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CAV block-column iterative method for computed tomographic (CT) problems

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Abstract

This study revolves around the investigation of tomography and its mathematical modelling. In this regard, the CAV block-column iterative algorithm is proposed to reconstruct a high-quality image of a specific object. Indeed, this algorithm is applied for solving the problem of image reconstruction associated with computed tomography (CT). Then, the effects of both the relaxation parameter and the number of blocks are investigated on the convergence speed and the control of the semi-convergence phenomenon in this kind of iterative algorithm. Results show that a significant improvement in convergence speed and relative error can be achieved by a suitable choice of the relaxation parameter value.

Keywords: Tomography, Component averaging (CAV) Iterative method, Relaxation parameter, Block-column algorithm, Semi-convergence phenomenon
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1 Introduction and mathematical modeling of tomography

The word “tomography” originates from two Greek words, “Tomos” and “Graphein”. “Tomos” means to slice and section, and “Graphein” means register and record. In the imaging process, Computed tomography (CT) refers to the process of gathering information from passing or penetrating waves through an object in different directions. This method is based on the direct measurement of the remaining energy of an X-ray after it passes through various parts of the body. During this process, a sensitive detector measures the remaining energy and provides the results for the computer. Then, the computer calculates the absorption number for all points of the body through which the X-ray has passed.

Computed tomography (CT) is a useful tool for diagnosing diseases and injuries. This machine applies a series of X-rays and a computer to produce a 3D image of the soft tissues and bones. Also, it has other applications such as radiology, archaeology, geology, mining, oceanography, material science, and physics [3]. The emergence of this technology was a revolution in medical diagnosis. The mathematical basis of computed tomography dates back to the early twentieth century. However, this method was practically implemented in the 1960s. In 1917, J. Radon, an Austrian mathematician, proved that it is possible to take an infinite number of images in different directions of a two- or three-dimensional (2D or 3D) object. This was the basis of today’s CT scan machines.

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In X-ray tomography, if the wave passes through an object, its energy level decreases exponentially [5]. It is assumed that the object is homogeneous with a damping ratio of. After passing through the object, the X-ray is described as

$$I = I_0 e^{-ux}, \tag{1.1}$$

where I_0 is the initial energy and x is the length of the path that the wave passes through the object. Generally, when the wave passes through multiple homogeneous surfaces with a damping ratio of and length, the energy level reduces to

$$I = I_0 e^{-\sum u_i x_i}. \tag{1.2}$$

If the number of homogeneous surfaces tends to infinity, then the summation will turn to integration as follows.

$$I = I_0 e^{-\int_L u(x)dx}. \tag{1.3}$$

where L is the length of the path passing by the wave and $u(x)$ is the linear attenuation, which is a function of displacement x. Having different applications, problems may be 1D, 2D or 3D in which u is considered as $u(x)$, $u(x, y)$, or $u(x, y, z)$, respectively. In this view, the length of the path is denoted by parameter S and

$$I = I_0 e^{-\int_L u(x)ds}. \tag{1.4}$$

Eq (1.4) is a fundamental formula for CT scan [27, 30]. The purpose of this study is to estimate the density function $u(x)$ based on various measurements of X-ray attenuation over the length of different paths inside the object. The discrete version of those problems gives rise to an ill-conditioned linear system of equations (often inconsistent and with noisy data) of the form:

$$Ax = b \tag{1.5}$$

where $A \in Rm \in n$ is a large and sparse matrix with $b \in Rm$. The numerical solution of (1.5) calls for the use of iterative methods because direct factorization methods involve a prohibitive amount of computation. Tomography analytic reconstruction algorithms, such as the filtered back projection algorithm [29], are efficient when we have enough data with ignorable noise in the data. The iterative methods carry an advantage over the filtered back projection method, e.g., when the problems are underdetermined. This happens in limited-angle applications, e.g., breast X-ray tomography and few-projection measurements, where the X-ray dose should be limited. Other advantages are the possibility of introducing constraints [31], and involving some important physical factors into iterative reconstruction methods to make an accurate approximation. The iterative reconstruction algorithms obtain proper and desired results even if the measured data contain noise [14, 15, 18] or there are insufficient data [19, 22]. In recent years, iterative methods have been investigated in many research works as [7, 8, 9, 10, 11, 17, 24]. Censor [6] coined the expression row-action methods for a specific class of algebraic iterative methods (ART), also called Kaczmarz's method [25], and Cimmino's method [12]. The simultaneous Numerical Algorithms algebraic reconstruction technique (SART) was proposed as a refinement and superior implementation of the ART [1]. The methods DROP [8] and CAV [10] were presented to improve the rate of convergence using the sparsity of the matrix A. The block version of the CAV algorithm, called BICAV, was proposed in [9]. The basic idea of all row-action methods is to partition the data A and b into row-blocks of equations. All row-action methods update all elements of the approximate solution (of (1.5) or its normal equation) in each iteration. Instead of row-action methods, we consider here the column-oriented algorithms. The main advantage of the column version is that it does not exhibit the cyclic convergence of the row version, but converges to a least squares solution. Another advantage is the possibility of saving computational work during the iterations. This kind of algorithm is closely related to a coordinate descent type algorithm [33]. The studies in [4, 13] explain how to base an iterative reconstruction algorithm on columns rather than on rows. Furthermore, a column-based reconstruction method using nonnegativity constraints and a two-parameter algorithm based on a block-column partitioning is considered in [2, 32], respectively. In a recent paper [16], the authors studied a stationary column-oriented version of algebraic iterative methods, which is called block-column-iteration (BCI). The BCI method is an iterative reconstruction algorithm based on column partitioning of A rather than row partitioning.

Continuing the iteration process often produces iteration vectors that are corrupted by noise; see [14] and [15]. This phenomenon was called semi-convergence by Natterer [29] and provides much better results than the CGLS case [28]. Finding suitable ways for accelerating the convergence and determining the number of iterations required for reaching a reasonable solution is an important issue for every iterative algorithm. Attempts to increase the convergence speed and decrease the relative error have attracted scientists to carefully choose the best value for the relaxation parameter. These attempts were comprised of finding and evaluating strategies that give optimal results in a variety of problems with different parameters. In [16], a general algorithm was presented, but the factors that may influence its results were not discussed. We are going to show the effect of the relaxation parameter on the convergence rate and the relative error value of the CAV algorithm with several numerical tests.

The rest of the paper is organized as follows. In section 2, a CAV block-column iterative algorithm is proposed for solving the problem of image reconstruction. In section 3, several numerical examples and simulations are considered to investigate the effects of the relaxation parameter and number of blocks on the relative error and convergence speed. Finally, a conclusion is presented in section **??**.

2 CAV block-column iterative algorithm

The tomography problem is discretized as a linear system $Ax = b$, where $A = (a_{i,j})$ is a large, ill-posed and sparse matrix with

$$a_{i,j} = \begin{cases} \text{lenght of ray } i \text{ pixel } j \\ 0, \quad \text{if ray } i \text{ does not intersect pixel } j. \end{cases}$$

In addition, b is the vector of data $b_i = \sum_j a_{i,j} x_j$ and x is an unknown image. How to pixelate and ray passing in tomography is demonstrated in Figure 1

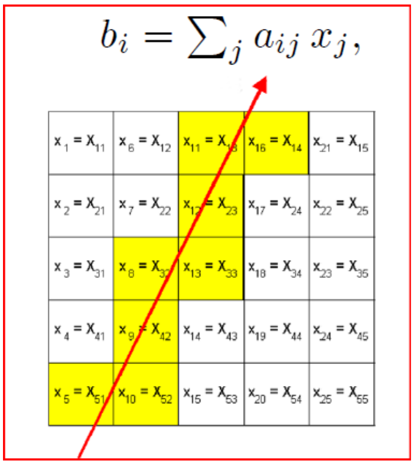

Figure 1: How to pixelate and ray passing in tomography

Let A be a $M \times N$ matrix, $x \in \mathbb{R}^N$ and $b \in \mathbb{R}^M$. Furthermore, assume that A and x are partitioned as follows:

$$A = (A_1, \ldots, A_q), \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix}, \tag{2.1}$$

where $A_i \in \mathbb{R}^{M \times N_i}, x_i \in \mathbb{R}^{N_i}$ for $i = 1, \ldots, q$. It should be noted that this blocking method and the general block-column algorithm have been presented in [16] for the first time. Block-column algorithm is described as follows:

Algorithm 1.

Initialization: $x^0 \in \mathbb{R}^N$ *is arbitrary*; $r^{0,1} = b - Ax^0$

For $k = 0, 1, 2, \ldots$ **(cycles or outer iterations)**

For $i = 1, \ldots, q$ **(inner iterations)**

$$x_i^{k+1} = x_i^k + \omega_i M_i A_i^T r^{k,i}$$

$$r^{k,i+1} = r^{k,i} - A_i\left(x_i^{k+1} - x_i^k\right)$$

End

$$r^{k+1,1} = r^{k,q+1}$$

End

Taking s_i^j as the number of non-zero elements of row$j$ from block A_i and vector a_i^j as column j from A_i, the weight matrix for the CAV method is defined as follows:

$$\begin{aligned} S_i &= \mathrm{Diag}(s_i^1, \dots, s_i^M), \|a\|_s = a^T sa, \\ M_i &= \mathrm{Diag}\left(\frac{1}{\|a_i^1\|_{s_i}^2}, \dots, \frac{1}{\|a_i^{N_i}\|_{s_i}^2}\right), \end{aligned} \tag{2.2}$$

The CAV block-column iterative algorithm is defined by considering the block format (2.2) and relaxation parameter. In algorithm 1, the weight matrices could be any given symmetric positive definite. Also, the relaxation parameter is a positive number in which some conditions are applied in [16] to the method that leads to the convergence of the above iterative method.

3 Discussion of convergence and Numerical results

In this section, through a variety of numerical experiments, we show that the number and condition of blocking, as well as the selection of the relaxation parameter, have a significant effect on the convergence rate. This, in turn, provides incentives to devise a strategy for the selection of the relaxation parameter. In the article [16], for the block-column algorithm, a theorem was expressed and proved for convergence, so that this method is convergent to the answer of the least squares if, for $0 < \varepsilon < 2$ and $i = 1, 2, \dots, q$.

$$0 < \varepsilon \leqslant \omega_i \leqslant \frac{2-\varepsilon}{\rho(A_i M_i A_i^T)}, \tag{3.1}$$

As we know, for CAV we have

$$\|A_i M_i A_i^T\|_2 \leqslant 1.$$

So $\omega_i \in (0, 2)$ [3].

Corollary 3.1. Assume that the iterates of (algorithm1) is convergent x^+, Then

$$A^T A x^+ = A^T b.$$

That is, x^+ is least squares solution of $Ax = b$ [16].

Here, we have tried to calculate and compare the relative error value on two problems taken from tomographic imaging, with different choices of relaxation parameters and considering the number of different blocks. In the first test, we use the SNARK09 software package [26]. We work with the standard head phantom from [21]. The phantom is discretized into 63×63 pixels, and 16 projections (evenly distributed between 0 and 174 degrees) with 99 rays per projection are used. The resulting matrix has dimensions of 1376×3969, so the system of equations is highly underdetermined. In this test for relaxation parameters, we considered the values from the convergence range mentioned in [16], and for blocking, we considered the number of columns of matrices equal and in four conditions: $nb = 1$, $nb = 9$, $nb = 27$ and $nb = 81$. The relative error histories of the block-CAV algorithm with different relaxation parameters and various blocking schemes are shown in Figure 2. The first simulation is implemented for each fixed blocking. In this case, different values are chosen for the relaxation parameter.

As shown in Figure 2, although it is allowed to take the value of the relaxation parameter in the range $(0, 2)$ based on the convergence theorem, the results are affected by the fluctuations for the values greater than or equal to 1. Therefore, these are unacceptable. All four curves in figure 2 have a smaller relative error with $\omega = 0.25$.

In Figure 3, different simulations have been run to reach the optimum value considering various blocking schemes. Also, the relative error is obtained for every blocking, considering the optimum value of the relaxation parameter. As shown in this figure, the highest convergence speed is achieved by using a number of 441 blocks, so that there are 9 columns in each block. Furthermore, it is deduced that a larger number of blocks leads to a better result.

Another example of the implementation of the CAV algorithm is studied by the AIR TOOLS software [20]. In this setting, the phantom is divided into 365×365 pixels. A total of 88 images (angle between zero and 179) are used, and 516 X-rays are passed from each image. Accordingly, the dimension of the matrix is 40892×13325. In this case, we follow some procedures like the previous case for determining different values of the relaxation parameters and number

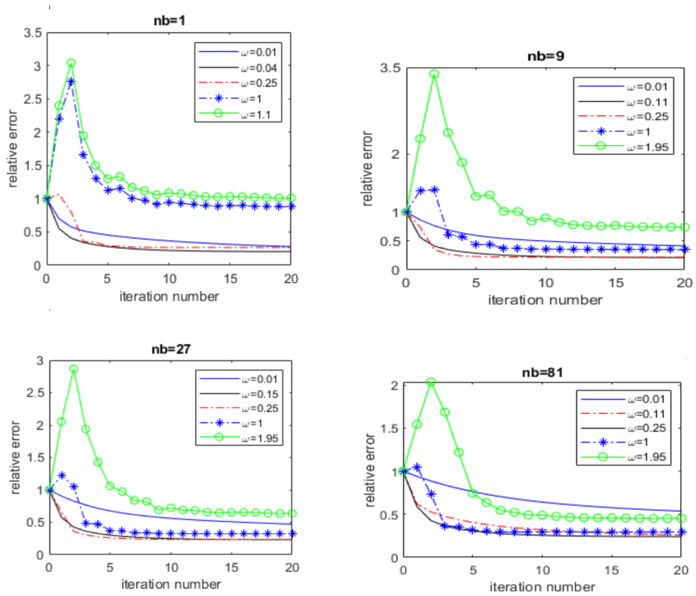

Figure 2: The relative error for different relaxation parameters and various blocking schemes

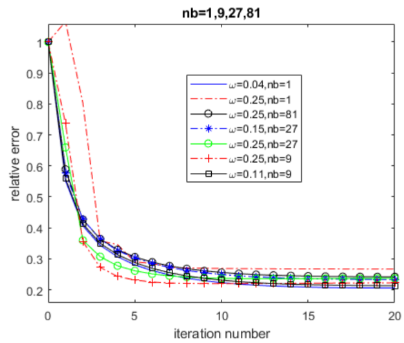

Figure 3: The relative error for different blocking schemes

of blocks, $b = 100$, $b = 532$, $b = 1000$. Similar to the previous case, the convergence speed is increased by considering a large number of blocks ($b = 5329$). The proper choice of relaxation parameter leads to a better convergence speed and a decrease in the relative error. In Figure 4, the relative error is shown for three different blocking schemes and two relaxation parameters.

In figure 5, the noise level of 0.5 is considered for the right vector b . The relative error is simulated for the CAV block-column method, CAV without blocking technique, and CGLS method in two states of noisy and free-noise [23]

4 Conclusion

In this paper, the CAV block-column iterative algorithm was studied to solve a system of linear equations which is sparse, ill-posed and large-scale. This algorithm is implemented by discretizing MRI and CT scan problems. This procedure is performed by using a simulation. The results demonstrated that this type of iterative method has several benefits for solving the system of linear equations, including suitable stability and acceptable relative error. As shown in Figure 5, in the noisy case, the CAV block-column method properly controls the semi-convergence phenomenon (initially, the iteration vectors approach a regularized solution). Nevertheless, continuing the iteration process often produces iteration vectors that are corrupted by noise; see [14] and [15]. This phenomenon was called semi-convergence by Natterer [29] and provides much better results than the CGLS case [23, 28]. This improvement arises from the stability feature, which belongs to this type of algorithm. Besides, it is shown that a significant improvement in the convergence speed and the relative error can be achieved by a suitable choice of the value of the relaxation parameter. It is concluded that, assuming the relaxation parameter in the range $(0.1, 0.5)$ has given reasonable results for various

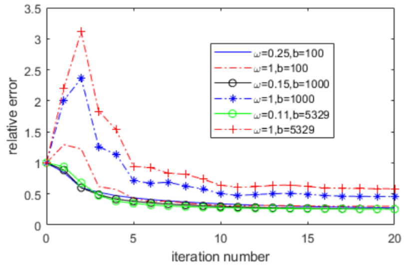

Figure 4: The comparison of relative error for different blocking schemes

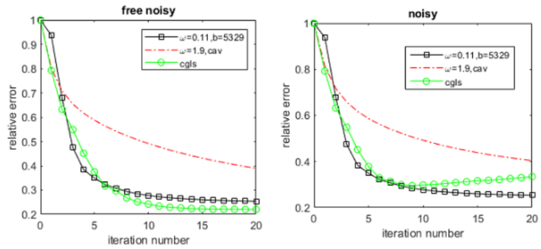

Figure 5: A comparison between the relative error of CAV method, CAV block–column method, and CGLS method

problems. It is indicated that choosing the type of blocking and the number of blocks has a positive effect on the results. As shown in Figure 5, the CAV algorithm has a lower relative error in the blocking mode. Additionally, this algorithm executes the computations in a much shorter time than others.

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