Int. J. Nonlinear Anal. Appl. In Press, 1–7
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2025.35301.5270

On T-neighborhoods of various classes of analytic functions

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(Communicated by Ali Ebadian)

Abstract

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathbb{U} = \{z : \ |z| < 1\}$ with the normalization conditions $f(0) = 0,\ f'(0) = 1$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\delta > 0$ are given, then the T_δ-neighborhood of the function f is defined as

$$TN_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} T_n \left| a_n - b_n \right| \le \delta \right\},$$

where $T = \{T_n\}_{n=2}^{\infty}$ is a sequence of positive numbers. In the present paper we investigate some problems concerning $T_\delta-$neighborhoods of analytic functions with $T = \left\{ \frac{n^2}{3^n n!} \right\}_{n=2}^{\infty}$. One of the considered problems is to find a number $\delta_T^*(A, B)$ such that

$$\delta_T^*(A, B) = \inf \left\{ \delta > 0 : B \subset TN_\delta(f) \text{ for all } f \in \mathcal{A} \right\},$$

where the sets $A, B \in \mathcal{A}$ are given.

Keywords: univalent, starlike, convex, close-to-convex, neighborhood, $T_\delta-$neighborhood, $T-$factor
2020 MSC: Primary 30C45; Secondary 30C80

1 Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathbb{U} = \{z : \ |z| < 1\}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ with the normalization conditions $f(0) = 0,\ f'(0) = 1$. Given a sequence $T = \{T_n\}_{n=2}^{\infty}$ consisting of positive numbers, the $T_\delta-$neighborhood ($\delta > 0$) of the function f is defined as

$$TN_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} T_n \left| a_n - b_n \right| \le \delta \right\}.$$

If $T = \{n\}_{n=2}^{\infty}$ then $T_\delta-$neighborhood becomes the $\delta-$neighborhood $N_\delta(f)$ introduced by St. Ruscheweyh [13]. He proved that if $f \in \mathcal{C}$ then $N_{1/4}(f) \in \mathcal{S}^*$, where $\mathcal{C}, \mathcal{S}^*$ denote the well known classes of convex and starlike functions, respectively. In this way he generalized the earlier result that $N_1(z) \in \mathcal{S}^*$. Some results of this type one can find in [8, 9, 10, 17]. The $T_\delta-$neighborhood was introduced in [15], where the authors considered the problem of finding a sufficient condition $f \in \mathcal{A}$ that implies the existence of $TN_\delta(f)$ being contained in a given subclass. U. Bednarz and J. Sokół considered and investigated $T_\delta-$neighborhood for various subclasses of analytic functions [18, 19]. Also

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Received: September 2024 *Accepted:* March 2025

a certain class of analytic functions was described by S. Shams et al by $T = \{2^{-n}n^{-2}\}_{n=2}^{\infty}$, see [16]. The convolution or Hadamard product of the functions f and g of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty}$ is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

An interesting problem of stability of convolution on certain classes by using the $\delta-$ neighborhoods was considered in [11, 12]. For work on this problem see also the papers [4, 5, 6]. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ of functions univalent in $\mathbb{U}$. Let us consider the following sequence of nonnegative reals

$$T = \left\{\frac{n^2}{3^n n!}\right\}_{n=2}^{\infty}. \tag{1.1}$$

In this paper we will use the above sequence to obtain the results about $T_\delta-$neighbor- hoods. The motivation of choice the sequence (1.1) is the convergence of the series $\sum_{n=2}^{\infty} T_n |a_n - b_n|$ for $|a_n| \le n$, $|b_n| \le n$, notice that Since

$$\sum_{n=0}^{\infty} \frac{n^2 x^n}{3^n n!} = \frac{3x + x^2}{3^2} e^{x/3},$$
$$\sum_{n=0}^{\infty} \frac{n^3 x^n}{3^n n!} = \frac{9x + 9x^2 + x^3}{3^3} e^{x/3},$$
$$\sum_{n=0}^{\infty} \frac{n^4 x^n}{3^n n!} = \frac{27x + 63x^2 + 18x^3 + x^4}{3^4} e^{x/3},$$

so we have

$$\sum_{n=0}^{\infty} \frac{n^2}{3^n n!} = \frac{4}{9} e^{1/3}, \tag{1.2}$$
$$\sum_{n=0}^{\infty} \frac{n^3}{3^n n!} = \frac{19}{27} e^{1/3}, \tag{1.3}$$
$$\sum_{n=0}^{\infty} \frac{n^4}{3^n n!} = \frac{109}{81} e^{1/3}. \tag{1.4}$$

Definition 1.1. [2] Let us consider the functions f that are meromorphic and univalent in $\mathbb{U}$, holomorphic at 0 and have the expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ If, in addition, the complement of $f(\mathbb{U})$ with respect to $\mathbb{C}$ is convex, then f is called a concave univalent function. The class of all concave functions is denoted by $\mathcal{C}o$.

It is well known [1], that if $f \in \mathcal{C}o$, then $|a_n| \ge 1$ for all $n > 1$ and equality holds if and only if $f(z) = \frac{z}{1-\mu z}$, $|\mu| = 1$ (see [1, 3]). The authors in [2] considered the class $\mathcal{C}o(p) \in \mathcal{C}o$ consisting of all concave functions that have a pole at the point p and are analytic in $|z| < |p|$. They proved that if $f \in \mathcal{C}o(1)$, then

$$\left| a_n - \frac{n+1}{2} \right| \le \frac{n-1}{2} \quad \text{for} \quad n \ge 2 \tag{1.5}$$

and equality holds only for the function f_θ defined by

$$f_\theta(z) = \frac{2z - (1 - e^{i\theta})z^2}{2(1-z)^2}, \ |z| < 1.$$

It is well known that if $f \in \mathcal{C}o(1)$, then the complement of $f(\mathbb{U})$ can be represented as the union of a set of mutually disjoint half-lines (the end point of one half-line can lie on the another half-line), so $f(\mathbb{U})$ is a linearly accessible domain in the strict sense, see [8, 17]. The authors in [7] also showed that $\mathcal{C}o(1) \subset \mathcal{K}$, where $\mathcal{K}$ is the set of close to-convex functions.

2 Main result

Throughout this section T will always be the sequence given by (1.1) unless otherwise stated.

Theorem 2.1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $|a_n| \leq n, |b_n| \leq n$, $n = 2, 3, 4, \cdots$, then $g \in TN_{\left\{\frac{38}{27}e^{\frac{1}{3}} - \frac{2}{3}\right\}}(f)$, where T is given in (1.1), The number $\frac{38}{27}e^{\frac{1}{3}} - \frac{2}{3} = 1.297\cdots$ is the best possible.

Proof . By (1.1) and (1.3), We have

$$\sum_{n=2}^{\infty} T_n |a_n - b_n| = \sum_{n=2}^{\infty} \frac{2n^3}{3^n n!} = 2((\frac{19}{27})e^{1/3} - \frac{1}{3}) = \frac{38}{27}e^{1/3} - \frac{2}{3} = 1.297\cdots .$$

For the functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + \sum_{n=2}^{\infty} nz^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n = z - \sum_{n=2}^{\infty} nz^n$$

we have

$$\sum_{n=2}^{\infty} T_n |a_n - b_n| = 2\sum_{n=2}^{\infty} \frac{n^3}{3^n n!} = 2((\frac{19}{27})e^{1/3} - \frac{1}{3}) = 1.297\cdots .$$

□

It is well known that if $\mathcal{S}, \mathcal{S}^*, \mathcal{C}$ and $\mathcal{K}$ denote the well-known classes of univalent, starlike, convex and close-to-convex functions respectively then $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$ and if $f \in \mathcal{S}^*$ then $|a_n| \leq n$ while if $f \in \mathcal{C}$ then $|a_n| \leq 1$. As a direct application of Theorem 2.1 we obtain $T_\delta-$neighborhood information for $\mathcal{S}^*$ and $\mathcal{K}$.

Corollary 2.2. If f belongs to one of the classes $\mathcal{S}^*, \mathcal{S}$, then

$$\mathcal{S} \subset TN_{(\frac{38}{27})e^{\frac{1}{3}} - \frac{2}{3}}(f),$$

where T is given in (1.1).

The constant $\frac{38}{27}e^{\frac{1}{3}} - \frac{2}{3} = 1.297\cdots$ seems not to be the best possible. An interesting open problem is to find the smallest constant σ such that for each $f \in \mathcal{S}$

$$\mathcal{S} \subset TN_\sigma(f),$$

where T is given in (1.1). For the Koebe function $f(z) = \frac{z}{(1-z)^2}$ and $g(z) = -f(-z)$, we have $f, g \in \mathcal{S}$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + \sum_{n=2}^{\infty} nz^n,$$

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n = z + \sum_{n=2}^{\infty} (-1)^{n-1} nz^n$$

so

$$\sum_{n=2}^{\infty} T_n |a_n - b_n| = 2\sum_{n=1}^{\infty} \frac{(2n)^3}{3^{(2n)}(2n)!} = 2(\frac{19}{54}e^{1/3} - \frac{1}{54}e^{-1/3}) = 0.9556\cdots .$$

Therefore, the number σ can not be smaller than $0.9556\cdots$. We conjecture that $\sigma = 0.9556\cdots$. The result will change if we consider the class of convex functions $\mathcal{C}$.

Corollary 2.3. Let $f \in \mathcal{C}$. Then $\mathcal{S} \subset TN_\beta(f)$ with $\beta = \frac{31}{27}e^{1/3} - \frac{2}{3} = 0.9357\cdots$.

Proof . If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, then $|a_n|\leq 1, n\geq 2$. Thus if $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{S}$, then by (1.2) and (1.3), we have

$$\begin{aligned}\sum_{n=2}^{\infty}T_n\,|a_n-b_n| &\leq \sum_{n=2}^{\infty}n^2\frac{n+1}{3^n n!}\\ &=\sum_{n=2}^{\infty}\frac{n^3}{3^n n!}+\sum_{n=2}^{\infty}\frac{n^2}{3^n n!}\\ &=\frac{19}{27}e^{1/3}-\frac{1}{3}+\frac{4}{9}e^{1/3}-\frac{1}{3}\\ &=\frac{31}{27}e^{1/3}-\frac{2}{3}\\ &=0.9357\cdots\\ &=\beta. \end{aligned} \tag{2.1}$$

□

In a similar way as in Corollary 2.2, the constant $\beta=\frac{31}{27}e^{1/3}-\frac{2}{3}=0.9357\cdots$ given in Corollary 2.3 is also not sharp but if the class $\mathcal{S}$ is replaced by the much larger class of normalized analytic function f such that $|a_n(f)|\leq n$ for $n\geq 2$, then β becomes sharp. The best possible constant in the case $f\in\mathcal{S}$ is not known. We conjecture that the sharp constant is attained by the functions

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n=\frac{z}{(1-z)^2}=z+\sum_{n=2}^{\infty}nz^n,$$

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n=\frac{z}{1+z}=z+\sum_{n=2}^{\infty}(-1)^{n-1}z^n.$$

It is clear that $f\in\mathcal{S}$ and $g\in\mathcal{C}$. Moreover,

$$\begin{aligned}\sum_{n=2}^{\infty}T_n\,|a_n-b_n| &= \sum_{n=2}^{\infty}\frac{n-(-1)^{n-1}}{3^n n!}n^2\\ &=\sum_{n=2}^{\infty}\frac{n+1}{3^n n!}n^2-\sum_{n=2}^{\infty}\frac{1+(-1)^{n-1}}{3^n n!}n^2\\ &=\sum_{n=2}^{\infty}\frac{n^3}{3^n n!}+\sum_{n=2}^{\infty}\frac{n^2}{3^n n!}-2\sum_{n=1}^{\infty}\frac{(2n+1)^2}{3^{(2n+1)}(2n+1)!}\\ &=\frac{31}{27}e^{1/3}-\frac{2}{3}-\frac{2}{9}(2e^{1/3}+e^{-1/3})\\ &=0.1562\cdots. \end{aligned} \tag{2.2}$$

Therefore, the smallest constant β such that $\mathcal{S}\in TN_\beta(f)$ for each $f\in\mathcal{C}$ lies between $0.1562\cdots$ and $0.9357\cdots$. We conjecture that it is the first number.

Theorem 2.4. Let f, g_1, g_2 be of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ g_1(z)=z+\sum_{n=2}^{\infty}c_nz^n,\ g_2(z)=z+\sum_{n=2}^{\infty}d_nz^n,$$

where $|a_n|\leq n, |c_n|\leq n, |d_n|\leq n, n=2,3,\cdots$.Then

$$g_1*g_2\in TN_{\left\{\frac{148}{81}e^{1/3}-\frac{2}{3}\right\}}(f).$$

The number $\frac{148}{81}e^{1/3}-\frac{2}{3}$ is the best possible.

Proof . Since

$$(g_1 * g_2)(z) = z + \sum_{n=2}^{\infty} c_n d_n z^n,$$

we have

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{n^2}{3^n n!} |c_n d_n - a_n| &\leq \sum_{n=2}^{\infty} \frac{n^2}{3^n n!}(n^2+n) \\ &= \frac{109}{81} e^{1/3} - \frac{1}{3} + \frac{19}{27} e^{1/3} - \frac{1}{3} \\ &= \frac{166}{81} e^{1/3} - \frac{2}{3} \\ &= 2.1934\cdots . \end{aligned} \tag{2.3}$$

The functions

$$f(z) = z - \sum_{n=2}^{\infty} n z^n, \qquad g_1(z) = g_2(z) = z + \sum_{n=2}^{\infty} n z^n$$

show that the number $\frac{166}{81} e^{1/3} - \frac{2}{3} = 2.1934\cdots$ is the best possible. Therefore the proof is completed. □

Definition 2.5. [7] Let A and B be arbitrary subset of the $\mathcal{A}$, and let T be a sequence of positive number, then $\delta_T^*(A, B)$ is defined by

$$\delta_T^*(A, B) = \inf \{\delta > 0 : B \subset TN_\delta(f) \text{ for all } f \in A\}.$$

Let us denote

$$T(f, g) = \sum_{n=2}^{\infty} T_n |a_n - b_n|. \tag{2.4}$$

Therefore, we can write

$$\begin{aligned}\delta_T^*(A, B) &= \inf \{\delta : T(f, g) < \delta \text{ for all} f \in A, g \in B\} \\ &= \sup \{T(f, g) : f \in A, g \in B\},\end{aligned}$$

where the condition $T(f, g) < \delta$ means that the series $T(f, g)$ is convergent and its sum is less than δ. Therefore, we see that $\delta_T^*(A, B) = \delta_T^*(B, A)$, and we will say that $\delta_T^*(A, B)$ is the $T-$factor with respect to the classes A and B. Making use of the above definition, Corollary 2.2 and the consideration below Corollary 2.2, we can state next corollary, where $T = \{T_n\}_{n=2}^{\infty}$ is again of the form (1.1).

Corollary 2.6. The $T-$factor with respect to the classes $\mathcal{S}$ and $\mathcal{S}$ satisfies the following inequality

$$0.9556\cdots \leq \delta_T^*(\mathcal{S}, \mathcal{S}) \leq 1.297\cdots . \tag{2.5}$$

It is well known that the Koebe function and all its rotations belong to each of the classes $\mathcal{S}, \mathcal{S}^*$ and $\mathcal{K}$ (univalent, starlike and close-to-convex functions respectively), then Corollary 2.6 follows the next corollary.

Corollary 2.7. Let A and B be one of the classes $\mathcal{S}, \mathcal{S}^*$ or $\mathcal{K}$. Then

$$0.9556\cdots \leq \delta_T^*(A, B) \leq 1.297\cdots . \tag{2.6}$$

In the same way as above, we can express Corollary 2.3 in terms T-factor. It is done in the next result.

Corollary 2.8. The $T-$ factor with respect to the classes $\mathcal{C}$ of convex functions and $\mathcal{S}$ satisfies the following inequality

$$0.1562\cdots \leq \delta_T^*(\mathcal{C}, \mathcal{S}) \leq 0.9357\cdots .$$

Remark 2.9. Now we consider the "central" function with respect to coefficient in the class $\mathcal{C}o(1)$ which is denoted by $f_c(z)$ and defined by

$$f_c(z) = \frac{1}{2}\left\{\frac{z}{1-z} + \frac{z}{(1-z)^2}\right\} = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n, \quad |z| < 1.$$

In [7] the authors showed that $f_c \in \mathcal{C}o(1)$.

Theorem 2.10. The following inclusion relation holds

$$\mathcal{C}o(1) \in TN_\delta(f_c),$$

where $\delta = \frac{1}{2}(\frac{7}{27}e^{1/3}) = 0.1809\cdots$

Proof . Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}o(1)$, then from (1.2) and (1.3), we obtain

$$\begin{aligned}
\sum_{n=2}^{\infty} T_n \left| a_n - \frac{n+1}{2} \right| &\leq \sum_{n=2}^{\infty} T_n \left| \frac{n-1}{2} \right| \\
&= \frac{1}{2} \sum_{n=2}^{\infty} \frac{(n-1)n^2}{3^n n!} \\
&= \frac{1}{2} \left[\sum_{n=2}^{\infty} \frac{n^3}{3^n n!} - \sum_{n=2}^{\infty} \frac{n^2}{3^n n!} \right] \\
&= \frac{1}{2} \left[\frac{19}{27} e^{1/3} - \frac{1}{3} - \frac{4}{9} e^{1/3} + \frac{1}{3} \right] \\
&= \frac{1}{2} \left(\frac{7}{27} e^{1/3} \right) \\
&= 0.1809\cdots \\
&= \delta. \qquad (2.7)
\end{aligned}$$

□

Acknowledgment

We would like to thank the referee for the valuable comments and suggestions on improving the paper.

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