b^2(x,s)|u(x,s)-u^{(N)}(x,s)|^2dxds\right)^{1/2}\\ &\le HM_b\left(\int_0^t |u(s)-u^{(N)}(s)|^2ds\right)^{1/2},\end{aligned}$$

which implies that

$$|u(t) - u^{(N+1)}(t)|^2 \le 2(HM_b)^2\int_0^t |u(s)-u^{(N+1)}(s)|^2ds + 2(HM_b)^2\int_0^t |u^{(N+1)}(s)-u^{(N)}(s)|^2ds.$$

Then, by Gronwall's inequality and (3.35), we find

$$\begin{aligned}|u(t) - u^{(N+1)}(t)|^2 &\le 2(HM_b)^2\exp(2(HM_b)^2t)\int_0^t |u^{(N+1)}(s)-u^{(N)}(s)|^2ds\\ &\le 2(HM_b)^2\exp(2(HM_b)^2t)\frac{(HM_b)^{2N}\mathcal{E}^2}{N!}\int_0^t s^N ds.\end{aligned}$$

Finally, we conclude that

$$\|u-u^{(N+1)}\|_{\mathcal{B}} \le \sqrt{2}\exp((HM_b)^2T)\mathcal{E}\frac{(HM_b)^{N+1}T^{\frac{N+1}{2}}}{\sqrt{(N+1)!}},$$

when $N\to\infty$, we obtain $u^{N+1}(t)\to u(t)$. Furthermore, we can find

$$\|a-a^{(N)}\|_{C[0,T]} \le E_m^{-1}M_b\|u-u^{(N)}\|_{\mathcal{B}},$$

then, we deduce that $a^{(N)}(t)\to a(t)$ when $N\to\infty$. As a consequence, $u\in\mathcal{B}$ and $a\in C[0,T]$. On other hand, the series expression (3.1) of the solution $u(x,t)$ gives

$$|u(x,t)| \le 2|u_0(t)| + 4\sum_{n\ge1}|u_{1,n}(t)| + 4\sum_{n\ge1}|u_{2,n}(t)|.$$

From Lemma 3.1 and Remark 3.1, we conclude by (3.6),(3.7) and (3.8) that the majorizing sums of the series (3.1) and its x-partial derivative $\sum_{n\ge1}\partial_x$ are absolutely convergent. Then, they are uniformly convergent and their sums are continuous in $\bar{D}_T$. In addition, the series of xx-partial derivative $\sum_{n\ge1}\partial_{xx}$ of (3.1) is uniformly convergent in $(0,1)\times[\varepsilon,T]$, for any $\varepsilon>0$. In addition, from the expressions of fractional derivative (3.2)-(3.3) and (3.4), we get the following inequalities

$$\begin{aligned}|{}^CD^\alpha_{0+}u_0(t)| &\le \|a\|_{C[0,T]}|u_0(t)| + \|F_0(.,u)\|_{C[0,T]},\\ |{}^CD^\alpha_{0+}u_{1,n}(t)| &\le \lambda_n|u_{1,n}(t)| + \|a\|_{C[0,T]}\||u_{1,n}(t)| + \|F_{1,n}(.,u)\|_{C[0,T]},\\ |{}^CD^\alpha_{0+}u_{2,n}(t)| &\le \lambda_n|u_{2,n}(t)| + 2\sqrt{\lambda_n}|u_{1,n}(t)| + \|a\|_{C[0,T]}|u_{2,n}(t)| + \|F_{2,n}(.,u)\|_{C[0,T]},\end{aligned}$$

due to Lemma 3.1 and Remark 3.1, we deduce the uniform convergence of the series $\sum_{n\ge1}\lambda_n|u_{1,n}(t)|$, $\sum_{n\ge1}\sqrt{\lambda_n}|u_{1,n}(t)|$ and $\sum_{n\ge1}\lambda_n|u_{2,n}(t)|$. Then, from the Weirstrass M-test, we show that the series $\sum_{n=1}^{\infty}{}^CD^\alpha_{0+}u_{1,n}(t)$ and $\sum_{n=1}^{\infty}{}^CD^\alpha_{0+}u_{2,n}(t)$ are convergent, also from (3.8) we observe that, ${}^CD^\alpha_{0+}u_0(t)$ is bounded. By virtue of Lemma 2.2, the α-partial derivative ${}^C\partial^\alpha_{0+,t}$ of the series (3.1) and the series $\sum_{n\ge1}{}^C\partial_{0+,t}$ are uniformly convergent in $(0,1)\times[\varepsilon,T]$, for any $\varepsilon>0$. Thus, $u\in C^{2,\alpha}(D_T)\cap C^{1,0}(\bar{D}_T)$ for arbitrary positive $a\in C[0,T]$.

Step 3: To prove the uniqueness result, we assume that problem (1.1)-(1.4) has two solution pairs $\{u(t),a_1(t)\}$, $\{v(t),a_2(t)\}$, first from (3.9) we have

$$\|a_1-a_2\|_{C[0,T]} \le E_m^{-1}M_b\|u-v\|_{\mathcal{B}}. \tag{3.36}$$

Next, as done previously, we have

$$|u\left(t\right)-v\left(t\right)|\leq HM_b\left(\int_0^t|u(s)-v(s)|^2ds\right)^{1/2}.$$

Applying **Step G**, we get $|u\left(t\right)-v\left(t\right)|\leq 0; t\in[0,T]$ and consequently $u\left(t\right)=v\left(t\right)$ on $[0,T]$. Furthemore, from (3.36) we deduce that $a_1\left(t\right)=a_2\left(t\right), t\in[0,T]$. □

3.2 Continuous dependence of $\{u, a\}$ upon the data

Theorem 3.2. Under assumption (A_1)-(A_3), the solution $\{u(x,t),a(t)\}$ of the inverse problem (1.1)-(1.4) depends continuously upon the data of $\{\varphi(x),E(t)\}$.

Proof . Let $\{u(x,t),a(t)\}$, $\{\bar{u}(x,t),\bar{a}(t)\}$ be two solutions of the inverse problem (1.1)-(1.4), corresponding two sets of the data $\{\varphi(x),E(t)\}$ and $\{\bar{\varphi}(x),\bar{E}(t)\}$ respectively. First, we have

$$|a(s)-\bar{a}(s)|\leq\left|\frac{{}^CD^\alpha\bar{E}(s)}{\bar{E}(s)}-\frac{{}^CD^\alpha E(s)}{E(s)}\right|+\int_0^1\left|\frac{F(x,s,u(x,s))}{E(s)}-\frac{F(x,s,\bar{u}(x,s))}{\bar{E}(s)}\right|dx. \tag{3.37}$$

Adding and subtracting $\frac{{}^CD^\alpha\bar{E}(s)}{E(s)}$ to the first term of the right sided of (3.37), we get

$$\begin{aligned}\left|\frac{{}^CD^\alpha\bar{E}(s)}{\bar{E}(s)}-\frac{{}^CD^\alpha E(s)}{E(s)}\right|&\leq\left|\frac{{}^CD^\alpha(\bar{E}(s)-E(s))}{E(s)}\right|+\left|{}^CD^\alpha\bar{E}(s)\left(\frac{1}{\bar{E}(s)}-\frac{1}{E(s)}\right)\right|\\&\leq\frac{D}{E_m}\|E-\bar{E}\|_{AC[0,T]}+\frac{D}{E_m^2}\|\bar{E}\|_{AC[0,T]}\|E-\bar{E}\|_{C[0,T]}.\end{aligned}$$

Next, adding and subtracting $\int_0^1\frac{F(x,s,\bar{u}(x,s))}{E(s)}dx$ and $\left(\frac{1}{E(s)}-\frac{1}{\bar{E}(s)}\right)\int_0^1F(x,s,0)dx$ to the second term of the right sided of (3.37), we get

$$\begin{aligned}\int_0^1\left|\frac{F(x,s,u(x,s))}{E(s)}-\frac{F(x,s,\bar{u}(x,s))}{\bar{E}(s)}\right|dx\leq&\frac{1}{E(s)}\int_0^1|F(x,s,u(x,s))-F(x,s,\bar{u}(x,s))|dx\\&+\left|\frac{1}{E(s)}-\frac{1}{\bar{E}(s)}\right|\int_0^1|F(x,s,\bar{u}(x,s))-F(x,s,0)|dx\\&+\left|\frac{1}{E(s)}-\frac{1}{\bar{E}(s)}\right|\int_0^1|F(x,s,0)|dx\\\leq&\frac{1}{E(s)}\int_0^1b\left(x,s\right)|u(x,s)-\bar{u}(x,s)|dx\\&+\left|\frac{1}{E(s)}-\frac{1}{\bar{E}(s)}\right|\int_0^1b\left(x,s\right)|\bar{u}(x,s))|dx+\left|\frac{1}{E(s)}-\frac{1}{\bar{E}(s)}\right|\int_0^1|F(x,s,0)dx\\\leq&E_m^{-1}\int_0^1b\left(x,s\right)|u(x,s)-\bar{u}(x,s)|dx+E_m^{-2}\|E-\bar{E}\|_{C[0,T]}\left[M_bM_{\bar{u}}+M_F\right]\end{aligned}$$

where, $M_b=\|b\|_{C[0,T]}$, $\quad M_{\bar{u}}=\|\bar{u}\|_{\mathcal{B}}$. Thus, we deduce that there exist positive constants $\mu_i, i=1,2$

$$|a(s)-\bar{a}(s)|\leq\mu_1\|E-\bar{E}\|_{AC[0,T]}+\mu_2\int_0^1b\left(x,s\right)|u(x,s)-\bar{u}(x,s)|dx \tag{3.38}$$

and hence

$$\|a-\bar{a}\|_{C[0,T]}\leq\mu_1\|E-\bar{E}\|_{AC[0,T]}+\mu_2M_b\|u-\bar{u}\|_{\mathcal{B}}. \tag{3.39}$$

Now, to estimate $u(x,t)-\bar{u}(x,t)$ we start with

$$|u_0(t)-\bar{u}_0(t)|\leq\|E-\bar{E}\|_{C[0,T]}.$$

Also, from (3.10) we have

$$\begin{aligned}|u_{1,n}\ (t)-\bar{u}_{1,n}(t)| &\leq |\varphi_{1,n}-\bar{\varphi_{1,n}}|+\int_0^t (t-s)^{\alpha-1}|a(s)-\bar{a}(s)||u_{1,n}(s)|ds\\ &+\int_0^t (t-s)^{\alpha-1}\bar{a}(s)|u_{1,n}(s)-\bar{u}_{1,n}(s)|ds\\ &+\left|\int_0^t\int_0^1 (t-s)^{\alpha-1}\left[F(x,s,u(x,s))-F(x,s,\bar{u}(x,s))\right]\cos(2\pi nx)dxds\right|.\end{aligned}$$

Then, by(3.38) and the same estimation techniques used previously in the **Step 2** of the proof Theorem 3.1, there exist positive constants $\eta_i, i=1,...,7$ such that

$$\sum_{n\geq 1}|u_{1,n}(t)-\bar{u}_{1,n}(t)|\leq \eta_1\sum_{n\geq 1}|\varphi_{1,n}-\bar{\varphi}_{1,n}|+\eta_2\|E-\bar{E}\|_{AC[0,T]}+\eta_3\left(\int_0^t\int_0^1 b^2(x,s)|u(x,s)-\bar{u}(x,s)|^2dxds\right)^{1/2}$$

and

$$\begin{aligned}\sum_{n\geq 1}|u_{2,n}(t)-\bar{u}_{2,n}(t)| \leq &\eta_4\sum_{n\geq 1}|\varphi_{2,n}-\bar{\varphi}_{2,n}|+\eta_5\|E-\bar{E}\|_{AC[0,T]}+\eta_6\sum_{n\geq 1}|\varphi_{1,n}-\bar{\varphi}_{1,n}|\\ &+\eta_7\left(\int_0^t\int_0^1 b^2(x,s)|u(x,s)-\bar{u}(x,s)|^2dxds\right)^{1/2}.\end{aligned}$$

Thus, we get for some positive constants $\rho_i, i=1,2,3$

$$|u\,(t)-\overline{u}\,(t)\,|\leq \rho_1\|\varphi-\bar{\varphi}\|_{C^4[0,1]}+\rho_2\|E-\bar{E}\|_{AC[0,T]}+\rho_3 M_b\left(\int_0^t|u(s)-\bar{u}(s)|^2dxds\right)^{1/2}. \tag{3.40}$$

Applying **Step G** to (3.40), we get

$$|u\,(t)-\overline{u}\,(t)\,|\leq \sqrt{2}\exp\left(\rho_3^2M_b^2T\right)\left[\rho_1\|\varphi-\bar{\varphi}\|_{C^4[0,1]}+\rho_2\|E-\bar{E}\|_{AC[0,T]}\right].$$

It follows that for some positive constants $\Pi_i, i=1,...,4$ and a large T, we get

$$\|u-\bar{u}\|_{\mathcal{B}}\leq \Pi_1\|\varphi-\bar{\varphi}\|_{C^4[0,1]}+\Pi_2\|E-\bar{E}\|_{AC[0,T]}$$

and by (3.39),

$$\|a-\bar{a}\|_{C[0,T]}\leq \Pi_3\|\varphi-\bar{\varphi}\|_{C^4[0,1]}+\Pi_4\|E-\bar{E}\|_{AC[0,T]}.$$

This implies that when $\|\varphi-\bar{\varphi}\|_{C^4[0,1]}\leq\varepsilon_1$ and $\|E-\bar{E}\|_{AC[0,T]}\leq\varepsilon_2$, then

$$\|u-\bar{u}\|_{\mathcal{B}}\leq \Pi_1\varepsilon_1+\Pi_2\varepsilon_2 \quad\text{ and }\quad \|a-\bar{a}\|_{C[0,T]}\leq \Pi_3\varepsilon_1+\Pi_4\varepsilon_2.$$

This induces the continuous dependence of $\{u(x,t),a\,(t)\}$ on the data of $\{\varphi(x),E(t)\}$. $\square$

4 Example

Consider problem (1.1)-(1.4) with

$$\varphi(x)=-2\pi\sin(2\pi x)+(1-x),\quad F(x,t,u)=(1-x)[\cos(2\pi x)e^{-t}u+4],\quad E(t)=\frac{1}{2}E_\alpha(-2\pi t^\alpha).$$

Clearly that (A_1)-(A_2) are satisfied with

$$\left|\frac{\partial^n}{\partial x^n}F(x,t,u)-\frac{\partial^n}{\partial x^n}F(x,t,\tilde{u})\right|\leq e^{x-t}|u-\tilde{u}|,\quad n=0,1,2$$

and

$$F_0(t,u) = \int_0^1 F(x,t,u)dx = 2 > 0, \quad \forall x \in (0,1).$$

Moreover, from (2.2) we have

$$^C D^\alpha E(t) = -\pi E_\alpha(-2\pi t^\alpha) < 0, \quad \forall t \in [0,T]$$

and

$$\int_0^1 \varphi(x)dx = \frac{1}{2} = E(0).$$

Therefore, all conditions of Theorem 3.1 are satisfied. Thus, this inverse problem has a unique classical solution $\{u(x,t), a(t)\}$ on $[0,T]$ for $\alpha \in (1/2, 1)$. In addition, Theorem 3.2 implies that the continuous dependence of $\{u(x,t), a\,(t)\}$ upon the data of $\{\varphi(x), E(t)\}$.

5 Conclusion

In this paper, we studied well-posedness of the solution of an inverse time potential coefficient problem involving a nonlinear time-fractional reaction-diffusion equation with nonlocal boundary and over-determination conditions by using the Fourier method with some bi-orthogonal system and the iteration method. The key point here, is the use of Gronwall's Lemma which makes that the obtained results hold on $[0,T]$ for a large T and a limited $\alpha \in (1/2, 1)$.

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