Int. J. Nonlinear Anal. Appl. In Press, 1–10
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.33780.5038

A comparative analysis of hypergroupoids derived from metric spaces

Kazem Hamidizadeh[a], Saeed Mirvakili[b,*], Rauofeh Manaviyat[a], Hossein Naraghi[a]

[a] *Department of Mathematics, Payame Noor University, Tehran, Iran*

[b] *Department of Mathematical Sciences, Yazd University, Yazd, Iran*

(Communicated by Ali Farajzadeh)

Abstract

This study investigates the correlation between various hypergroupoids and a specified metric space, examining the distinctive characteristics of this hyperstructure. Through rigorous analysis, our inquiry establishes the potential of these hypergroupoids to function as commutative quasihypergroups. Additionally, we delineate specific conditions under which these hypergroupoids exhibit hypergroup properties, shedding light on the complex interplay of these mathematical structures within the context of the given metric space. Finally, a metric space obtained from a given hypergroupoid is introduced.

Keywords: metric space, hypergroupoid, Hypergroup, H_v-group
2020 MSC: 20N20, 54E35

1 Introduction

The theory of hyperstructures originated in 1934 with Marty's introduction of the concept of a hypergroup. Numerous papers and books, such as [4, 5, 7], have since been published on this subject.

The connection between algebra and metric spaces can be traced back to the early 20th century, when mathematicians began to study the properties of abstract spaces using algebraic tools. One of the key figures in this development was Hausdorff [15], who introduced the concept of a topological space in 1914. This allowed mathematicians to study the properties of spaces without relying on specific geometric or metric structures. In 1930, mathematicians such as John Von Neumann and Gelfand began to develop the theory of Banach spaces, which are complete normed vector spaces [12, 16]. This theory combined algebraic and metric concepts to provide a framework for studying linear operators on infinite-dimensional spaces. Another important development in the connection between algebra and metric spaces was the introduction of functional analysis by Stefan Banach [1] in 1930. This field uses algebraic techniques to study the properties of functions on metric spaces, and has applications in many areas of mathematics and physics. In the latter half of the 20th century, the study of algebraic structures on metric spaces continued to grow, with the development of theories such as C^*-algebras and von Neumann algebras. These theories provide a powerful framework for studying quantum mechanics and other areas of mathematical physics. Overall, the connection between algebra

*Corresponding author
Email addresses: k.hamidizadeh@pnu.ac.ir (Kazem Hamidizadeh), saeedmirvakili@yazd.ac.ir (Saeed Mirvakili), r.manaviyat@pnu.ac.ir (Rauofeh Manaviyat), ho.naraghi@pnu.ac.ir (Hossein Naraghi)

Received: April 2024 *Accepted:* July 2024

and metric spaces has been an important area of research in mathematics for over a century, with applications in many different fields.

The connection between algebraic hyperstructures and metric spaces was first established by Zarei and Mirvakili [18] and Mirvakili and Manaviyat [14] by focusing on open neighborhoods of elements in metric spaces and building a hyperoperation based on the community of two neighborhoods. In this paper, we derive a class of hypergroupoids based on the concept of open and closed balls in a metric space. We examine the necessary and sufficient conditions for the associativity of the constructed hyperoperation. We also illustrate this connection with various examples.

Algebraic hyperstructures provide a fitting extension of classical algebraic structures. Whereas the composition of two elements in a classical algebraic structure results in an element, a composition of two elements in an algebraic hyperstructure yields a set. Let $P^*(X)$ be the set of all non-empty subsets of a given set X. A *hypergroupoid* is a pair $(X, \circ)$, where X is a non-empty set and $\circ$ is a *hyperoperation*, i.e.,

$$\circ : X \times X \longrightarrow P^*(X), \quad (x, y) \mapsto x \circ y.$$

If $A, B \in P^*(X)$, then we define $A \circ B = \bigcup \{a \circ b \mid a \in A, b \in B\}$, $x \circ B = \{x\} \circ B$ and $A \circ y = A \circ \{y\}$. If $A = \emptyset$ or $B = \emptyset$ we define $A \circ B = \emptyset$.

A hypergroupoid $(X, \circ)$ is called *semihypergroup* if the associative axiom is valid, i.e., $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in X$ and it is called *reproductive* if $x \circ X = X \circ x = X$, for all $x \in X$. A *hypergroup* is a reproductive semihypergroup. A commutative hypergroup $(X, \circ)$ (i.e. $x \circ y = y \circ x$ for all $x, y \in X$) is called a *join space* if the following implication holds for all elements a, b, c, d of X:

$$a/b \bigcap c/d \neq \emptyset \Longrightarrow a \circ d \bigcap b \circ c \neq \emptyset,$$

where $a/b = \{x \mid a \in x \circ b\}$. Connections between hypergraphs and hyperstructures are studied by many authors, for example, see [6].

Let X be any set. Then a function $d : X \times X \to \mathbb{R}$ is said to be a metric on X if it has the following properties for all $x, y, z \in X$:

(M1) $d(x, y) \geq 0$;

(M2) $d(x, y) = 0$ if and only if $x = y$;

(M3) $d(x, y) = d(y, x)$;

(M4) $d(x, y) + d(y, z) \geq d(x, z)$.

The real number $d(x, y)$ is called the distance between x and y, and the set X together with a metric d is called a metric space (X, d) [17]. Given a metric space (X, d) and any real number $r > 0$, the open ball of radius r and center a is the set $B_r(a) \subseteq X$ defined by

$$B_r(a) = \{x \in X | d(x, a) < r\}.$$

Also, we set

$$\overline{B}_r(a) = \{x \in X | d(x, a) \leq r\}.$$

2 Main Results

Let $\mathcal{X} = (X, d)$ be a metric space. Then $\mathcal{X}_r = (X, \circ_r)$ and $\overline{\mathcal{X}}_r = (X, \overline{\circ}_r)$ are two hypergroupoids where the hyperoperations $\circ_r$ and $\overline{\circ}_r$ are defined by

$$x \circ_r y = B_r(x) \bigcup B_r(y), \quad \forall (x, y) \in X^2,$$

when $r > 0$ and

$$x \overline{\circ}_r y = \overline{B}_r(x) \bigcup \overline{B}_r(y), \quad \forall (x, y) \in X^2,$$

where $r \geq 0$.

Example 2.1. The real numbers with the distance function $d(x,y) = |y - x|$ given by the absolute difference form a metric space. For any $r > 0$ and $x \in \mathbb{R}$, we have $B_r(x) = (x - r, x + r)$ and $\overline{B}_r(x) = [x - r, x + r]$. Then

$$x \circ_r y = (x - r, x + r) \cup (y - r, y + r), \quad \forall (x,y) \in X^2,$$

and

$$x \overline{\circ}_r y = [x - r, x + r] \cup [y - r, y + r], \quad \forall (x,y) \in X^2.$$

Lemma 2.2. For all $x, y \in X$ and $r > 0$, $x \circ_r y$ is an open set and $x \overline{\circ}_r y$ is a closed set.

Theorem 2.3. The hypergroupoids $\mathcal{X}_r = (X, \circ_r)$ and $\overline{\mathcal{X}}_r = (X, \overline{\circ}_r)$ satisfy for each $(x,y) \in H^2$:

(1) $x \circ_r y = x \circ_r x \cup y \circ_r y$;

(1') $x \overline{\circ}_r y = x \overline{\circ}_r x \cup y \overline{\circ}_r y$;

(2) $x \in x \circ_r x$;

(2') $x \in x \overline{\circ}_r x$;

(3) $y \in x \circ_r x \Leftrightarrow x \in y \circ_r y$;

(3') $y \in x \overline{\circ}_r x \Leftrightarrow x \in y \overline{\circ}_r y$.

Proof . It obtains from definitions of $\circ_r$ and $\overline{\circ}_r$. □

Theorem 2.4. A hypergroupoid X satisfying (1), (2), (3) of the Theorem 2.3 also satisfies

(4) $\{x, y\} \subseteq x \circ_r y$,

(5) $x \circ_r y = y \circ_r x$,

(6) $x \circ_r X = X$,

(7) $(x \circ_r x) \circ_r x = \cup_{z \in x \circ_r x} z \circ_r z$,

(8) $(x \circ_r x) \circ_r (x \circ_r x) = x \circ_r x \circ_r x$, where $x \circ_r x \circ_r x = (x \circ_r x) \circ_r x = x \circ_r (x \circ_r x)$.

Proof . It is straightforward. □

By (5) and (6) of Theorem 2.4 we obtain:

Corollary 2.5. A hypergroupoid $\mathcal{X}_r = (X, \circ_r)$ is a commutative quasihypergroup.

Proof . Commutativity obtain from Theorem 2.4 part (5) and reproduce law obtain from Theorem 2.4 part (6). □

Corollary 2.6. A hypergroupoid $\mathcal{X}_r = (X, \circ_r)$ is a commutative Hv-group.

Proof . By Theorem 2.4 part (4), for every $x, y, z \in X$, we have

$$\{x, y, z\} \subseteq (x \circ_r y) \circ_r z \cap x \circ_r (y \circ_r z).$$

This show that proof is complete. □

Theorem 2.7. A hypergroupoid X satisfying $(1')$, $(2')$, $(3')$ of the Theorem 2.3 also satisfies

(4') $\{x, y\} \subseteq x \overline{\circ}_r y$,

(5') $x \overline{\circ}_r y = y \overline{\circ}_r x$,

(6') $x \overline{\circ}_r X = X$,

(7') $(x\overline{\circ}_r x)\overline{\circ}_r x = \cup_{z\in x\overline{\circ}_r x} z\overline{\circ}_r z$,

(8') $(x\overline{\circ}_r x)\overline{\circ}_r(x\overline{\circ}_r x) = x\overline{\circ}_r x\overline{\circ}_r x$.

Proof . It is straightforward. □

By $(5')$ and $(6')$ of Theorem 2.7 we obtain:

Corollary 2.8. A hypergroupoid $\overline{\mathcal{X}}_r = (X, \overline{\circ}_r)$ is a commutative quasihypergroup. Moreover, $\overline{\mathcal{X}}_r = (X, \overline{\circ}_r)$ is a commutative H_v-group.

Proof . The proof is similar to proof of Corollaries 2.5 and 2.6. □

Theorem 2.9. [4] A hypergroupoid X satisfying (1), (2), (3) of the Theorem 2.3 is a hypergroup if and only if

$$\forall (a,c)\in X^2,\quad c\circ_r c\circ_r c - c\circ_r c \subseteq a\circ_r a\circ_r a.$$

where $c\circ_r c\circ_r c = (c\circ_r c)\circ_r c \cup c\circ_r (c\circ_r c)$.

Lemma 2.10. (i) Let $x\in X$, then $x\circ_r x\circ_r x = B_r^2(x)$, where $B_r^2(x) = \bigcup_{z\in B_r(x)} B_r(z)$.

(ii) Let $x\in X$, then $x\overline{\circ}_r x\overline{\circ}_r x = \overline{B^2}_r(x)$.

Proof . (i) By part (8) of Theorem 2.4, we have $(x\circ_r x)\circ_r (x\circ_r x) = x\circ_r x\circ_r x$. Then

$$\begin{aligned} x\circ_r x\circ_r x &= B_r(x)\circ_r B_r(x) \\ &= \bigcup_{y,z\in B_r(x)} y\circ_r z \\ &= \bigcup_{y,z\in B_r(x)} (B_r(z)\cup B_r(y)) \\ &= \bigcup_{z\in B_r(x)} B_r(z) = B^2(R). \end{aligned}$$

(ii) By part (8') of Theorem 2.7, we have $(x\overline{\circ}_r x)\overline{\circ}_r(x\overline{\circ}_r x) = x\overline{\circ}_r x\overline{\circ}_r x$. Then

$$\begin{aligned} x\overline{\circ}_r x\overline{\circ}_r x &= \overline{B}_r(x)\overline{\circ}_r\overline{B}_r(x) \\ &= \bigcup_{y,z\in \overline{B}_r(x)} y\overline{\circ}_r z \\ &= \bigcup_{y,z\in \overline{B}_r(x)} (\overline{B}_r(z)\cup \overline{B}_r(y)) \\ &= \bigcup_{z\in \overline{B}_r(x)} \overline{B}_r(z) = \overline{B}^2(R). \end{aligned}$$

Therefore the proof is complete. □

Corollary 2.11. A hypergroupoid $\mathcal{X}_r = (X, \circ_r)$ is a hypergroup if and only if for every $x, y\in X$ we have

$$B_r^2(x) - B_r(x) \subseteq B_r^2(y),$$

where $B_r^2(x) = \bigcup_{z\in B_r(x)} B_r(z)$.

Corollary 2.12. A hypergroupoid $\mathcal{X}_r = (X, \circ_r)$ is a hypergroup if and only if for every $x, y\in X$ we have

$$B_r^2(x) - B_r(x) \subseteq B_r^2(y),$$

where $B_r^2(x) = \bigcup_{z\in B_r(x)} B_r(z)$.

Proof . It obtains from Theorem 2.9 and Lemma 2.10 part (i). □

Corollary 2.13. A hypergroupoid $\overline{\mathcal{X}}_r = (X, \overline{\circ}_r)$ is a hypergroup if and only if for every $x, y \in X$ we have

$$\overline{B^2}_r(x) - \overline{B}_r(x) \subseteq \overline{B^2}_r(y),$$

where $\overline{B^2}_r(x) = \bigcup_{z \in \overline{B}_r(x)} \overline{B}_r(z)$.

Proof . It obtains from Theorem 2.9 and Lemma 2.10 part (ii). □

Corollary 2.14. If for every $x \in X$, $B_r^2(x) = B_r(x)$, then the hypergroupoid $\mathcal{X}_r = (X, \circ_r)$ is a hypergroup.

Proof . Since $B^2{}_r(x) - B_r(x) = \emptyset$, we have $\mathcal{X}_r = (X, \circ_r)$ is a hypergroup by Corollary 2.12 . □

Corollary 2.15. If for every $x \in X$, $\overline{B^2}_r(x) = \overline{B}_r(x)$ then the hypergroupoid $\overline{\mathcal{X}}_r = (X, \overline{\circ}_r)$ is a hypergroup.

Proof . We obtain $\overline{B^2}_r(x) - \overline{B}_r(x) = \emptyset$ and by Corollary 2.13 we have $\overline{\mathcal{X}}_r = (X, \overline{\circ}_r)$ is a hypergroup. □

Example 2.16. Let (X, d) be the metric space where d is discrete metric, that $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise. Then for every $r \in \mathbb{R}^+$ and for every $x \in X$, we have $B_r^2(x) = B_r(x)$. So, by Corollary 2.14, the hypergroupoid $\mathcal{X}_r = (X, \circ_r)$ is a hypergroup.

Also, by Corollary 2.14, the hypergroupoid $\overline{\mathcal{X}}_r = (X, \overline{\circ}_r)$ is a hypergroup.

Example 2.17. If G is an undirected connected graph, then the set V of vertices of G can be turned into a metric space by defining $d(x, y)$ to be the length of the shortest path connecting the vertices x and y.

The r-ball $B_r(x)$ of center x and radius $r \geq 0$ consists of all vertices of G at distance at most r from x: In particular, if $0 \leq r \leq 1$, the ball $B_r(x) = \{x\}$ and if $1 < r \leq 2$, the ball $B_r(x)$ comprises x and $N(x)$, where $N(x)$ is the neighborhood of vertex x in graph G.

Therefore, the hyperoperation $\circ_r$ is to coincide with the hyperoperation $\circ$ in [6], when for every $x, y \in G$,

$$x \circ y = N(x) \cup N(y).$$

Theorem 2.18. If the hypergroupoid $(X, \circ_r)$ is a hypergroup then it is a join space.

Proof . Let $a/b = \{x \mid a \in x \circ_r b\}$. Suppose that $a/b \bigcap c/d \neq \emptyset$ then there exists $x \in X$ such that $a \in x \circ_r b$ and $c \in x \circ_r d$. So $a \in B_r(x) \cup B_r(b)$ and $c \in B_r(x) \cup B_r(d)$. We have one of the four following cases:

(1) $a \in B_r(x)$ and $c \in B_r(x)$, then $x \in B_r(a)$ and $x \in B_r(c)$ and so $x \in a \circ_r d \cap b \circ_r c$.

(2) $a \in B_r(x)$ and $c \in B_r(d)$, then $c \in a \circ_r d \cap b \circ_r c$.

(3) $a \in B_r(b)$ and $c \in B_r(x)$, then $a \in a \circ_r d \cap b \circ_r c$.

(4) $a \in B_r(b)$ and $c \in B_r(d)$, then $a \in a \circ_r d \cap b \circ_r c$.

□

Theorem 2.19. If the hypergroupoid $\overline{\mathcal{X}}_r = (X, \overline{\circ}_r)$ is a hypergroup then it is a join space.

Proof . It is similar to the proof of Theorem 2.18. □

In general, we have $\circ_r \subseteq \overline{\circ}_r$. Also, there are many examples in which we have $\circ_r \subsetneq \overline{\circ}_r$. Moreover, $\circ_r \neq \overline{\circ}_s$ and $\overline{\circ}_r \neq \circ_s$ in the general state. Now, the following two questions arise:

Question 1. For every $r > 0$, is there $s > 0$ such that $\overline{\circ}_r = \circ_s$?

Question 2. For every $r > 0$, is there $s > 0$ such that $\circ_r = \overline{\circ}_s$?

The next example shows that there is a metric space where the answers to the above questions can not be correct.

Example 2.20. Let $X = \mathbb{R}$ with the usual metric. Then for every $x \in \mathbb{N}$ and $r > 0$, there is no $s > 0$ such that $B_r(x) = \overline{B}_s(x)$. Also, for every $x \in \mathbb{N}$ and $r > 0$, there is no $s > 0$ such that $\overline{B}_r(x) = B_s(x)$.

The next example shows that there is a metric space where the answers to the above questions are correct.

Example 2.21. Consider $\mathbb{N} = \{1, 2, 3, \ldots\}$ with the usual metric. Then for every $x \in \mathbb{N}$ and $r > 0$ we have $B_r(x) = \overline{B}_s(x)$, where $s = \begin{cases} r & r \notin \mathbb{N} \\ r-1 & r \in \mathbb{N} \end{cases}$. Therefore $(X, \circ_r) = (X, \overline{\circ}_s)$.

Moreover, for every $x \in \mathbb{N}$ and $r > 0$ we have $\overline{B}_r(x) = B_s(x)$, where $s = \begin{cases} r & r \notin \mathbb{N} \\ r+1 & r \in \mathbb{N} \end{cases}$. Therefore $(X, \overline{\circ}_r) = (X, \circ_s)$.

Example 2.22. Consider the set $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ with the usual metric. Then every closed ball in X is open in X; but the open ball $B_1(1)$ is not closed in X.

In this example, for every $x \in X$ and $r > 0$ we have $\overline{B}_r(x) = B_s(x)$, for some $s > 0$. So $(X, \overline{\circ}_r) = (X, \circ_s)$.

But, for every $r > 0$, $1 \circ_1 1 = B_1(1) = X - \{0\} \neq 1 \overline{\circ}_r 1 \ni 0$. So for every $r > 0$, $\overline{\circ}_r \neq \circ_1$.

Example 2.23. Consider the Hilbert cube $H := [0,1]^{\mathbb{N}}$ with metric

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}.$$

For each $i \in N$, let e_i denote the element of H which is all 0s except for a 1 in the ith slot. So, $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, etc. I'll also write 0 for the all-zeroes element $(0, 0, 0, \ldots)$

Let $Y \subset H$ be given by $Y = \{0, e_1, e_2, \ldots\}$. Then Y will be the desired example. That is, I claim that 1) every open ball in Y is closed but 2) that there is a closed ball in Y which is not open.

As a preliminary observation, note that for $i \neq j$, that $d(e_i, e_j) = \frac{1}{2^i} + \frac{1}{2^j} > \max\{\frac{1}{2^i}, \frac{1}{2^j}\}$, and that $d(e_i, 0) = \frac{1}{2^i}$. It follows that each e_i is isolated, with, e.g., $B_{\frac{1}{2^i}}(e_i) = \{e_i\}$ being an open ball containing just the one point e_i. Moreover, for each e_i, 0 is the unique closest point to it.

Here's the proof of 1) To begin with, note first that every open ball centered at 0 is closed because the complement can only possible contain some of the $\{e_i\}$ which are all isolated.

So, consider $B_r(e_i)$, an open ball centered at e_i of radius $r > 0$. Since all the e_i are isolated, the only case we need to consider is if $0 \notin B_r(e_i)$. But 0 is the closest point to e_i, so if 0 is not in $B_r(e_i)$, then neither are any e_j with $i \neq j$. Thus, any ball around 0 which doesn't contain e_i witnesses the fact that the complement of B is open. E.g., one can take $B_{\frac{1}{2^i}}(0)$. This concludes the proof of 1).

We now prove 2), that there is a closed ball which is not open. To that end, consider $\overline{B_{\frac{1}{2}}}(e_1)$. This set clearly contain e_1 and 0; since 0 is the unique closest point to e_1, it follows that $\overline{B_{\frac{1}{2}}}(e_1) = \{e_1, 0\}$. So, to show $\overline{B_{\frac{1}{2}}}(e_1)$ is not open, it's enough to show that every ball around 0 contains an e_i other than e_1.

But $d(e_i, 0) = \frac{1}{2^i} \to 0$ as $i \to \infty$, so any open ball around 0 contains all but finitely many of the e_i.

In this example, for every $x \in X$ and $r > 0$ we have $B_r(x) = \overline{B}_s(x)$, for some $s > 0$. So $(X, \overline{\circ}_s) = (X, \circ_r)$.

But, for every $r > 0$, $1 \circ_1 1 = B_1(1) = X - \{0\} \neq 1 \overline{\circ}_r 1 \ni 0$. So for every $r > 0$, $\overline{\circ}_r \neq \circ_1$.

Definition 2.24. Let $(H, \circ)$ and $(H, \star)$ be two hypergroupoids. We say that $(H, \circ) \sqsubseteq (H, \star)$ if for every $x, y \in H$, $x \circ y \subseteq x \star y$.

Theorem 2.25. For every $0 < r < s$, we have $(X, \circ_r) \sqsubseteq (X, \circ_s)$ and $0 \leq r < s$, $(X, \overline{\circ}_r) \sqsubseteq (X, \overline{\circ}_s)$.

Proof . We have $\overline{B}_r(x) \subseteq \overline{B}_s(x)$ and $B_r(x) \subseteq B_s(x)$ for every $x \in X$. So proof is complete. $\square$

Example 2.26. the Euclidean space $R^n = R \times \ldots \times R$ is the set of all ordered n-tuples or vectors over the real numbers R (when $n = 1$ we refer to the vectors as scalars). We denote a vector in R^n when $n > 1$ in bold and refer to its scalar components via subscripts. For example the vector $x = (x_1, \ldots, x_n)$ has n scalar components $x_i \in R$, $i = 1, \ldots, n$. Together with the Euclidean distance

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (xi - yi)^2},$$

the Euclidean space is a metric space (R^n, d).

Then for every $0 < r < s$, $(X, \circ_r) \sqsubseteq (X, \circ_s)$ and $(X, \bar{\circ}_r) \sqsubseteq (X, \bar{\circ}_s)$ while $(X, \circ_r) \neq (X, \circ_s)$ and $(X, \bar{\circ}_r) \neq (X, \bar{\circ}_s)$

Example 2.27. Consider $\mathbb{N} = \{1, 2, 3, \ldots\}$ with the usual metric. Then for every $0 < r < s < 1$ we have $(X, \bar{\circ}_r) = (X, \bar{\circ}_s) = (X, \circ_r) = (X, \circ_s)$.

Definition 2.28. Let X and Γ be two non-empty sets. Then X is called a Γ-H_v-group if for each $\gamma \in \Gamma$ is a hyperoperation on X such that (X, γ) is an H_v-group and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in X$ we have

$$x\alpha(y\beta z) \cap (x\alpha y)\beta z \neq \emptyset.$$

In fact, Γ has a mutually weak associative law.

Theorem 2.29. Let (X, d) be a metric space.

(1) Let $\Gamma \subseteq \{\circ_r | r \in (0, \infty]\}$. Then (X, Γ) is a Γ-H_v-group.

(2) Let $\Gamma \subseteq \{\bar{\circ}_r | r \in [0, \infty]\}$. Then (X, Γ) is a Γ-H_v-group.

Proof . (1) For every $x, y, z \in X$ and $r, r' > 0$, we have

$$\{x, y, z\} \subseteq x \circ_r (y \circ_{r'} z) \cap (x \circ_r y) \circ_{r'} z.$$

Since $(X, \circ_r)$ is an H_v-group, (X, Γ) is a Γ-H_v-group.

(2) It is proved by the same method. □

Now, we will build a metric space from the structure of a hypergroupoid.

Definition 2.30. Let $H = (X, \circ)$ be a hypergroupoid. Then define $d_H : X \times X \to \mathbb{Z}$ as follows,

(i) $d(x, x) = 0$, for every $x \in X$

(ii) $d(x, y) = r$, where $r = \min\ D$ where

$$D = \{n \mid \exists y_0, \ldots, y_n,\ s.t.\ y_i \in y_{i-1} \circ y_{i-1}\ or\ y_{i-1} \in y_i \circ y_i\ \forall\ 1 \leq i \leq n\ and\ y_0 = x,\ y_n = y\}$$

for every $x, y \in X$, if D is empty, then let $d(x, y) = \infty$.

Theorem 2.31. Let $H = (X, \circ)$ be a hypergroupoid. Then (X, d_H) forms a metric space, when d_H is defined as above.

Proof . The item M_2 is clear by Definition 2.30 part(i). Since $D \subset \mathbb{N} \cup \{0\}$, the item M_1 is satisfied. Let $x, y \in X$. If there exists $y_0, \ldots, y_n$ such that $y_i \in y_{i-1} \circ y_{i-1}$ or $y_{i-1} \in y_i \circ y_i$ for all $1 \leq i \leq n$ where $y_0 = x,\ y_n = y$, then $y'_0, \ldots, y'_n$ is a sequence from y to x such that $y'_i \in y'_{i-1} \circ y'_{i-1}$ or $y'_{i-1} \in y'_i \circ y'_i$ where $y'_i = y_{n-i}$ for all $1 \leq i \leq n$ and so $d_H(x, y) = d_H(y, x)$. For $x, y, z \in X$, if $d_H(x, y) = \infty$ or $d_H(y, z) = \infty$, then clearly $d_H(x, y) + d_H(y, z) \geq d_H(x, z)$. Now, let $y_0 = x, \ldots, y_n = y$ such that $y_i \in y_{i-1} \circ y_{i-1}$ or $y_{i-1} \in y_i \circ y_i$ and $y'_0 = y, \ldots, y'_m = z$ such that $y'_j \in y'_{j-1} \circ y'_{j-1}$ or $y'_{j-1} \in y'_j \circ y'_j$. Then $y_0 = x, \ldots, y_n = y = y'_0, \ldots, y'm = z$ is a desired sequence with length at most $n + m$ from x to z. Thus we have $d_H(x, y) + d_H(y, z) \geq d_H(x, z)$, item M_4 is satisfied and the proof is complete. □

Theorem 2.32. Let $H = (X, \circ)$ be a hypergroupoid satisfying (1), (2) and (3) of Theorem 2.3. Then there is a metric space $\mathcal{X} = (X, d)$ such that $(X, \circ_r)$ is isomorphic to H.

Proof . Let $d = d_H$ and $r = 1$ as defined in Definition 2.30. Then for all $z \in X$

$$E(z) = \bigcup_{z \in x \circ x} \{x, z\} = \bigcup_{x \in z \circ z} \{z, x\} = z \circ z.$$

Then for all $x, y \in X$ $x \circ y = x \circ x \cup y \circ y = E(x) \cup E(y)$, so $(X, \circ_1)$ is isomorphic to H and the result follows. □

Theorem 2.33. Let (X, d) be a metric space. Set $H_0 = (X, \circ_r^0)$, $H_1 = (X, \circ_r^1), \dots, H_k = (X, \circ_r^k), \dots$ the sequence of hypergroupoid obtained by setting for all $x, y \in X$, $x \circ_r^0 y = x \circ_r y$, $x \circ_r^{k+1} x = x \circ_r^k x \circ_r^k x$, $x \circ_r^{k+1} y = x \circ_r^{k+1} x \cup x \circ_r^{k+1} y$. Then for all $k \geq 0$,

(i) The hyperoperation $\circ_r^k$ satisfies (1), (2) and (3) of Theorem 2.3.

(ii) $((x \circ_r^k x \circ_r^k x) \circ_r^k (x \circ_r^k x \circ_r^k x)) \circ_r^k (x \circ_r^k x \circ_r^k x) = x \circ_r^{k+2} x$.

Proof .

(i) We prove it by induction on k. So assume that $\circ_r^k$ satisfies (1), (2) and (3) of Theorem 2.3. We that the same items are satisfied by $\circ_r^{k+1}$.

(1) holds by definition of $\circ_r^{k+1}$.

(2) for all $x \in X$, $x \in x \circ_r^k x \subset x \circ_r^k x \circ_r^k x = x \circ_r^{k+1} x$ by the hypothesis of induction.

(3) assume that $y \in x \circ_r^{k+1} x = x \circ_r^k x \circ_r^k x$, then there is $z \in x \circ_r^k x$ such that $y \in z \circ_r^k x = z \circ_r^k x \cup x \circ_r^k x$, by the hypothesis of induction. If $y \in x \circ_r^k x$, then by the inductive hypothesis

$$x \in y \circ_r^k y \subset y \circ_r^{k+1} y.$$

If $y \in z \circ_r^k z$, then $z \in y \circ_r^k y$. Since $z \in x \circ_r^k x$, we have $x \in z \circ_r^k z$. So by Theorem 2.4(7),

$$x \in (y \circ_r^k y) \circ_r^k (y \circ_r^k y) = y \circ_r^k y \circ_r^k y = y \circ_r^{k+1} y$$

(ii) For any $S \subset X$

$$S \circ_r S = \bigcup_{(y,z) \in S \times S} y \circ_r z = \bigcup_{(y,z) \in S \times S} (y \circ_r y \cup z \circ_r z) = \bigcup_{x \in S \times S} x \circ_r x.$$

So, for any $S \subset X$, by Theorem 2.4(8),

$$\begin{aligned} S \circ_r S \circ_r S \subset & (S \circ_r S) \circ_r (S \circ_r S) \\ &= \bigcup_{y \in S \circ_r S} y \circ_r y \\ &= \bigcup_{x \in S} (\bigcup_{y \in x \circ_r x} y \circ_r y) \\ &= \bigcup_{x \in S} (x \circ_r x) \circ_r (x \circ_r x) \\ &= \bigcup_{x \in S} x \circ_r x \circ_r x \subset S \circ_r S \circ_r S, \end{aligned}$$

thus we have

$$S \circ_r S \circ_r S = \bigcup_{x \in S} x \circ_r x \circ_r x.$$

Now, let

$$S = x \circ_r^k x \circ_r^k x = x \circ_r^{k+1} x.$$

So, by Theorem 2.4(7), we have

$$\begin{aligned} S \circ_r^k S \circ_r^k S &= \bigcup_{z\in S} z \circ_r^k z \circ_r^k z \\ &= \bigcup_{z\in x\circ_r^{k+1}x} z \circ_r^{k+1} z \\ &=(x \circ_r^{k+1} x) \circ_r^{k+1} (x \circ_r^{k+1} x) \\ &=x \circ_r^{k+1} x \circ_r^{k+1} x \\ &=x \circ_r^{k+2} x \end{aligned}$$

and the proof is complete.

□

By the similar way we have:

Theorem 2.34. Let (X, d) be a metric space. Set $\overline{H}_0 = (X, \overline{\circ}_r^0)$, $\overline{H}_1 = (X, \overline{\circ}_r^1), \ldots, \overline{H}_k = (X, \overline{\circ}_r^k), \ldots$ the sequence of hypergroupoid obtained by setting for all $x, y \in X$, $x\overline{\circ}_r^0 y = x\overline{\circ}_r y$, $x\overline{\circ}_r^{k+1}x = x\overline{\circ}_r^k x\overline{\circ}_r^k x$, $x\overline{\circ}_r^{k+1}y = x\overline{\circ}_r^{k+1}x \cup x\overline{\circ}_r^{k+1}y$. Then for all $k \geq 0$,

(i) The hyperoperation $\overline{\circ}_r^k$ satisfies (1), (2) and (3) of Theorem 2.3.

(ii) $((x\overline{\circ}_r^k x\overline{\circ}_r^k x)\overline{\circ}_r^k(x\overline{\circ}_r^k x\overline{\circ}_r^k x))\overline{\circ}_r^k(x\overline{\circ}_r^k x\overline{\circ}_r^k x) = x\overline{\circ}_r^{k+2}x$.

3 Conclusion

For the first time, Mirvakili and et. al. introduced the connection between algebraic hyperstructures and metric spaces [14, 18] by focusing on open neighborhoods of elements in metric spaces and constructing a hyperoperation based on the intersection of two neighborhoods. In this article, the authors examined the necessary and sufficient conditions for the associativity of the constructed hyperoperation. Additionally, they illustrated this connection through various examples.

For future work, constructing non-commutative hypergroupoids based on metric spaces and hyperstructures based on the concept of paths in metric spaces can be considered. Additionally, examining the idea of constructing hyperstructures in metric spaces for topological spaces and uniform topology could also be beneficial.

References

[1] S. Banach, *Théorie des Opérations Linéaires*, Monografje Matematyczne, TOM I, 1932.

[2] C. Berge, *Graphs et Hypergraphs*, Dunod, Paris, 1970.

[3] C. Berge, *Hypergraphs*, Elsevier Science Publishers B.V., 1989.

[4] P. Corsini and V. Leoreanu, *Applications of Hyperstructures Theory*, Advances in Mathematics, Kluwer Academic Publishers, 2003.

[5] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani editor, Second edition, 1993.

[6] P. Corsini, *Hypergraphs and hypergroups*, Algebra Univer. **35** (1996), 548–555.

[7] B. Davvaz and T. Vougiouklis, *A Walk Through Weak Hyperstructures: H_v-Structure*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2019.

[8] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA, 2007.

[9] B. Davvaz, *Polygroup theory and related systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[10] M. Farshi, B. Davvaz, and S. Mirvakili, *Degree hypergroupoids associated with hypergraphs*, Filomat **28** (2014), 119–129.

[11] M. Farshi, B. Davvaz, and S. Mirvakili, *Hypergraphs and hypergroups based on a special relation*, Commun. Algebra **42** (2014), 3395–3406.

[12] I.M. Gelfand, *Normierte ringe*, Rec. Math. (Mat. Sbornik) **51** (1941), 3–24.

[13] F. Marty, *Sur une generalization de la notion de group*, 8th Cong. Math. Scand., 1934, pp. 45–49.

[14] S. Mirvakili and R. Manaviyat, *r-hypergroupoids associated with metric spaces*, 3^{rd} Seminar Harmonic Anal. Appl., University of Yazd, Iran, 2015, pp. 46.

[15] F. Hausdorff, *Punktmengen in allgemeinen räumen, grundzäge der mengenlehre*, Göschens Lehrbücherei/Gruppe I, Reine und Angewandte Mathematik Series, Leipzig: Von Veit, 2011, pp. 211. [In German]

[16] J. Von Neumann, *Einige sätze über messbare abbildungen*, Ann. Math. **33** (1932), no. 3, 574–586.

[17] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1976.

[18] M.A. Zarei and S. Mirvakili, *Hypergroupoids associated with a metric space*, 52^{nd} Ann. Iran. Math. Conf., Shahid Bahonar University of Kerman, Iran, 2021, pp. 673–676.