Int. J. Nonlinear Anal. Appl. In Press, 1–8
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.35153.5253

Existence of solution for a $\varphi(\chi)$-Kirchhoff equation by Neumann condition

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(Communicated by Mohammad Bagher Ghaemi)

Abstract

The present article deals with a variational method, named the Mountain Pass Theorem. We prove the existence of nontrivial weak solutions for the problem of the following form

$$\begin{cases} -(\alpha-\beta\int_\Omega \dfrac{1}{\varphi(\chi)}|\nabla\upsilon|^{\varphi(\chi)}d\chi)\Delta_{\varphi(\chi)}\upsilon+\ |\upsilon|^{\psi(\chi)-2}\upsilon=\lambda\ \eta(\chi,\upsilon) & \chi\in\Omega,\\ (\alpha-\beta\int_{\partial\Omega}\dfrac{1}{\varphi(\chi)}|\nabla\upsilon|^{\varphi(\chi)}d\chi)|\nabla\upsilon|^{\varphi(\chi)-2}\dfrac{\partial\upsilon}{\partial\nu}=0 & \chi\in\ \partial\Omega,\end{cases}$$

where $\alpha\geq\beta>0, \Delta_{\varphi(\chi)}\upsilon$ is the $\varphi(\chi)$-Laplacian operator, Ω is a smooth bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and ν is the outer unit normal to $\partial\Omega$, $\varphi(\chi),\psi(\chi)\in C(\bar{\Omega})$ with $1<\varphi(\chi)<N, \varphi(\chi)<\psi(\chi)<\varphi^*(\chi):=\dfrac{N\varphi(\chi)}{N-\varphi(\chi)}$, $\lambda>0$ is a real parameter and $\eta(\chi,t)\in C(\bar{\Omega}\times\mathbb{R},\mathbb{R})$.

Keywords: generalized Lebesgue-Sobolev spaces, weak solution, Mountain pass theorem
2020 MSC: 35J60, 35J20

1 Introduction

In this article, we consider the following problem

$$\begin{cases} -(\alpha-\beta\int_\Omega \dfrac{1}{\varphi(\chi)}|\nabla\upsilon|^{\varphi(\chi)}d\chi)\Delta_{\varphi(\chi)}\upsilon+|\upsilon|^{\psi(\chi)-2}\upsilon=\lambda\ \eta(\chi,\upsilon) & \chi\in\Omega,\\ (\alpha-\beta\int_{\partial\Omega}\dfrac{1}{\varphi(\chi)}|\nabla\upsilon|^{\varphi(\chi)}d\chi)|\nabla\upsilon|^{\varphi(\chi)-2}\dfrac{\partial\upsilon}{\partial\nu}=0 & \chi\in\partial\Omega,\end{cases}\tag{1.1}$$

where $\alpha\geq\beta>0$, $\Delta_{\varphi(\chi)}\upsilon$ is the $\varphi(\chi)$- Laplacian operator, defined as

$$\Delta_{\varphi(\chi)}\upsilon:=\operatorname{div}(|\nabla\upsilon|^{\varphi(\chi)-2}\nabla\upsilon)=\sum_{i=1}^N(|\nabla\upsilon|^{\varphi(\chi)-2}\frac{\partial\upsilon}{\partial\chi_i}),$$

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Received: September 2024 *Accepted:* October 2024

Ω is a smooth bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and ν is the outer unit normal to $\partial\Omega$ and $\varphi(\chi), \psi(\chi) \in C(\bar{\Omega})$ with $1 < \varphi(\chi) < N$, $\varphi(\chi) < \psi(\chi) < \varphi^*(\chi) := \dfrac{N\varphi(\chi)}{N - \varphi(\chi)}$, $\lambda > 0$ is a real parameter. We define $\varphi_{_l}$ and φ_s for convenience as follows: $\varphi_{_l} := \varphi^- = \inf_\Omega \varphi(\chi)$ and $\varphi_s := \varphi^+ = \sup_\Omega \varphi(\chi)$, for all $\varphi(\chi) \in C(\bar{\Omega})$ and $\eta(\chi, t) \in C(\bar{\Omega}\times\mathbb{R}, \mathbb{R})$ that, the function η has mentioned in the following sextet conditions.

(η_1) : $|\eta(\chi, t)| \leq c(1 + |t|^{r(\chi)-1})$, $\quad \forall(\chi, t) \in \Omega \times \mathbb{R}$ where $c > 0$ and $\varphi(\chi) < r(\chi) < \varphi^*(\chi)$,

(η_2) : $\lim_{t\to 0} \dfrac{\eta(\chi, t)}{|t|^{\varphi(\chi)-2}t} = 0$,

(η_3) : there exist $\theta \in \left(\psi_{_s}, \dfrac{2\varphi_l^2}{\varphi_s}\right)$ and $T > 0$ so that $0 < \theta\hbar(\chi, t) \leq t\,\eta(\chi, t)$ for all $|t| \geq T, \chi \in \Omega$ where $\hbar(\chi, t) = \int_0^t \eta(\chi, s)ds$.

In addition to the conditions that, given for η, the functions $\varphi(\chi), \psi(\chi), r(\chi)$ must apply to the following condition, which we call the $(\varphi\psi r)_c$-condition:

$$c < \varphi_{_l} < \varphi(\chi) < \varphi_s < \psi_{_l} < \psi(\chi) < \psi_s < 2\varphi_{_l} < r_{_l} < r(\chi) < r_s < \varphi^*(\chi),$$

in the above condition, $c = 1$ or $c = 2$.

Kirchhoff [29] studied an equation in 1883, marking a turning point in the emergence of a branch of differential problems known as Kirchhoff-type equations. These equations are used in vibrational physics[28, 31], elastic mechanics [42], and electrorheological fluids [1].

In recent years, the variable exponent Sobolev space $W^{1,\varphi(\chi)}(\Omega)$ has been very attractive and many writers and researchers entered the field [2, 4, 5, 6, 7, 12, 17, 18, 22]. Working on problems in such spaces due to their inhomogeneity and nonlinearity has always been related to difficulties, which have made work in this field more attractive (see [11, 13, 26]). One of the instructive and very useful theorems in solving such problems is the Sobolev embedding [20].

In the way of proving our theorem, as well as inequalities, and other useful theorems such as Hölder and Poincaré's inequality, will benefit from this theorem very useful. The Sobolev embedding theorem has greatly blurred the dividing borders of Sobolev spaces and L^p spaces.

Recently, nonlocal problems have been a very interesting subject. The interest in $p(x)$-Kirchhoff type problems arises from their ability to model a wide range of physical phenomena where nonlocal effects and heterogeneous media are involved (see [3, 8, 9, 10, 16, 19, 24, 25, 26, 27, 30, 32, 34, 36, 38, 39, 40, 41]). These problems are particularly relevant in areas such as elasticity theory, quantum mechanics, and fluid dynamics, where the properties of the material or medium may change from point to point. In the present work, we generally bring up a Kirchhoff problem containing the $\varphi(\chi)$-Laplacian operator, which contains the expression $(\alpha - \beta \int_\Omega |\nabla\upsilon|^p d\chi)$.

Now we state the main result:

Theorem 1.1. Suppose that $(\varphi\psi r)_1$-condition and $(\eta_1) - (\eta_3)$ hold, so our problem has at least a nontrivial weak solution, for each $\lambda > 0$.

2 Preliminary Results

In this section, we recall some important definitions and essential characteristics of the generalized Lebesgue-Sobolev spaces $L^{\varphi(\chi)}(\Omega)$ and $W^{1,\varphi(\chi)}(\Omega)$ where $\Omega \subset \mathbb{R}^N$ is an open set. In this regard, we refer the readers to the book of Musielak [33] and the papers [15, 14, 35]. Set $C_+(\bar{\Omega}) := \{h : h \in C(\bar{\Omega}), h(\chi) > 1 \text{ for all } \chi \in \bar{\Omega}\}$. For each $\varphi(\chi) \in C_+(\bar{\Omega})$ denote

$$L^{\varphi(\chi)}(\Omega) = \left\{\upsilon : \text{a measurable real-valued function such that } \int_\Omega |\upsilon(\chi)|^{\varphi(\chi)}d\chi < \infty\right\}$$

which is the definition of variable exponent Lebesgue space, that by mentioned the norm (so-called Luxemburg norm) are reflexive and separable Banach spaces

$$\|\upsilon\|_{\varphi(\chi)} := \inf\left\{\mu > 0; \int_\Omega \left|\frac{\upsilon(\chi)}{\mu}\right|^{\varphi(\chi)} d\chi \leq 1\right\}.$$

These spaces are similar to classical Lebesgue spaces in many aspects:

a) if $0 < |\Omega| < \infty$ and $\varphi_1(\chi), \varphi_2(\chi)$ are variable exponents so that $\varphi_1(\chi) \leq \varphi_2(\chi)$ a.e. $\chi \in \Omega$ then there is a continuous embedding
$$L^{\varphi_2(\chi)}(\Omega) \hookrightarrow L^{\varphi_1(\chi)}(\Omega),$$

b) the Hölder inequality holds, if $L^{\varphi'(\chi)}(\Omega)$ is a conjugate of $L^{\varphi(\chi)}(\Omega)$, where $\frac{1}{\varphi(\chi)} + \frac{1}{\varphi'(\chi)} = 1$, we have
$$\left| \int_\Omega \upsilon v \, d\chi \right| \leq \left(\frac{1}{\varphi_l} + \frac{1}{\varphi_i'} \right) \|\upsilon\|_{\varphi(\chi)} \|v\|_{\varphi'(\chi)}, \quad \forall \upsilon \in L^{\varphi(\chi)}(\Omega), \ \forall v \in L^{\varphi'(\chi)}(\Omega).$$

Modular plays an essential role in manipulating the $L^{\varphi(\chi)}$ spaces and defined by the following relation, $\rho_{\varphi(\chi)} : L^{\varphi(\chi)} \to \mathbb{R}$;
$$\rho_{\varphi(\chi)}(\upsilon) = \int_\Omega |\upsilon|^{\varphi(\chi)} d\chi.$$

Proposition 2.1 ([21]). If $\upsilon, \upsilon_n \in L^{\varphi(\chi)}(\Omega)$ and $\varphi_s < +\infty$, then the following relations hold

1. $\|\upsilon\|_{\varphi(\chi)} > 1 \Rightarrow \|\upsilon\|_{\varphi(\chi)}^{\varphi_l} < \rho_{\varphi(\chi)}(\upsilon) \leq \|\upsilon\|_{\varphi(\chi)}^{\varphi_s}$;
2. $\|\upsilon\|_{\varphi(\chi)} < 1 \Rightarrow \|\upsilon\|_{\varphi(\chi)}^{\varphi_s} < \rho_{\varphi(\chi)}(\upsilon) \leq \|\upsilon\|_{\varphi(\chi)}^{\varphi_l}$;
3. $\|\upsilon\|_{\varphi(\chi)} < 1(\textit{respectively}, = 1; > 1) \iff \rho_{\varphi(\chi)}(\upsilon) < 1(\textit{respectively}, = 1; > 1)$;
4. $\|\upsilon_n\|_{\varphi(\chi)} \to 0(\textit{respectively}, \to +\infty) \iff \rho_{\varphi(\chi)}(\upsilon) = 0 \ (\textit{respectively}, \to +\infty)$;
5. $\lim_{n\to\infty} \|\upsilon_n - \upsilon\|_{\varphi(\chi)} = 0 \iff \lim_{n\to\infty} \rho_{\varphi(\chi)}(\upsilon_n - \upsilon) = 0$;
6. For $\upsilon \neq 0, \|\upsilon\|_{\varphi(\chi)} = \lambda \iff \rho\left(\frac{\upsilon}{\lambda}\right) = 1$.

Definition 2.2 ([20]). Let $\Omega \subset \mathbb{R}^N$. The Sobolev space with variable exponent $W^{1,\varphi(\chi)}(\Omega)$ is defined as
$$W^{1,\varphi(\chi)}(\Omega) := \{\upsilon : \Omega \to \mathbb{R} : \upsilon \in L^{\varphi(\chi)}(\Omega), |\nabla \upsilon| \in L^{\varphi(\chi)}(\Omega)\},$$
endowed with the following norm
$$|||\upsilon||| := \|\upsilon\|_{W^{1,\varphi(\chi)}(\Omega)} = \|\upsilon\|_{\varphi(\chi)} + \|\nabla \upsilon\|_{\varphi(\chi)}$$
or equivalently
$$|||\upsilon||| = \inf \left\{ \mu > 0, \int_\Omega (|\frac{\upsilon}{\mu}|^{\varphi(\chi)} + |\frac{\nabla \upsilon}{\mu}|^{\varphi(\chi)}) d\chi \leq 1 \right\}.$$

Denote by $W_0^{1,\varphi(\chi)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi(\chi)}(\Omega)$. As we know, $\|\nabla \upsilon(\chi)\|_{\varphi(\chi)}$ is an equivalent norm in $W_0^{1,\varphi(\chi)}(\Omega)$.

Proposition 2.3 (Poincare inequality [20]). There exists a positive constant c so that
$$\|\upsilon\|_{\varphi(\chi)} \leq c \|\nabla \upsilon\|_{\varphi(\chi)}, \quad \forall \upsilon \in W_0^{1,\varphi(\chi)}(\Omega). \tag{2.1}$$

Proposition 2.4 (Sobolev embedding [20]). If $\varphi(\chi), \psi(\chi) \in C_+(\bar{\Omega})$ and $1 \leq \psi(\chi) \leq \varphi^*(\chi)$ for each $\chi \in \bar{\Omega}$, then there exists a continuous embedding
$$W^{1,\varphi(\chi)}(\Omega) \hookrightarrow L^{\psi(\chi)}(\Omega). \tag{2.2}$$

If $1 < \psi(\chi) < \varphi^*(\chi)$, the continuous embedding is compact.

In the sequel, we denote by $X = W^{1,\varphi(\chi)}(\Omega); X^* = (W^{1,\varphi(\chi)}(\Omega))^*$, the dual space; $\langle \cdot, \cdot \rangle$, the dual pair.

Proposition 2.5 ([20]). Let

$$J(\upsilon)=\int_\Omega \frac{1}{\varphi(\chi)}|\nabla \upsilon|^{\varphi(\chi)}d\chi, \quad \forall \upsilon\in X,$$

then $J(\upsilon)\in C^1(X,R)$ and the derivative operator J' of J is

$$\langle J'(\upsilon),\vartheta\rangle=\int_\Omega |\nabla \upsilon|^{\varphi(\chi)-2}\nabla\upsilon\nabla\vartheta d\chi,\ \ \forall \upsilon,\vartheta\in X,$$

and the following relations hold:

1. J is a convex functional.
2. $J':X\to X^*$ is a strictly monotone operator and bounded homeomorphism.
3. J' is a mapping of type (S_+), it means, $\upsilon_n\rightharpoonup \upsilon$ (weakly) and $\lim_{n\to+\infty}\sup\langle J'(\upsilon),\upsilon_n-\upsilon\rangle\leq 0$, imply $\upsilon_n\to\upsilon$ (strongly) in X.

Definition 2.6. $\upsilon\in X$ is a weak solution of problem (1.1), if

$$\begin{aligned}&\Big(\alpha-\beta\int_\Omega \frac{1}{\varphi(\chi)}|\nabla \upsilon|^{\varphi(\chi)}d\chi\Big)\int_\Omega |\nabla \upsilon|^{\varphi(\chi)-2}\nabla\upsilon\nabla\vartheta d\chi+\int_\Omega |\upsilon|^{\psi(\chi)-2}\upsilon\vartheta d\chi\\ &=\lambda\int_\Omega \eta(\chi,\upsilon)\vartheta d\chi, \quad \forall\vartheta\in X.\end{aligned}$$

The energy functional related to our problem, $J_\lambda:X\to\mathbb{R}$ such that

$$J_\lambda(\upsilon)=\alpha\int_\Omega \frac{1}{\varphi(\chi)}|\nabla \upsilon|^{\varphi(\chi)}d\chi-\frac{\beta}{2}\left(\int_\Omega \frac{1}{\varphi(\chi)}|\nabla \upsilon|^{\varphi(\chi)}d\chi\right)^2+\int_\Omega \frac{1}{\psi(\chi)}|\upsilon|^{\psi(\chi)}d\chi-\lambda\int_\Omega \hbar(\chi,\upsilon)d\chi, \quad \forall \upsilon\in X, \tag{2.3}$$

which is also well defind and of class C^1 in $(X,\mathbb{R})$.

Now we define J'_λ as the derivative operator of J_λ in the weak sense, by the following formula,

$$\langle J'_\lambda(\upsilon),\vartheta\rangle=(\alpha-\beta\int_\Omega \frac{1}{\varphi(\chi)}|\nabla \upsilon|^{\varphi(\chi)}d\chi)\int_\Omega |\nabla \upsilon|^{\varphi(\chi)-2}\nabla\upsilon\nabla\vartheta d\chi+\int_\Omega |\upsilon|^{\psi(\chi)-2}\upsilon\vartheta d\chi-\lambda\int_\Omega \eta(\chi,\upsilon)\vartheta d\chi, \tag{2.4}$$

for all $\upsilon,\vartheta\in X$. A critical point of J_λ is clearly a weak solution of our problem.

Definition 2.7. If $J_\lambda\in C^1(X,\mathbb{R})$ and $c\in\mathbb{R}$, J_λ ensures the Palais-Smale condition at level c ($(\mathcal{PS})_c$ in short), if for each $\{\upsilon_n\}\subset X$ satisfying $J_\lambda(\upsilon_n)\to c$ and $J'_\lambda(\upsilon_n)\to 0$ as $n\to\infty$ has a convergent subsequence.

3 Proof of main results

Lemma 3.1. Suppose that the conditions $(\eta_1)-(\eta_3)$ are true, then the functional J_λ holds in the $(\mathcal{PS})_c$ condition.

Proof . First, we consider the boundary condition for $\{\upsilon_n\}$, let $\{\upsilon_n\}\subset X$ be a $(\mathcal{PS})_c$ sequence related to the J_λ, so that,

$$J_\lambda(\upsilon_n)\to c \text{ and } J'_\lambda(\upsilon_n)\to 0,\ \ as\ \ n\to\infty. \tag{3.1}$$

From (3.1) and (η_3), we have for n large enough,

$$\begin{aligned}c+|||\upsilon_n|||\geq&\theta J_\lambda(\upsilon_n)-\langle J'_\lambda(\upsilon_n),\upsilon_n\rangle\\ \geq&\theta\left(\alpha\int_\Omega \frac{1}{\varphi(\chi)}|\nabla \upsilon_n|^{\varphi(\chi)}d\chi-\frac{\beta}{2}\left(\int_\Omega \frac{1}{\varphi(\chi)}|\nabla \upsilon_n|^{\varphi(\chi)}d\chi\right)^2+\int_\Omega \frac{1}{\psi(\chi)}|\upsilon_n|^{\psi(\chi)}d\chi-\lambda\int_\Omega \hbar(\chi,\upsilon_n)d\chi\right)\\ &-\left(\left[\alpha-\beta\int_\Omega \frac{1}{\varphi(\chi)}|\nabla \upsilon_n|^{\varphi(\chi)}d\chi\right]\int_\Omega |\nabla \upsilon_n|^{\varphi(\chi)}d\chi+\int_\Omega |\upsilon_n|^{\psi(\chi)}d\chi-\lambda\int_\Omega \eta(\chi,\upsilon_n)\upsilon_n d\chi\right)\\ \geq&\alpha\left(\frac{\theta}{\varphi_s}-1\right)\int_\Omega |\nabla \upsilon_n|^{\varphi(\chi)}d\chi+\beta\left(\frac{-\theta}{2\varphi_\iota^2}+\frac{1}{\varphi_s}\right)\left(\int_\Omega |\nabla \upsilon_n|^{\varphi(\chi)}d\chi\right)^2+\left(\frac{\theta}{\psi_s}-1\right)\int_\Omega |\upsilon_n|^{\psi(\chi)}d\chi-c_1\lambda\int_\Omega d\chi.\end{aligned}$$

By Poincare inequality, we have

$$c + |||\upsilon_n||| \geq \alpha\left(\frac{\theta}{\varphi_s} - 1\right)|||\upsilon_n|||^{\varphi_l} + \beta\left(\frac{-\theta}{2\varphi_l^2} + \frac{1}{\varphi_s}\right)|||\upsilon_n|||^{2\varphi_l} - c_1\lambda|\Omega|.$$

By dividing the previous inequality sides on the positive value $|||\upsilon_n|||^{\varphi_l}$ and considering (η_3) and since $\varphi_l < 2\varphi_l$, we can infer that $\{\upsilon_n\}$ is bounded in X, then

$$\upsilon_n \rightharpoonup \upsilon \text{ in } X, \tag{3.2}$$

by Sobolev embedding (2.2), we have

$$X \hookrightarrow L^{s(\chi)}(\Omega), \text{ for } 1 \leq s(\chi) < \varphi^*(\chi), \tag{3.3}$$

is compact. From (3.2) and (3.3) we can infer that

$$\upsilon_n \rightharpoonup \upsilon \text{ in } X, \upsilon_n \to \upsilon \text{ in } L^{s(\chi)}(\Omega), \ \upsilon_n(\chi) \to \upsilon(\chi), \text{ a.e. in } \Omega. \tag{3.4}$$

By Hölder's inequality and (3.4) we have

$$\left|\int_\Omega |\upsilon_n|^{\psi(\chi)-2}\upsilon_n(\upsilon_n - \upsilon)d\chi\right| \leq \int_\Omega |\upsilon_n|^{\psi(\chi)-1}|\upsilon_n - \upsilon|d\chi \leq \||\upsilon_n|^{\psi(\chi)-1}\|_{\frac{\psi(\chi)}{\psi(\chi)-1}}\|\upsilon_n - \upsilon\|_{\psi(\chi)} \to 0 \ \ as \ n \to \infty,$$

thus

$$\int_\Omega |\upsilon_n|^{\psi(\chi)-2}\upsilon_n(\upsilon_n - \upsilon)d\chi \to 0, \quad as \ n \to \infty. \tag{3.5}$$

By (η_1) and (η_2), we have that for each $\varepsilon \in (0,1)$, there is $c_\varepsilon > 0$ so that

$$|\eta(\chi, \upsilon_n)| \leq \varepsilon|\upsilon_n|^{\varphi(\chi)-1} + c_\varepsilon|\upsilon_n|^{r(\chi)-1}. \tag{3.6}$$

By Sobolev embedding (2.2) and Hölder's inequality and (3.6), we have

$$\begin{aligned}\left|\int_\Omega \eta(\chi, \upsilon_n)(\upsilon_n - \upsilon)d\chi\right| &\leq \int_\Omega (\varepsilon|\upsilon_n|^{\varphi(\chi)-1}|\upsilon_n - \upsilon| + c_\varepsilon|\upsilon_n|^{r(\chi)-1}|\upsilon_n - \upsilon|)d\chi \\ &\leq \varepsilon\||\upsilon_n|^{\varphi(\chi)-1}\|_{\frac{\varphi(\chi)}{\varphi(\chi)-1}}\|\upsilon_n - \upsilon\|_{\varphi(\chi)} + c_\varepsilon\varepsilon\||\upsilon_n|^{r(\chi)-1}\|_{\frac{r(\chi)}{r(\chi)-1}}\|\upsilon_n - \upsilon\|_{r(\chi)} \to 0,\end{aligned}$$

as $n \to \infty$. Therefore

$$\int_\Omega \eta(\chi, \upsilon_n)(\upsilon_n - \upsilon)d\chi \to 0, \quad as \quad n \to \infty. \tag{3.7}$$

From (3.1) we have $\langle J'_\lambda(\upsilon_n), \upsilon_n - \upsilon\rangle \to 0$, as $n \to \infty$, so we can infer that

$$\begin{aligned}&\left(\alpha - \beta\int_\Omega \frac{1}{\varphi(\chi)}|\nabla\upsilon_n|^{\varphi(\chi)}d\chi\right)\int_\Omega |\nabla\upsilon_n|^{\varphi(\chi)-2}\nabla\upsilon_n(\nabla\upsilon_n - \nabla\upsilon)d\chi + \int_\Omega |\upsilon_n|^{\psi(\chi)-2}\upsilon_n(\upsilon_n - \upsilon)d\chi \\ &- \lambda\int_\Omega \eta(\chi, \upsilon_n)(\upsilon_n - \upsilon)d\chi \to 0.\end{aligned} \tag{3.8}$$

From (3.5) and (3.7) and (3.8), we can write

$$\left(\alpha - \beta\int_\Omega \frac{1}{\varphi(\chi)}|\nabla\upsilon_n|^{\varphi(\chi)}d\chi\right)\int_\Omega |\nabla\upsilon_n|^{\varphi(\chi)-2}\nabla\upsilon_n(\nabla\upsilon_n - \nabla\upsilon)d\chi \to 0, \quad as \ \ n \to \infty. \tag{3.9}$$

Since $\{\upsilon_n\}$ is bounded in X, therefore, it is necessary for the following positive sequence to converge to a non-negative value such as υ_p, which means,

$$\int_\Omega \frac{1}{\varphi(\chi)}|\nabla\upsilon_n|^{\varphi(\chi)}d\chi \to \upsilon_p \geq 0, \quad as \ \ n \to \infty.$$

Similar to the proof of Lemma 3.1 in [23], we have the sequence $\left\{\alpha - \beta \int_\Omega \frac{1}{\varphi(\chi)}|\nabla\upsilon_n|^{\varphi(\chi)}d\chi\right\}$ is bounded, when n is large enough, so, it follows from (3.9) that

$$\int_\Omega |\nabla\upsilon_n|^{\varphi(\chi)-2}\nabla\upsilon_n(\nabla\upsilon_n - \nabla\upsilon)d\chi \to 0.$$

So by the (S_+) property (see lemma 2.5), $|||\upsilon_n||| \to |||\upsilon|||$ strongly in X, that means J_λ ensures the $(\mathcal{PS})_c$-condition. □

Lemma 3.2. Assume that η ensures $(\eta_1)-(\eta_3)$, then the functional J_λ ensures the Mountain pass geometry, it means,

(A) there exist $a > 0$ and $R > 0$ such that $J_\lambda(\upsilon) \geq a > 0$, for each $\upsilon \in X$ so that $|||\upsilon||| = R$.

(B) there exists $e \in X$ with $|||e||| > R$ such that $J_\lambda(e) < 0$.

Proof . By conditions (η_1) and (η_2), we can infer that for each $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|\hbar(\chi,\upsilon)| \leq \frac{\varepsilon}{\varphi(\chi)}|\upsilon|^{\varphi(\chi)} + \frac{c_\varepsilon}{r(\chi)}|\upsilon|^{r(\chi)}. \tag{3.10}$$

Let $\upsilon \in X$ be such that $|||\upsilon||| = R \in (0,1)$. Using (2.3), (3.10), Poincare inequality and Proposition 2.1, we have

$$\begin{aligned}
J_\lambda(\upsilon) =&\alpha\int_\Omega \frac{1}{\varphi(\chi)}|\nabla\upsilon|^{\varphi(\chi)}d\chi - \frac{\beta}{2}\left(\int_\Omega \frac{1}{\varphi(\chi)}|\nabla\upsilon|^{\varphi(\chi)}d\chi\right)^2 + \int_\Omega \frac{1}{\psi(\chi)}|\upsilon|^{\psi(\chi)}d\chi - \lambda\int_\Omega \hbar(\chi,\upsilon)d\chi\\
\geq&\alpha\int_\Omega \frac{1}{\varphi(\chi)}|\nabla\upsilon|^{\varphi(\chi)}d\chi - \frac{\beta}{2}\left(\int_\Omega \frac{1}{\varphi(\chi)}|\nabla\upsilon|^{\varphi(\chi)}d\chi\right)^2 - \lambda\varepsilon\int_\Omega \frac{|\upsilon|^{\varphi(\chi)}}{\varphi(\chi)}d\chi - \lambda c_\varepsilon\int_\Omega \frac{|\upsilon|^{r(\chi)}}{r(\chi)}d\chi\\
\geq&\left(\alpha - \frac{\lambda\varepsilon}{c_1}\right)\int_\Omega \frac{1}{\varphi(\chi)}|\nabla\upsilon|^{\varphi(\chi)}d\chi - \frac{\beta}{2}\left(\int_\Omega \frac{1}{\varphi(\chi)}|\nabla\upsilon|^{\varphi(\chi)}d\chi\right)^2 - \frac{\lambda c_2 c_\varepsilon}{r_l}\int_\Omega |\nabla\upsilon|^{r(\chi)}d\chi\\
\geq&\frac{1}{\varphi_s}\left(\alpha - \frac{\lambda\varepsilon}{c_1}\right)\rho_{\varphi(\chi)}(\nabla\upsilon) - \frac{\beta}{2\varphi_l^2}(\rho_{\varphi(\chi)}(\nabla\upsilon))^2 - \frac{\lambda c_2 c_\varepsilon}{r_l}\rho_{r(\chi)}(\nabla\upsilon)\\
\geq&\frac{1}{\varphi_s}\left(\alpha - \frac{\lambda\varepsilon}{c_1}\right)|||\upsilon|||^{\varphi_s} - \frac{\beta}{2\varphi_l^2}|||\upsilon|||^{2\varphi_l} - \frac{\lambda c_2 c_\varepsilon}{r_l}|||\upsilon|||^{r_l},
\end{aligned}$$

where c_1 and c_2 are Sobolev embedding constants. Now, let $0 < \varepsilon < \frac{1}{\lambda}\alpha c_1$. Considering $(\varphi\psi r)_c$-condition, we deduce that

$$J_\lambda(\upsilon) \geq \left(\frac{1}{\varphi_s}(\alpha - \frac{\lambda\varepsilon}{c_1}) - \frac{\beta}{2\varphi_l^2}|||\upsilon|||^{2\varphi_l-\varphi_s} - \frac{\lambda c_2 c_\varepsilon}{r_l}|||\upsilon|||^{r_l-\varphi_s}\right)|||\upsilon|||^{\varphi_s}.$$

By selecting R adequately small (i.e. R is such that $\frac{1}{\varphi_s}(\alpha - \frac{\lambda\varepsilon}{c_1}) - \frac{\beta}{2\varphi_l^2}R^{2\varphi_l-\varphi_s} - \frac{\lambda c_2 c_\varepsilon}{r_l}R^{r_l-\varphi_s} > 0$), so that

$$J_\lambda(\upsilon) \geq R^{\varphi_s}\left(\frac{1}{\varphi_s}(\alpha - \frac{\lambda\varepsilon}{c_1}) - \frac{\beta}{2\varphi_l^2}R^{2\varphi_l-\varphi_s} - \frac{\lambda c_2 c_\varepsilon}{r_l}R^{r_l-\varphi_s}\right) =: a > 0.$$

Therefore there is $a > 0$ so that for each $\upsilon \in X$ with $|||\upsilon||| = R$ we have $J_\lambda(\upsilon) \geq a > 0$. Also, by (η_3), we can infer that

$$\forall T > 0,\ \exists c_T > 0;\ \ \hbar(\chi,\upsilon) \geq T|\upsilon|^\theta - c_T, \quad \forall(\chi,\upsilon) \in \Omega\times\mathbb{R}. \tag{3.11}$$

Let $\vartheta \in C_0^\infty(\Omega), \vartheta > 0$ and $m > 1$. By (3.11) we have

$$\begin{aligned}
J_\lambda(m\vartheta) =&\alpha\int_\Omega \frac{1}{\varphi(\chi)}|m\nabla\vartheta|^{\varphi(\chi)}d\chi - \frac{\beta}{2}\left(\int_\Omega \frac{1}{\varphi(\chi)}|m\nabla\vartheta|^{\varphi(\chi)}d\chi\right)^2 + \int_\Omega \frac{1}{\psi(\chi)}|m\vartheta|^{\psi(\chi)}d\chi - \lambda\int_\Omega \hbar(x,m\vartheta)d\chi\\
\leq&\alpha\int_\Omega \frac{1}{\varphi(\chi)}|m\nabla\vartheta|^{\varphi(\chi)}d\chi - \frac{\beta}{2}\left(\int_\Omega \frac{1}{\varphi(\chi)}|m\nabla\vartheta|^{\varphi(\chi)}d\chi\right)^2 + \int_\Omega \frac{1}{\psi(\chi)}|m\vartheta|^{\psi(\chi)}d\chi - \lambda\int_\Omega (T|m\vartheta|^\theta - c_T)d\chi,\\
\leq&\frac{am^{\varphi_s}}{\varphi_l}\int_\Omega |\nabla\vartheta|^{\varphi(\chi)}d\chi - \frac{bm^{2\varphi_l}}{2\varphi_s^2}\left(\int_\Omega |\nabla\vartheta|^{\varphi(\chi)}d\chi\right)^2 + \frac{m^{\psi_s}}{\psi_l}\int_\Omega |\vartheta|^{\psi(\chi)}d\chi - \lambda T m^\theta\int_\Omega |\vartheta|^\theta d\chi + c_T|\Omega|.
\end{aligned}$$

Since $\varphi_s < \psi_s < \theta$, we obtain $J_\lambda(m\vartheta) \to -\infty$ as $m \to +\infty$, then for $m > 1$ large enough, by selecting $e = m\vartheta$ so that $||e|| > R$ and $J_\lambda(e) < 0$.

Proof of Theorem 1.1

By Lemmas 3.1 and 3.2 and J_λ ensures, $J_\lambda(0) = 0$ and the Mountain pass theorem (see [37]), thus our problem has at least a nontrivial weak solution. □

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