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Weak solution to a class of nonlinear degenerate weighted elliptic p(u)-Laplacian problem

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Abstract

In this work, our objective is to prove the existence and uniqueness of weak solutions to a class of nonlinear degenerate weighted elliptic $p(u)$-Laplacian problem with Dirichlet-type and L^∞ data. For this, we utilise some results from Sobolev spaces with weighted and variable exponents, as well as theorems such as the Minty-Browder theorem.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ be an open bounded domain with a connected Lipschitz boundary $\partial\Omega$ and $p(x) \in (1, \infty)$ for all $x \in (0, \infty)$. Our goal in this work is to demonstrate the existence and uniqueness of weak solutions to the nonlinear degenerate elliptic problem:

$$\begin{cases} -div(\omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u)) + \omega|u|^{p(u)-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $p(.)$ is a continuous function defined on $\overline{\Omega}$ with $p(x) > 1$ for all $x \in \overline{\Omega}$, ω is a measurable positive and a.e finite function defined in $\mathbb{R}^N$ and the datum f is in L^∞. The nonlinear elliptic equation (1.1) can be written as follows:

$$\begin{cases} Au = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with A is the Leray-Lions operator defined on $W^{1,p(.)}(\Omega, \omega)$ to its dual $W^{-1,p'(.)}(\Omega, \omega)$, several examples to this type of operator are already treated, for example,

$$\begin{cases} -div(\omega|\nabla u|^{p(u)-2}\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

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and,

$$\begin{cases} -div(\omega|\nabla u|^{p(u)-2})\nabla u + \omega\alpha(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In recent years, the study of partial differential equations (PDEs) and variational problems has gained considerable momentum due to its wide-ranging applications in mathematical physics and applied sciences. These applications span diverse fields such as elastic mechanics, electrorheological fluid dynamics, and image processing, highlighting the versatility of these mathematical frameworks.

Moreover, the development of mathematical models for degenerate phenomena has become a focal area, driven by challenges in oceanography, turbulent fluid flows, induction heating, and electrochemical processes. Such phenomena often introduce unique mathematical and computational challenges, necessitating innovative approaches to modelling and analysis. Notable references in this context include [3] and [5], which provide foundational insights into these complex systems. This paper aims to establish the existence and uniqueness of weak solutions by using the properties of Sobolev spaces with weighted and variable exponents. To do this, we use the Minty-Browder theorem, which helps us to prove the existence of the weak solution by showing the subjectivity of the operator A. Notably, we have a wealth of articles dedicated to exploring the existence and uniqueness of the equation (1.3) in [3], as well as the equation (1.4) in [2], but with a constant value of p. To give a value to our basic problem (1.3), it is interesting to note that this is the origin of some native problems, namely the following two basic problems noted in Ruzicka [15] and Bay et al. [3]:

- **Model 1. The magneto-quasi-static approximation.** In induction heating processes, the frequencies involved can vary widely, ranging from 50Hz (low frequency used for even heating) to several hundred MHz (high frequency employed for heat treatment purposes). When applying the magneto-quasi-static approximation, we disregard the displacement currents $\frac{\partial D}{\partial t}$ in the Maxwell-Ampere equation, effectively ignoring propagation phenomena. This assumption is valid when the distances between the source locations and the points where the electromagnetic field is calculated are shorter than the wavelength. In the industrial setups where frequencies are typically below 109 Hz, this condition holds, which can be expressed by the following mathematical equation:

$$div(\frac{1}{\mu}\nabla E) = -\frac{\partial J}{\partial t},$$

  where E is the magnetic field, J is the current density and μ is the permeability of space.

- **Model 2. Fluid flow through porous media**. This model is governed by the following equation,

$$\frac{\partial \theta}{\partial t} - div(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e) = 0,$$

  where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydrostatic potential and e is the unit vector in the vertical direction.

Over the past few years, elliptic equations involving variable exponents takes the attention several researcher, and especially the type when will treated in this paper, for example and for $\theta = 0$, $\omega(x) = 1$ and $f \in L^\infty$ data, Chipot and Oliveira in [5] using the Schauder fixed-point theorem to prove the existence of weak solutions for some p(u)-Laplacian problems. With the same condition and $f \in L^1$ data, C. Zhang and X. Zhang in [19] proved the existence of entropy solutions to problem (1.1). M. Sanchon and J.M. Urbano in [18] generalize the proof of C. Zhang and X. Zhang with θ are real functions defined on $\mathbb{R}$ to $\mathbb{R}^N$ and $b(u) = u$. The case when p(.) be a constant and $f \in L^\infty$ is already treated by Sabri et al. [17], they prove the existence and uniqueness of weak solutions.

2 Preliminaries and notations

In the present section, we give some definitions, notations and results which well be used in this work. Let Ω be a bounded open domain in $\mathbb{R}^N$, we consider the following set

$$C^+(\Omega) = \{p : \Omega \to \mathbb{R}^+ \text{ continuous and such that } 1 < p^- < p^+ < \infty\}, \tag{2.1}$$

where $p^- = \min_{x\in\Omega} p(x)$ and $p^+ = \max_{x\in\Omega} p(x)$. Let ω be a measurable positive and a.e finite function defined in $\mathbb{R}^N$ and satisfied the following integrability conditions.

(H_1) $\omega \in L^1_{Loc}(\Omega)$ and $\omega^{\frac{-1}{p(x)-1}} \in L^1_{Loc}(\Omega)$,

(H_2) $\omega^{-s(x)} \in L^1_{Loc}(\Omega)$, where $s(x) \in (\frac{N}{p(x)}, \infty) \cap (\frac{1}{p(x)-1}, \infty]$,

for $p(.) \in C^+(\Omega)$, we define the weighted Lebesgue with variable exponent $L^{p(.)}(\Omega, \omega)$ by

$$L^{p(.)}(\Omega, \omega) = \left\{ g : \Omega \to \mathbb{R} : \text{g is measurable and} \int_\Omega |g|^{p(x)} \omega(x) dx < \infty \right\}.$$

Endowed with the Luxemburg norm

$$\|g\|_{L^{p(.)}(\Omega,\omega)} = \inf \left\{ \lambda > 0, \int_\Omega |\frac{g(x)}{\lambda}|^{p(x)} \omega(x) dx \leq 1 \right\}.$$

We denote by $L^{p'(.)}(\Omega, \omega)$ the conjugate space of $L^{p(.)}(\Omega, \omega)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

and where

$$\omega^*(x) = \omega^{1-p'(x)} \text{ for all } x \in \Omega.$$

On the space $L^{p(.)}(\Omega, \omega)$, we consider the function $\varrho_{p(.),\omega} : L^{p(.)}(\Omega, \omega) \to \mathbb{R}^+$ defined by

$$\varrho_{p(.),\omega}(u) = \varrho_{L^{p(.)}(\Omega,\omega)}(u) = \int_\Omega |u(x)|^{p(x)} \omega(x) dx.$$

The connection between $\varrho_{p(.),\omega}$ and $\|.\|_{L^{p(.)}(\Omega,\omega)}$ is established by the next result.

Proposition 2.1. Let u be an element of $L^{p(.)}(\Omega, \omega)$ and hypothesis (H_1) be satisfied. Then, the following assertions hold:

i) $\|u\|_{p(.),\omega} < 1$ $(respectively >, = 1) \Leftrightarrow \varrho_{p(.),\omega}(u) < 1$ $(respectively >, = 1)$,

ii) If $\|u\|_{p(.),\omega} < 1$ then $\|u\|^{p^+}_{p(.),\omega} \leq \varrho_{p(.),\omega}(u) \leq \|u\|^{p^-}_{p(.),\omega}$,

iii) If $\|u\|_{p(.),\omega} > 1$ then $\|u\|^{p^-}_{p(.),\omega} \leq \varrho_{p(.),\omega}(u) \leq \|u\|^{p^+}_{p(.),\omega}$,

iv) $\|u\|_{p(.),\omega} \to 0 \Leftrightarrow \varrho_{p(.),\omega}(u) \to 0$ and $\|u\|_{p(.),\omega} \to \infty \Leftrightarrow \varrho_{p(.),\omega}(u) \to \infty$.

Proposition 2.2. Let $u \in L^{p(.)}(\Omega, \omega)$, $v \in L^{p'(.)}(\Omega, \omega)$ functions and assuming that hypothesis (H_1) is satisfied, we have

$$\int_\Omega \omega |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(.),\omega} \|v\|_{p'(.),\omega} \leq 2 \|u\|_{p(.),\omega} \|v\|_{p'(.),\omega}.$$

The weighted Sobolev space with variable exponent is defined by

$$W^{1,p(.)}(\Omega, \omega) = \left\{ g \in L^{p(.)}(\Omega, \omega) \text{ such that } \nabla g \in L^{p(.)}(\Omega, \omega) \right\}$$

with the norm

$$\|u\|_{1,p(.),\omega} = \|u\|_{p(.),\omega} + \|\nabla u\|_{p(.),\omega} \text{ for all } x \in W^{1,p(.)}(\Omega, \omega).$$

In the following of this paper, the space $W_0^{1,p(.)}(\Omega, \omega)$ denote the closure of C_0^∞ in $W^{1,p(.)}(\Omega, \omega)$ with respect the norm $\|.\|_{1,p(.),\omega}$. Let $p(.), s(.)$ are two elements of space where the function $s(.)$ satisfies the hypothesis (H_2), we define the following function

$$p^*(x) = \frac{Np(x)}{N - p(x)} \text{ for } p(x) < N,$$

$$p_s(x) = \frac{p(x)s(x)}{1 + s(x)} < p(x),$$

$$p_s^*(x) = \begin{cases} \frac{p(x)s(x)}{(1+s(x))N-p(x)s(x)} & \text{if } N > p_s(x), \\ +\infty & \text{if } N \leq p_s(x), \end{cases}$$

for all most all $x \in \Omega$.

Proposition 2.3 ([17]). Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ a open set of $\mathbb{R}^N$, $p(.) \in C^+(\overline{\Omega})$ and let hypothesis (H_1) be satisfied, we have

$$L^{p(.)}(\Omega, \omega) \hookrightarrow L^1_{Loc}(\Omega)$$

Proposition 2.4 ([17]). Let hypothesis (H_1) be satisfied and $p(.) \in C^+(\overline{\Omega})$, the space $(W^{1,p(.)}(\Omega, \omega), \|.\|_{1,p(.),\omega})$ is a separable and reflexive Banach space.

Proposition 2.5 ([17]). Assume that hypotheses (H_1) and (H_2) hold and $p(\cdot)$; $s(\cdot) \in C^+(\Omega)$, then we have the continuous embedding

$$W^{1,p(.)}(\Omega, \omega) \hookrightarrow W^{1,p_s(.)}(\Omega, \omega).$$

Moreover, we have the compact embedding

$$W^{1,p(.)}(\Omega, \omega) \hookrightarrow\hookrightarrow W^{1,r(.)}(\Omega, \omega),$$

provided that $r \in C^+(\overline{\Omega}), 1 \leq r < p^*$ for all $x \in \Omega$.

Proposition 2.6 ([6]). Suppose that $\Omega \subset \mathbb{R}^N$ be a bounded set. Let the exponent $p(.) \in C^+(\overline{\Omega})$. Then, for all $u \in W_0^{1,p(.)}(\Omega, \omega)$, the inequality

$$\|u\|_{p(.),\omega} \leq C_0 \|\nabla u\|_{p(.),\omega},$$

is satisfied where the constant C_0 depends on the exponent $p(.), diam(\Omega)$ and the dimension N.

Definition 2.7. Given a constant $k > 0$; we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| < k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases}$$

Lemma 2.8 ([1]). For $\xi, \eta \in \mathbb{R}^N$ and $1 < p(.) < \infty$, we have:

$$\frac{1}{p(.)}|\xi|^{p(.)} - \frac{1}{p(.)}|\eta|^{p(.)} \leq |\xi|^{p(.)-2}\xi.(\eta - \xi).$$

where a dot denote the Euclidian scalar product in $\mathbb{R}^N$.

Lemma 2.9 ([1]). For $a > 0$, $b > 0$ and $1 \leq p(.) < \infty$, we have:

$$(a + b)^{p(.)} \leq 2^{p(.)-1}(a^{p(.)} + b^{p(.)}).$$

Lemma 2.10 ([16]). Let $p(.)$ and $p'(.)$ two reals numbers such that $p > 1$, $p' > 1$, and $\frac{1}{p(.)} + \frac{1}{p'} = 1$. There existed a positive constant m such that

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta|^{p'} \leq m\{(\xi - \eta)(|\xi|^{p-2}\xi - |\xi|^{p-2}\xi)\}^{\frac{\beta}{2}}\{\xi^p + \eta^{p'}\}^{1-\frac{\beta}{2}},$$

for all $\xi, \eta \in \mathbb{R}^N$, $\beta = 2$ if $1 < p \leq 2$, and $\beta = p'$ if $p > 2$.

Definition 2.11 ([8]). Let Y be a reflexive Banach space and let A be an operator from Y to its dual Y'. We say that A is *monotone* if

$$\langle Au - Av, u - v \rangle \geq 0 \;\; \forall u, v \in Y.$$

Theorem 2.12 ([8]). Let Y be a reflexive real Banach space and $A : Y \to Y'$ be a bounded operator, hemi-continuous, coercive and monotone on space Y . Then the equation $Au = v$ has at least one solution $u \in Y$ for each $v \in Y'$.

3 Assumptions and main result

In this section, we introduce the concept of a weak solution to problem (1.1) and state the existence results for such solutions. In addition to the hypotheses $(H_1)$and (H_2) listed earlier, we also assume the following assumptions:

(H_3) θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$ such that $\theta(0)=0$ and for all real numbers x, y we have is a real constant $|\theta(x)-\theta(y)| < \lambda_0|x-y|$ where λ_0 is a real constant such that $0<\lambda_0<\left(\frac{1}{p_+}\frac{1}{2^{p^+-1}}\frac{p_-}{2C_0^{p_2}}\right)^{\frac{1}{p^-}}$ with $p_2=p^+$ if $C_0>1$ else $p_2=p^-$.

(H_4) $f\in L^\infty(\Omega)$.

Definition 3.1. A function $u\in W_0^{1,p(.)}(\Omega,\omega)$ is a weak solution of degenerate elliptic problem (1.1) if and only if

$$\int_\Omega \omega\Phi(\nabla u-\theta(u))\nabla\varphi+\int_\Omega \omega|u|^{p(u)-2}u\varphi=\int_\Omega f\varphi \tag{3.1}$$

for all $\varphi\in W_0^{1,p(.)}(\Omega,\omega)\cap L^\infty(\Omega)$. Where $\Phi(\xi)=|\xi|^{p(\xi-2}\xi,\ \forall\xi\in\mathbb{R}^N$.

The central result of this work is encapsulated in the following theorem:

Theorem 3.2. Let hypotheses (H_1), (H_2), (H_3) and (H_4) be satisfied. Then, the problem (1.1) has a unique weak solution.

Proof . Let the operator $T: W_0^{1,p(.)}(\Omega,\omega)\to(W_0^{1,p(.)}(\Omega,\omega))'$, where $(W_0^{1,p(.)}(\Omega,\omega))'$ is the dual space of $W_0^{1,p(.)}(\Omega,\omega)$.

$$T(u)=A^1(u)+A^2(u)-L,\ and\ A(u)=A^1(u)+A^2(u).$$

For $u,v\in W_0^{1,p(.)}(\Omega,\omega)$

$$\begin{aligned}
\langle A^1u,v\rangle &= \int_\Omega \omega\Phi(\nabla u-\theta(u))\nabla v dx,\\
\langle A^2u,v\rangle &= \int_\Omega \omega|u|^{p(u)-2}uv dx,\\
\langle L,v\rangle &= \int_\Omega fv dx.
\end{aligned}$$

We must use the Theorem (2.12) to prove the existence of the weak solution. for that it is necessary to show that the operator T is bounded, monotone coercive and hemi continuous.

Step 1: The operator T is bounded. We use Holder inequality, lemma (2.9) and hypothesis (H_3), for any $u,\varphi\in W_0^{1,p(.)}(\Omega,\omega)$ we have

$$\begin{aligned}
|\langle Au,\varphi\rangle| &\le \int_\Omega \omega|\nabla u-\theta(u)|^{p(u)-1}|\nabla\varphi|dx+\int_\Omega\omega|u|^{p(u)-1}|\varphi|dx\\
&\le \int_\Omega 2^{p(u)-2}\omega(|\nabla u|^{p(u)-1}+|\theta(u)|^{p(u)-1})|\nabla\varphi|dx+\int_\Omega\omega|u|^{p(u)-1}|\varphi|dx\\
&\le 2^{p^+-2}\int_\Omega \omega(|\nabla u|^{p(u)-1}+|\theta(u)|^{p(u)-1})|\nabla\varphi|dx+\int_\Omega\omega|u|^{p(u)-1}|\varphi|dx\\
&\le 2^{p^+-2}\left(\int_\Omega\omega(|\nabla u|^{p(u)-1}|\nabla\varphi|dx+\int_\Omega\lambda_0^{p^*-1}|u|^{p(u)-1})|\nabla\varphi|dx\right)+2\|u\|_{p(.),\omega}^{p_1-1}\|\varphi\|_{p'(.),\omega}\\
&\le 2^{p^+-2}\left(2\|\nabla u\|_{p(.),\omega}^{p_1-1}\|\nabla\varphi\|_{p'(.),\omega}+2\lambda_0^{p^*-1}\|u\|_{p(.),\omega}^{p_1-1}\|\nabla\varphi\|_{p'(.),\omega}\right)\\
&\le 2^{p^+-2}\left(\|\nabla u\|_{p(.),\omega}^{p_1-1}\|\nabla\varphi\|_{p'(.),\omega}+2\alpha\|\nabla u\|_{p(.),\omega}^{p_1-1}\|\nabla\varphi\|_{p'(.),\omega}\right)+2\|u\|_{p(.),\omega}^{p_1-1}\|\varphi\|_{p'(.),\omega},\\
&\le 2^{p^+-1}\|u\|_{1,p(.),\omega}^{p_1-1}\|\varphi\|_{1,p'(.),\omega}+2^{p^+-1}\alpha\|u\|_{1,p(.),\omega}^{p_1-1}\|\varphi\|_{1,p'(.),\omega}+2\|u\|_{1,p(.),\omega}^{p_1-1}\|\varphi\|_{1,p'(.),\omega}\\
&\le C\|u\|_{1,p(.),\omega}^{p_1-1}\|\varphi\|_{1,p'(.),\omega},
\end{aligned}$$

where

$$C = 2^{p^+-2}(1+\alpha)+2 \qquad \lambda_0^{p^*-1} = \max(\lambda_0^{p^--1}, \lambda_0^{p^+-1}), \qquad \alpha = {C_0}^{p_1}\lambda_0^{p^*-1}, \quad \text{and} \quad p_1 = \begin{cases} p^- & \text{if } \|u\|_{1,p(.),\omega} < 1, \\ p^+ & \text{else.} \end{cases}$$

Then we get immediately the boundedness of L. Hence, T is bounded.

Step 2: The operator T is coercive. For any $u \in W_0^{1,p(.)}(\Omega,\omega)$, remark that by application hypothesis (H_3), and by using Holder's inequality and hypothesis (H_4), there exists a positive constant C_3 such that

$$\int_\Omega f u dx \leq C_3 \|f\|_{p'(.)}\|u\|_{1,p(.),\omega}.$$

Consequently, for u large enough and by lemma 2, we get

$$\begin{aligned} \langle Au, u\rangle &= \int_\Omega \omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla u dx + \int_\Omega \omega |u|^{p(u)}dx \\ &\geq \int_\Omega \omega\frac{1}{p(u)}|\nabla u - \theta(u)|^{p(u)}dx - \int_\Omega \omega \frac{1}{p(u)}|\theta(u)|^{p(u)}dx \int_\Omega \omega|u|^{p(u)}dx \end{aligned}$$

and by lemma 3 we have

$$\tfrac{1}{2^{p_+-1}}|\nabla u|^{p(u)}dx - |\theta(u)|^{p(u)} \leq |\nabla u - \theta(u)|^{p(u)}$$

$$\begin{aligned} \langle Au, u\rangle &\geq \int_\Omega \omega\frac{1}{p(u)}\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}dx - \int_\Omega \omega\frac{2}{p(u)}|\theta(u)|^{p(u)}dx + \int_\Omega \omega|u|^{p(u)} \\ &\geq \frac{1}{p_+}\frac{1}{2^{p_+-1}}\int_\Omega \omega|\nabla u|^{p(u)}dx - \frac{2}{p-}\int_\Omega \omega|\theta(u)|^{p(u)}dx + \int_\Omega \omega|u|^{p(u)}dx. \end{aligned}$$

Now, using H_3 we have $|\theta(u)| \leq \lambda_0|u|$ and $\displaystyle\int_\Omega \omega|u|^{p(u)}dx \geq 0$. So, we obtain

$$\langle Au, u\rangle \geq \frac{1}{p_+}\frac{1}{2^{p_+-1}}\int_\Omega \omega|\nabla u|^{p(u)}dx - \frac{2}{p_-}\int_\Omega \omega\lambda_0^{p(u)}|u|^{p(u)}dx.$$

We denote $\lambda_1 = \min\left(\lambda_0^{p-}, \lambda_0^{p+}\right)$.. Furthermore using (H_3), we obtain

$$\begin{aligned} \langle Au, u\rangle &\geq \frac{1}{p_+}\frac{1}{2^{p_+-1}}\int_\Omega \omega|\nabla u|^{p(u)}dx - \frac{2}{p_-}\lambda_1\int_\Omega \omega|u|^{p(u)}dx \\ &\geq \frac{1}{p_+}\frac{1}{2^{p^+-1}}\int_\Omega \omega|\nabla u|^{p(u)}dx - \frac{2}{p_-}{C_0}^{p_2}\lambda_1\int_\Omega \omega|\nabla u|^{p(u)}dx \\ &\geq \left(\frac{1}{p_+}\frac{1}{2^{p^+-1}} - \frac{2}{p_-}{C_0}^{p_2}\lambda_1\right)\int_\Omega \omega|\nabla u|^{p(u)}dx \\ &\geq \left(\frac{1}{p_+}\frac{1}{2^{p^+-1}} - \frac{2}{p_-}{C_0}^{p_2}\lambda_1\right)\|\nabla u\|^{p_3}_{L^{p(.)}(\Omega,\omega)} \\ &\geq M\|u\|^{p_3}_{1,p(.),\omega}, \end{aligned}$$

where

$$M = \left(\frac{1}{p_+}\frac{1}{2^{p^+-1}} - \frac{2}{p_-}{C_0}^{p_2}\right)\frac{1}{(1+C_0)^{p^+}} \quad \text{and} \quad p_3 = \begin{cases} p^+ & \text{if } \|\nabla u\|_{L^{p(.)}(\Omega,\omega)} < 1, \\ p^- & \text{else.} \end{cases}$$

Then

$$\frac{\langle Au, u\rangle}{\|u\|_{1,p(.),\omega}} \longrightarrow +\infty \qquad \text{and} \qquad \|u\|_{1,p(.),\omega} \longrightarrow +\infty.$$

Then A is coercive finally the operator T is coercive.

Step 3 : The operator T is monotonous. First we have T is coercive so there exists $M_1 > 0$ such that $\langle Tu, u\rangle \geq M_1\|u\|^{p_3}_{1,p(.),\omega}$

$$\begin{aligned}
\langle Tu - Tv, u - v\rangle &= \langle Tu, u\rangle + \langle Tv, v\rangle - \langle Tu, v\rangle - \langle Tv, u\rangle \\
&\geq M_1\left(\|u\|^{p_3}_{1,p(.),\omega} + \|v\|^{p_3}_{1,p(.),\omega}\right) - C'\left(\|u\|^{p_3-1}_{1,p(.),\omega}\|v\|_{1,p(.),\omega} + \|v\|^{p_3-1}_{1,p(.),\omega}\|u\|_{1,p(.),\omega}\right) \\
&\geq M_2[\|u\|^{p_3}_{1,p(.),\omega} + \|v\|^{p_3}_{1,p(.),\omega} - \|v\|_{1,p(.),\omega}\|u\|^{p_3-1}_{1,p(.),\omega} - \|v\|^{p_3-1}_{1,p(.),\omega}\|u\|_{1,p(.),\omega}] \\
&\geq M_2(\|u\|^{p_3-1}_{1,p(.),\omega} - \|v\|^{p_3-1}_{1,p(.),\omega}) \times \left(\|u\|_{1,p(.),\omega} - \|v\|_{1,p(.),\omega}\right)
\end{aligned}$$

with $M_2 = \min(M_1, C')$. Finally, the operator T is monotonous.

Step 4: The operator T is is hemi continues. Let $(u_n)_{n\in\mathbb{N}} \subset W_0^{1,p(.)}(\Omega,\omega)$ and $u \in W_0^{1,p(.)}(\Omega,\omega)$ such that $u_n \to u$ strongly in $W_0^{1,p(.)}(\Omega,\omega)$. Firstly, we will prove that A_1 is continuous on $W_0^{1,p(.)}(\Omega,\omega)$. Indeed, we have for $\psi \in W_0^{1,p(.)}(\Omega,\omega)$

$$|\langle A_1u_n - A_1u, \psi\rangle| = |\int_\Omega \omega|\nabla u_n - \theta(u_n)|^{p(u_n)-2}(\nabla u_n - \theta(u_n))\nabla\psi dx - \int_\Omega \omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla\psi dx|$$

$$\begin{aligned}
\langle A_1u_n - A_1u, \psi\rangle \leq &|\int_\Omega \omega|\nabla u_n - \theta(u_n)|^{p(u_n)-2}(\nabla u_n - \theta(u_n))\nabla\psi dx - \int_\Omega \omega|\nabla u_n - \theta(u_n)|^{p(u)-2}(\nabla u_n - \theta(u_n))\nabla\psi dx| \\
&+ |\int_\Omega \omega|\nabla u_n - \theta(u_n)|^{p(u)-2}(\nabla u_n - \theta(u_n))\nabla\psi dx - \int_\Omega \omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla\psi dx|
\end{aligned}$$

we denote that

$$F_n|\nabla u_n - \theta(u_n)|^{p(u_n)-2}(\nabla u_n - \theta(u_n)), \quad G_n = |\nabla u_n - \theta(u_n)|^{p(u)-2}(\nabla u_n - \theta(u_n)), \quad F = |\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u)).$$

This implies that

$$\begin{aligned}
\langle A_1u_n - A_1u, \psi\rangle &\leq \int_\Omega |F_n - G_n| + |G_n - F|dx \\
&\leq \int_\Omega |F_n||\nabla u_n - \theta(u_n)|^{p(u_n)-p(u)} - 1||\nabla\psi|dx + \int_\Omega |G_n - F||\nabla\psi|dx.
\end{aligned}$$

We have $u_n \to u$ strongly in $W_0^{1,p(.)}(\Omega,\omega)$, then $\theta(u_n)$ and ∇u_n bounded. Moreover, p is continuous this implies that $p(u_n)$ bounded. Finally, we have F_n and $|\nabla u_n - \theta(u_n)|$ bounded. Consequently the exists K_0 and K_1 such that

$$\langle A_1u_n - A_1u, \psi\rangle \leq \int_\Omega |K_0|^{p(u)}|K_1^{p(u_n)-p(u)} - 1||\nabla\psi|dx + \int_\Omega |G_n - F||\nabla\psi|dx$$

and we have α reel number such that

$$|G_n - F| \leq ||\nabla u_n - \theta(u_n)|^{\alpha-2}(\nabla u_n - \theta(u_n)) - |\nabla u - \theta(u)|^{\alpha-2}(\nabla u - \theta(u))|$$

and we denote

$$F_n' = |\nabla u_n - \theta(u_n)|^{\alpha-2}(\nabla u_n - \theta(u_n)), \qquad \text{and} \qquad F' = |\nabla u - \theta(u)|^{\alpha-2}(\nabla u - \theta(u)).$$

For that we have

$$\langle A_1u_n - A_1u, \psi\rangle \leq \int_\Omega |K_0|^{p_+}|K_1^{\epsilon_n} - 1||\nabla\psi|dx + \|F_n' - F'\|_{1,p,\omega}\|\psi\|_{1,p',\omega}$$

since $u_n \to u$ strongly in $W_0^{1,p(.)}(\Omega,\omega)$ then

$$F_n' \to F' \quad \text{strongly in } W_0^{-1,p'(.)}(\Omega,\omega)^N$$

and

$$K_1^{\varepsilon_n} - 1 \to 0 \quad \text{strongly in } W_0^{-1,p'(.)}(\Omega, \omega)^N$$

with $\varepsilon_n = p(u_n) - p(u)$ and by the Dominated Convergence Theorem, we have

$$A_1 u_n \to A_1 u \quad \text{strongly in } (W_0^{1,p(.)}(\Omega, \omega))'$$

This implies that A_1 is continuous on $W_0^{1,p(.)}(\Omega, \omega)$. and we get immediately the continuity of A_2 and L, Therefore T is hemi-continuous on $W_0^{1,p(.)}(\Omega, \omega)$. Finally by Theorem 1, there exists a weak solution to problem (1).

Step 5: Uniqueness.

Lemma 3.3. Let hypotheses (H_1), (H_2), (H_3) and (H_4) be satisfied, if u is an weak solution of (1). then
(1) $\lim\limits_{h\to+\infty} \lim\limits_{k\to 0} \frac{1}{k} \int_{\{h<|u|<h+k\}} \omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla u dx = 0.$
(2) $\lim\limits_{h\to+\infty} \lim\limits_{k\to 0} \frac{1}{k} \int_{\{h<|u|<h+k\}} \omega|\nabla u|^{p(u)} dx = 0.$
(3) $\lim\limits_{h\to+\infty} \lim\limits_{k\to 0} \frac{1}{k} \int_{\{h<|u|<h+k\}} \omega|\nabla u - \theta(u)|^{p(u)} dx = 0.$

First we prove that there exists a positive constant β such that for all $k > 0$ we have

$$meas\{|u| > k\} \leq \frac{M}{\beta k^{p^- - 1}}$$

where $M = \|f\|_{L^\infty(\Omega)}$. Choosing $\varphi = T_k(u)$ in equality (3.1), we obtain

$$\int_\Omega \omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla T_k(u) + \int_\Omega \omega|u|^{p(u)-2} u T_k(u) = \int_\Omega f T_k(u), \tag{3.2}$$

since $\int_\Omega \omega|u|^{p(u)-2} u T_k(u) dx \geq 0$.

Remark 3.4. If $|u| < k$. Then we have:

$$\int_\Omega \omega|u|^{p(u)-2} u T_k(u) = \int_\Omega \omega|u|^{p(u)} \geq 0,$$

else $uT_k(u) \geq 0$.

This implies that

$$\int_\Omega \omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla T_k(u) dx = \int_{|u|>k} \omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla u dx$$

$$\leq k\|f\|_{L^\infty(\Omega)}$$

It may be obtained similarly as coercivity of operator A^1 , there exists a constant $\beta > 0$ such that

$$\int_\Omega \omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla T_k(u) \geq \beta \int_{|u|>k} \omega|\nabla u|^{p(u)} dx$$

Therefore

$$\int_{|u|>k} \omega|\nabla u|^{p(u)} dx \leq \frac{k}{\beta}\|f\|_{L^\infty(\Omega)}.$$

Then

$$\int_\Omega \omega|\nabla T_k(u)|^{p(u)} dx \leq \frac{k}{\beta}\|f\|_{L^\infty(\Omega)}.$$

This implies that for all $k > 0$,

$$\frac{1}{k}\int_{\Omega}\omega|\nabla T_k(u)|^{p(u)}dx \leq \frac{M}{\beta}$$

by the Markov inequality, we have

$$meas\{|u| > k\} = meas\{|T_k(u)| > k\} \leq \frac{1}{k^{p^-}}\int_{\Omega}\omega|T_k(u)|^{p(u)}dx \leq \frac{M}{\beta k^{p^- -1}} \quad (3.3)$$

Let k and h be two real numbers such that $0 < k < h$. Taking $\psi = T_k(u - T_h(u))$ in equality (3.1), we get

$$\int_{\Omega}\omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla T_k(u - T_h(u))dx + \int_{\Omega}\omega|u|^{p(u)-2}uT_k(u - T_h(u))dx = \int_{\Omega}fT_k(u - T_h(u))dx. \quad (3.4)$$

Firstly, we have

$$\int_{\Omega}\omega|u|^{p(u)-2}uT_k(u - T_h(u))dx = \int_{|u|>h}\omega|u|^{p(u)-2}uT_k(u - hsign(u))dx$$

then

$$\int_{\Omega}\omega|u|^{p(u)-2}uT_k(u - T_h(u))dx \geq 0.$$

Therefore, equality (6) becomes

$$\int_{h<|u|<h+k}\omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla T_k(u - T_h(u)))dx \leq \int_{|u|>h}fdx.$$

Now, using (3.3), we deduce that $meas\{|u| > h\}$ tends to zero as h goes to infinity. So

$$\lim_{h\to+\infty}\int_{|u|>h}|f|dx = 0.$$

This implies that

$$\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<h+k\}}\omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla udx = 0.$$

Now, using the coercivity of operator A, we have:

$$\frac{1}{p_+}\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)} - \frac{2}{p_-}\lambda_0^{p(u)}|u|^{p(u)} \leq |\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla u,$$

then

$$\frac{1}{p_+}\frac{1}{2^{p_+-1}}\int_{\Omega_h^k}\omega|\nabla u|^{p(u)}dx - \frac{2\lambda_1}{p_-}\int_{\Omega_h^k}\omega|u|^{p(u)}dx \leq \int_{\Omega_h^k}\omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla udx,$$

where $\Omega_h^k = \{h < |u| < h + k\}$, and by using Proposition (2.5) , we obtain

$$\frac{1}{p_+}\frac{1}{2^{p_+-1}}\int_{\Omega_h^k}\omega|\nabla u|^{p(u)}dx - \frac{2\lambda_1 C_0^{p_3}}{p_-}\int_{\Omega_h^k}\omega|\nabla u|^{p(u)}dx \leq \int_{\Omega_h^k}\omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla udx$$

$$p_3 = \begin{cases} p_+ & \text{if } C_0 < 1, \\ p_- & \text{else.} \end{cases}$$

Then

$$\int_{\Omega_h^k}\omega|\nabla u|^{p(u)}dx \leq K\int_{\Omega_h^k}\omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla udx$$

where K is a constant positive. Finally we have

$$\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<h+k\}}\omega|\nabla u|^{p(u)}dx=0$$

(3) We have by Lemma (2.8)

$$\frac{1}{p(u)}|\nabla u-\theta(u)|^{p(u)}-\frac{1}{p(u)}|\theta(u)|^{p(u)}\leq|\nabla u-\theta(u)|^{p(u)-2}(\nabla u-\theta(u))\nabla u$$

then

$$\begin{aligned}\frac{1}{p^-}\int_{\Omega_h^k}|\nabla u-\theta(u)|^{p(u)}dx &\leq \int_{\Omega_h^k}|\nabla u-\theta(u)|^{p(u)-2}(\nabla u-\theta(u))\nabla u dx+\frac{1}{p^+}\int_{\Omega_h^k}|\theta(u)|^{p(u)}dx\\ &\leq \int_{\Omega_h^k}|\nabla u-\theta(u)|^{p(u)-2}(\nabla u-\theta(u))\nabla u dx+\frac{1}{p^+}\int_{\Omega_h^k}\lambda_1|u|^{p(u)}dx\\ &\leq \int_{\Omega_h^k}|\nabla u-\theta(u)|^{p(u)-2}(\nabla u-\theta(u))\nabla u dx+\frac{\lambda_1 C_0^{p_4}}{p^+}\int_{\Omega_h^k}|u|^{p(u)}dx\end{aligned}$$

$p_3=p_+$ if $C_0>1$ else $p_3=p_-$, We apply the previous results (1) and (2) and we get that

$$\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<h+k\}}\omega|\nabla u-\theta(u)|^{p(u)}dx=0$$

Let u and v be two weak solutions of degenerate elliptic problem (1.1) and let h, k be two positive real numbers such that $1<k<h$. For the solution u, we take $\varphi=T_k(u-T_h(v))$ in equality (3.1), and for the solution v, we take $\varphi=T_k(v-T_h(u))$ as test function. We have

$$\int_\Omega\omega\phi(\nabla u-\theta(u))\nabla T_k(u-T_h(v))dx+\int_\Omega\omega|u|^{p(u)-2}uT_k(u-T_h(v))dx=\int_\Omega fT_k(u-T_h(v))dx$$

and

$$\int_\Omega\omega\Phi(\nabla v-\theta(v))\nabla T_k(v-T_h(u))dx+\int_\Omega\omega|v|^{p(v)-2}vT_k(v-T_h(u))dx=\int_\Omega fT_k(v-T_h(u))dx.$$

By letting h go to infinity and k to 0, we find $T_k(u-T_h(v))=u-v$ and $T_k(v-T_h(u))=v-u$ and by summing up the two above inequalities and applying Dominated Convergence, we have

$$||g(u)-g(v)||_1+\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\mathrm{I}(h;k)=0, \tag{3.5}$$

where $g(u)=|u|^{p(u)-2}u$ and

$$\mathrm{I}(h;k)=\int_\Omega\omega\Phi(\nabla u-\theta(u))\nabla T_k(u-T_h(v))dx+\int_\Omega\omega\Phi(\nabla v-\theta(v))\nabla T_k(v-T_h(u))dx.$$

to prove $u=v$ first should be prove that $\lim\limits_{h\to+\infty}\lim\limits_{k\to 0}\frac{1}{k}\mathrm{I}(h;k)\geq 0$. Consider the following decomposition:

$$\begin{aligned}\Omega_1(h)=\{|u|\leq h;|v|\leq h\};\quad \Omega_2(h)=\{|u|\leq h;|v|>h\}\\ \Omega_3(h)=\{|u|>h;|v|\leq h\};\quad \Omega_4(h)=\{|u|>h;|v|>h\}\end{aligned}$$

we have $\Omega=\cup_{i=1}^4\Omega_i(h)$ and for $i=1,..,4$,

$$\mathcal{I}_i(h;k)=\int_{\Omega_i(h)}\omega\Phi(\nabla u-\theta(u))\nabla T_k(u-v)dx+\int_\Omega\omega\Phi(\nabla v-\theta(v))\nabla T_k(v-u)dx.$$

Firstly, we have

$$\begin{aligned}\mathcal{I}_1(h;k) &= \int_{\Omega_1^k(h)}\omega(\phi(\nabla u-\theta(u))-\phi(\nabla v-\theta(v)))\nabla T_k(v-u)dx\\ &= \mathcal{I}_1^1(h;k)+\mathcal{I}_1^2(h;k)\end{aligned}$$

where

$$\Omega_1^k(h) = \{|u-v| \leq k;\ |u| \leq h;\ |v| \leq h\},$$

$$\mathcal{I}_1^1(h;k) = \int_{\Omega_1^k(h)} \omega(\Phi(\nabla u - \theta(u)) - \Phi(\nabla v - \theta(v)))\Psi^1(u;v)dx,$$

$$\mathcal{I}_1^2(h;k) = \int_{\Omega_1^k(h)} \omega(\Phi(\nabla u - \theta(u)) - \Phi(\nabla v - \theta(v)))\Psi^2(u;v)dx,$$

and

$$\Psi^1(u;v) = (\nabla u - \theta(u)) - (\nabla v - \theta(v)) \text{ and } \Psi^2(u;v) = (\theta(u) - \theta(v)).$$

Our goal now is to prove that $\lim\limits_{h\to+\infty}\lim\limits_{k\to 0}\frac{1}{k}\mathcal{I}_1(h;k) \geq 0$.

$$\mathcal{I}_1(h;k) = \int_{\Omega_1^k(h)} \omega(\Phi(\nabla u - \theta(u)) - \Phi(\nabla v - \theta(v)))((\nabla u - \theta(u)) - (\nabla v - \theta(v)))dx \geq 0$$

Case 1: $1 < p(.) \leq 2$. Let $\epsilon > 0$. We apply Young's inequality, we find

$$\begin{aligned}\mathcal{I}_1^2(h,k) &\leq \int_{\Omega_k^{1,h}} \frac{\varepsilon}{p(u)}\omega|\Phi(\nabla u - \theta(u)) - \Phi(\nabla v - \theta(v))|^{p(u)}dx + \int_{\Omega_k^{1,h}} \frac{1}{\varepsilon p'(u)}\omega|\theta(u) - \theta(v)|^{p'(u)}dx \\ &\leq \frac{\varepsilon}{p^-}\int_{\Omega_k^{1,h}} \omega|\Phi(\nabla u - \theta(u)) - \Phi(\nabla v - \theta(v))|^{p_0}dx + \frac{1}{\varepsilon p'^-}\int_{\Omega_k^{1,h}} \omega|\theta(u) - \theta(v)|^{p_0}dx.\end{aligned}$$

Now, using H_3, we have

$$|\theta(u) - \theta(v)|^{p_0} \leq \lambda_0^{p_0}|u-v|^{p_0} \qquad \text{and} \qquad |u-v| \leq k.$$

Then

$$\begin{aligned}\int_{\Omega_k^{1,h}} \omega|\theta(u) - \theta(v)|^{p_0}dx &\leq \lambda_0^{p_0}\int_{\Omega_k^{1,h}} \omega|u-v|^{p_0}dx \\ &\leq meas(\Omega_k^{1,h})k^{p_0} \\ &\leq C_5k^{p_0}.\end{aligned}$$

Now, we use Lemma (2.9), we find

$$\begin{aligned}\int_{\Omega_k^{1,h}} \omega|\Phi(\nabla u - \theta(u)) - \Phi(\nabla v - \theta(v))|^{p_0}dx &\leq m\int_{\Omega_k^{1,h}} \omega|\Phi(\nabla u - \theta(u)) - \Phi(\nabla v - \theta(v))|\Psi^1(u;v)dx \\ &\leq C_4\mathcal{I}_1^1(h,k).\end{aligned}$$

Then

$$\mathcal{I}_1^2(h,k) \leq (\varepsilon C_4\mathcal{I}_1^1(h,k) + \frac{C_5k^{p_0}}{\varepsilon}) \tag{3.6}$$

$$\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^2(h,k) \leq \lim_{k\to 0}\varepsilon C_4\frac{1}{k}\mathcal{I}_1^1(h,k). \tag{3.7}$$

If $\lim\limits_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(h,k) = 0$. Then

$$\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1(h,k) = 0.$$

If $0 < \lim\limits_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(h,k) < \infty$, we pose in (3.6), $\varepsilon = \frac{1}{h\lim\limits_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(h,k)}$. Then

$$\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1(h,k) \geq 0.$$

If $\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(h,k)=+\infty$, using (H_3), (3.3),used to prove the coercivity of A^1 for solution u and v, we find that

$$\begin{aligned}
\mathcal{I}_1^2(h,k) &\leq k\lambda_0\int_{\Omega_k^{1,h}}\omega|\Phi(\nabla u-\theta(u))-\Phi(\nabla v-\theta(v))|dx,\\
&\leq k\lambda_0\int_{\Omega_k^{1,h}}\omega|\nabla u-\theta(u)|^{p(u)}dx+\int_{\Omega_k^{1,h}}\omega|\nabla v-\theta(v)|^{p(v)}dx.\\
\frac{1}{k}\mathcal{I}_1^2(h,k) &\leq \lambda_0\int_{\Omega_k^{1,h}}\omega|\nabla u-\theta(u)|^{p(u)}dx+\int_{\Omega_k^{1,h}}\omega|\nabla v-\theta(v)|^{p(v)}dx.
\end{aligned}$$

For the solution u we take $\varphi=T_k(u)$ in equality (1.1) and we find:

$$\int_{\{|u|\leq k\}}\omega\phi(\nabla u-\theta(u))\nabla u dx\leq kC_9$$

This implies that

$$\int_{\Omega_k^{1,h}}\omega|\nabla u-\theta(u)|^{p(u)}dx\leq k^{p_1}C_{10}$$

and

$$\int_{\Omega_k^{1,h}}\omega|\nabla v-\theta(v)|^{p(v)}dx\leq k^{p_1}C_{11}$$

Therefore

$$\begin{aligned}
\frac{1}{k}|\mathcal{I}_1^2(h,k)| &\leq \lambda_0(h+k)^{p_1}C_{12}.\\
\lim_{k\to 0}\frac{1}{k}|\mathcal{I}_1^2(h,k)| &\leq \lambda_0 h^{p_1}C_{12}.\\
\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(h,k)+\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^2(h,k) &= \infty.
\end{aligned}$$

Then

$$\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(h,k)=\infty.$$

This implies that

$$\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1(h,k)\geq 0.$$

Case 2: $p(.)>2$. By similar method in case 1 and by use Young's inequality we get

$$\begin{aligned}
\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^2(h,k) &\leq \lim_{k\to 0}\left(C_7\frac{\varepsilon(k+h)}{kp'^-}+C_8\frac{k^{p^--1}}{\varepsilon p^-}\right),\\
&\leq \lim_{k\to 0}C_{11}\frac{(k+h)}{h^2p'^-}.
\end{aligned}\tag{3.8}$$

taking $\varepsilon=\frac{k}{h^2}$, we get

$$\lim_{h\to+\infty}\lim_{k\to 0}\tfrac{1}{k}\mathcal{I}_1^2(h,k)=0$$

This implies that

$$\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1(h,k)\geq 0.$$

On the other hand, we have

$$\begin{aligned}
\mathcal{I}_2(h;k) &= \int_{\Omega_2^k(h)}\omega\phi(\nabla u-\theta(u))\nabla T_k(v-u)dx+\int_{\Omega_2^k(h)}\omega\phi(\nabla u-\theta(u))\nabla T_k(u-hsign(v))dx\\
&= \mathcal{I}_2^1(h;k)+\mathcal{I}_2^2(h;k)
\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_2^1(h;k) &= \int_{\Omega_2^k(h)} \omega\phi(\nabla u-\theta(u))\nabla T_k(v-u)dx \\ &= \int_{\Omega_{2,1}^k(h)} \omega\phi(\nabla v-\theta(v))\nabla v dx - \int_{\Omega_{2,1}^k(h)} \omega\phi(\nabla v-\theta(v))\nabla u dx\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_2^2(h;k) &= \int_{\Omega_2^k(h)} \omega\phi(\nabla u-\theta(u))\nabla T_k(u-hsign(v))dx \\ &= \int_{\Omega_{2,2}^k(h)} \omega\phi(\nabla u-\theta(u))\nabla u dx\end{aligned}$$

where

$$\Omega_{2,1}^k(h)=\{|u-v|\le k;\ |u|\le h;\ |v|>h\}$$
$$\Omega_{2,2}^k(h)=\{|u-hsign(v)|\le k;\ |u|\le h;\ |v|>h\}$$

we use the result of the coercivity part and we get

$$\mathcal{I}_2^2(h;k)\ge 0$$

and moreover we have

$$\textstyle\int_{\Omega_{2,1}^k(h)} \omega\phi(\nabla v-\theta(v))\nabla v dx \ge 0.$$

On the other hand, by Holder's inequality, we have

$$\int_{\Omega_{2,1}^k(h)} \omega\phi(\nabla v-\theta(v))\nabla u dx \le C\left(\int_{\Omega_{2,1}^k(h)} \omega|\nabla v-\theta(v)|^{p(u)}dx\right)^{\frac{p^+-1}{p^-}}\left(\int_{\Omega_{2,1}^k(h)} \omega|\nabla u|^{p(u)}dx\right)^{1/p^-}$$

Now, using Lemma (2.4), we get

$$\lim_{h\to+\infty}\lim_{k\to 0}\tfrac{1}{k}\textstyle\int_{\Omega_{2,1}^k(h)} \omega\phi(\nabla v-\theta(v))\nabla u dx = 0.$$

Finally, we have $\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_2^1(h;k)\ge 0$. Then

$$\lim_{h\to+\infty}\lim_{k\to 0}\tfrac{1}{k}\mathcal{I}_2(h;k)\ge 0.$$

If we replace u by v we get $\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_3(h;k)\ge 0$. Now, we will prove that

$$\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_4(h;k)\ge 0.$$

Then, we have

$$\begin{aligned}\mathcal{I}_2(h;k) &= \int_{\Omega_4^k(h)} \omega\phi(\nabla u-\theta(u))\nabla T_k(u-hsign(v)dx + \int_{\Omega_4^k(h)} \omega\phi(\nabla v-\theta(v))\nabla T_k(v-hsign(u))dx \\ &= \int_{\Omega_4^k(h)} \omega\phi(\nabla u-\theta(u))\nabla u dx + \int_{\Omega_2^k(h)} \omega\phi(\nabla v-\theta(v))\nabla v dx \ge 0\end{aligned}$$

where

$$\Omega_{4,1}^k(h)=\{|u-hsign(v)|\le k;\ |u|>h;\ |v|>h\}$$
$$\Omega_{4,2}^k(h)=\{|v-hsign(u)|\le k;\ |u|>h;\ |v|>h\}$$

Finally, we have $\lim_{h\to+\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}(h;k)\ge 0$. Therefore, inequality (3) becomes $||g(u)-g(v)||_1=0.$. This implies that $u=v$ in Ω. This completes the proof of Theorem. □

Conclusions

In this work, we study the question of existence and Uniqueness of weak solutions for the elliptic Problem (1.1) in weighted Sobolev space with Dirichlet type boundary condition. Other questions are still being processed, it is the question existence and Uniqueness of weak solutions for the parabolic Problem.

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