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Zero divisor graphs of classes of five radical zero commutative completely primary finite rings

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Abstract

This paper provides a characterization for zero divisor graphs of a completely primary finite ring R satisfying the conditions $(Z(R))^5 = (0); (Z(R))^4 \neq (0)$ where $Z(R)$ is its subset of all zero divisors (including zero). This has been achieved through Anderson and Livingston's zero divisor graphs by precisely determining the graph invariants, including diameter, girth and the binding number, and graph characteristics including completeness, connectedness and partiteness.

Keywords: Completely Primary Finite Ring, Zero Divisor Graphs
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1 Introduction

In this section, we summarize some well-known results on the zero-divisor graphs of a completely primary finite ring. A completely primary finite ring is a ring R with identity $1 \neq 0$ whose subset of all its zero divisors forms a unique maximal ideal. The study of zero-divisor graphs of a finite ring encapsulates properties of the zero divisors of the completely primary finite rings. A plethora of such have been witnessed since Beck in [4] proposed the notion of zero divisor graphs of a commutative ring with identity. According to Beck, any two distinct vertices x and y in the graph of the ring R, $\Gamma(R) = R$ are adjacent if and only if $xy = 0$. Beck's graph was later simplified by Anderson and Livingston [3] by considering nonzero zero divisors as the vertices of the graph $\Gamma(R)$ and the adjacency concept corresponded to that of Beck. Mulay [5] and Redmond [10] discovered zero divisor graphs based on equivalence classes and ideal-based zero divisor graphs, respectively. Oduor [6] characterized the zero divisor graphs of a Galois ring $R_0 = GR(p^{nr}, p^n)$ based on zero divisor graphs defined by Anderson and Livingston [3] and Mulay [5]. The diameter, girth, connectedness and binding number for the $\Gamma(R_0)$ of the Galois ring R_0 were among the properties studied. Since the discovery of completely primary finite rings through idealization of R_0-modules, much work has been done on the zero-divisor graphs for such commutative rings with identity. For more, see [7, 8, 1, 2]. Throughout this paper, R denotes a completely primary finite ring, $Z(R)$ denotes the set of all zero divisors (including zero), $\Gamma(R)$ denotes the graph of the ring R, $|V(\Gamma(R))|$ denotes the number of vertices in the zero divisor graph of the ring R, $b(\Gamma(R))$ denotes the binding number of the $\Gamma(R)$, $gr(\Gamma(R))$ denotes the girth of $\Gamma(R)$ and $diam(\Gamma(R))$ denotes the diameter of $\Gamma(R)$.

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In section 2, we state some results that are important in this work and give a construction of five radical zero commutative completely primary finite rings. In section 3, we investigate the zero divisor graphs of such rings based on the zero divisor graphs determined by Anderson and Livingston in [3]. Section 4 gives a conclusion with a recommendation for further research.

2 Preliminaries

The following results are fundamental in this paper and their proofs can be obtained from the cited references.

Theorem 2.1. ([9, Theorem 2]). Let R be a finite ring with multiplicative identity $1 \neq 0$, whose zero divisors form an additive group J. Then

(i) J is the Jacobson radical of R;
(ii) $|R| = p^{nr}$ and $|J| = p^{(n-1)r}$ for some prime p, and some integers $n,\ r$;
(iii) $J^n = (0)$;
(iv) The characteristic of the ring R is p^k for some integer k with $1 \leq k \leq n$; and
(v) If the characteristic is p^n, then R will be commutative.

Theorem 2.2. ([11, Proposition 2.2]). Let R be a completely primary finite ring of characteristic p^k with radical J such that $R/J \equiv GF(p^r)$. Then there exists an independent generating set $u_1, \ldots, u_h \in J$ of U as an R_0-module such that each $R_0u_i,\quad i = 1, \ldots, h$ is an R_0-submodule of U and $R = R_0 \oplus U = R_0 \oplus R_0u_1 \oplus \cdots \oplus R_0u_h$.

We now give a specific case of the general construction of five radical zero commutative completely primary finite rings where the automorphism is the identity in R_0 and the ring R is commutative. Let $R_0 = GR(p^{kr}, p^k)$ be a Galois ring of order p^{kr} and characteristic p^k where p is a prime integer, $1 \leq k \leq 5$ and $r \in \mathbb{Z}^+$. Suppose U, V, W and Y are R_0/pR_0-spaces considered as R_0-modules generated by e, f, g and h elements respectively, such that the corresponding generating sets are $\{u_1, \ldots, u_e\}$, $\{v_1, \ldots, v_f\}$, $\{w_1, \ldots, w_g\}$ and $\{y_1, \ldots, y_h\}$ so that $R = R_0 \oplus U \oplus V \oplus W \oplus Y$ is an additive abelian group. Then, on the additive group, we define multiplication by the following relations:

(i) If $k = 1$, then

$$u_iu_{i'} = u_{i'}u_i = v_j, \quad u_iv_j = v_ju_i = w_k, \quad u_iw_k = w_ku_i = y_l, \quad u_iy_l = y_lu_i = 0, \quad v_jv_{j'} = v_{j'}v_j = y_l,$$
$$v_jw_k = w_kv_j = 0, \quad v_jy_l = y_lv_j = 0, \quad w_kw_{k'} = w_{k'}w_k = 0, \quad w_ky_l = y_lw_k = 0, \quad y_ly_{l'} = y_{l'}y_l = 0.$$

(ii) If $k = 2$, then

$$u_iu_{i'} = u_{i'}u_i = pr_0 + pu_i + v_j, \quad u_iv_j = v_ju_i = pu_i + w_k, \quad u_iw_k = w_ku_i = pu_i + y_l, \quad u_iy_l = y_lu_i = pu_i,$$
$$v_jv_{j'} = v_{j'}v_j = y_l,\ v_jw_k = w_kv_j = 0,\ v_jy_l = y_lv_j = 0,\ w_kw_{k'} = w_{k'}w_k = 0,\ w_ky_l = y_lw_k = 0,\ y_ly_{l'} = y_{l'}y_l = 0.$$

(iii) If $k = 3$, then

$$u_iu_{i'} = u_{i'}u_i = p^2r_0 + pu_i + v_j, \quad u_iv_j = v_ju_i = p^2r_0 + pu_i + pv_j + w_k, \quad u_iw_k = w_ku_i = p^2r_0 + pu_i + pw_k + y_l,$$
$$u_iy_l = y_lu_i = p^2r_0+pu_i,\ v_jv_{j'} = v_{j'}v_j = p^2r_0+pv_j+y_l,\ v_jw_k = w_kv_j = p^2r_0+pv_j+pw_k,\ v_jy_l = y_lv_j = p^2r_0+pv_j,$$
$$w_kw_{k'} = w_{k'}w_k = p^2r_0 + pw_k, \quad w_ky_l = y_lw_k = p^2r_0 + pw_k, \quad y_ly_{l'} = y_{l'}y_l = p^2r_0.$$

(iv) If $k = 4$, then

$$u_iu_{i'} = u_{i'}u_i = p^2r_0 + pu_i + v_j, \quad u_iv_j = v_ju_i = p^2r_0 + pu_i + pv_j + w_k, \quad u_iw_k = w_ku_i = p^2r_0 + pu_i + pw_k + y_l,$$
$$u_iy_l = y_lu_i = p^2r_0+pu_i,\ v_jv_{j'} = v_{j'}v_j = p^2r_0+pv_j+y_l,\ v_jw_k = w_kv_j = p^2r_0+pv_j+pw_k,\ v_jy_l = y_lv_j = p^2r_0+pv_j,$$
$$w_kw_{k'} = w_{k'}w_k = p^2r_0 + pw_k, \quad w_ky_l = y_lw_k = p^2r_0 + pw_k, \quad y_ly_{l'} = y_{l'}y_l = p^2r_0.$$

(v) If $k = 5$, then

$$u_iu_{i'} = u_{i'}u_i = p^2r_0 + pu_i + v_j, \quad u_iv_j = v_ju_i = p^2r_0 + pu_i + pv_j + w_k, \quad u_iw_k = w_ku_i = p^2r_0 + pu_i + pw_k + y_l,$$
$$u_iy_l = y_lu_i = p^2r_0+pu_i,\ v_jv_{j'} = v_{j'}v_j = p^2r_0+pv_j+y_l,\ v_jw_k = w_kv_j = p^2r_0+pv_j+pw_k,\ v_jy_l = y_lv_j = p^2r_0+pv_j,$$
$$w_kw_{k'} = w_{k'}w_k = p^2r_0 + pw_k, \quad w_ky_l = y_lw_k = p^2r_0 + pw_k, \quad y_ly_{l'} = y_{l'}y_l = p^2r_0.$$

Further $u_iu_{i'}u_{i''}u_{i'''}u_{i^{iv}} = 0$, $u_ir_0 = r_0u_i$, $v_jr_0 = r_0v_j$, $w_kr_0 = r_0w_k$, $y_lr_0 = r_0y_l$, where $r_0 \in R_0$ and $1 \le i,\ i' \le e$, $1 \le j,\ j' \le f$, $1 \le k,\ k' \le g$, $1 \le l,\ l' \le h$.

Lemma 2.3. From the given multiplication in R, we see that if $r_0 + \sum_{i=1}^{e} r_iu_i + \sum_{j=1}^{f} s_jv_j + \sum_{k=1}^{g} t_kw_k + \sum_{l=1}^{h} z_ly_l$ and $r'_0 + \sum_{i=1}^{e} r'_iu_i + \sum_{j=1}^{f} s'_jv_j + \sum_{k=1}^{g} t'_kw_k + \sum_{l=1}^{h} z'_ly_l$ are any two elements of R, then this multiplication turns R into a commutative ring with identity 1.

Proposition 2.4. The ring R constructed in this section is completely primary of characteristic p^n with Jacobson radical $Z(R)$:

1) If $n = 1$, then

$$\begin{aligned}
Z(R) &= \sum_{i=1}^{e} R_0u_i \oplus \sum_{j=1}^{f} R_0v_j \oplus \sum_{k=1}^{g} R_0w_k \oplus \sum_{l=1}^{h} R_0y_l\\
(Z(R))^2 &= \sum_{j=1}^{f} R_0v_j \oplus \sum_{k=1}^{g} R_0w_k \oplus \sum_{l=1}^{h} R_0y_l\\
(Z(R))^3 &= \sum_{k=1}^{g} R_0w_k \oplus \sum_{l=1}^{h} R_0y_l\\
(Z(R))^4 &= \sum_{l=1}^{h} R_0y_l\\
(Z(R))^5 &= (0)
\end{aligned}$$

2) If $n = 2$, then

$$\begin{aligned}
Z(R) &= pR_0 \oplus \sum_{i=1}^{e} R_0u_i \oplus \sum_{j=1}^{f} R_0v_j \oplus \sum_{k=1}^{g} R_0w_k \oplus \sum_{l=1}^{h} R_0y_l\\
(Z(R))^2 &= pR_0 \oplus p\sum_{i=1}^{e} R_0u_i \oplus \sum_{j=1}^{f} R_0v_j \oplus \sum_{k=1}^{g} R_0w_k \oplus \sum_{l=1}^{h} R_0y_l\\
(Z(R))^3 &= p\sum_{i=1}^{e} R_0u_i \oplus p\sum_{j=1}^{f} R_0v_j \oplus \sum_{k=1}^{g} R_0w_k \oplus \sum_{l=1}^{h} R_0y_l\\
(Z(R))^4 &= p\sum_{j=1}^{f} R_0v_j \oplus \sum_{l=1}^{h} R_0y_l\\
(Z(R))^5 &= (0)
\end{aligned}$$

3) If $n = 3$, then

$$\begin{aligned}
Z(R) &= pR_0 \oplus \sum_{i=1}^{e} R_0u_i \oplus \sum_{j=1}^{f} R_0v_j \oplus \sum_{k=1}^{g} R_0w_k \oplus \sum_{l=1}^{h} R_0y_l\\
(Z(R))^2 &= p^2R_0 \oplus p\sum_{i=1}^{e} R_0u_i \oplus \sum_{j=1}^{f} R_0v_j \oplus \sum_{k=1}^{g} R_0w_k \oplus \sum_{l=1}^{h} R_0y_l\\
(Z(R))^3 &= p^2\sum_{i=1}^{e} R_0u_i \oplus p\sum_{j=1}^{f} R_0v_j \oplus \sum_{k=1}^{g} R_0w_k \oplus \sum_{l=1}^{h} R_0y_l\\
(Z(R))^4 &= p^2\sum_{j=1}^{f} R_0v_j \oplus p\sum_{k=1}^{g} R_0w_k \oplus \sum_{l=1}^{h} R_0y_l\\
(Z(R))^5 &= (0)
\end{aligned}$$

4) If $n = 4$, then

$$Z(R) = pR_0 \oplus \sum_{i=1}^{e} R_0 u_i \oplus \sum_{j=1}^{f} R_0 v_j \oplus \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l$$

$$(Z(R))^2 = p^2 R_0 \oplus p \sum_{i=1}^{e} R_0 u_i \oplus \sum_{j=1}^{f} R_0 v_j \oplus \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l$$

$$(Z(R))^3 = p^3 R_0 \oplus p^2 \sum_{i=1}^{e} R_0 u_i \oplus p \sum_{j=1}^{f} R_0 v_j \oplus \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l$$

$$(Z(R))^4 = p^3 \sum_{i=1}^{e} R_0 u_i \oplus p^2 \sum_{j=1}^{f} R_0 v_j \oplus p \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l$$

$$(Z(R))^5 = (0)$$

5) $n = 5$, then

$$Z(R) = pR_0 \oplus \sum_{i=1}^{e} R_0 u_i \oplus \sum_{j=1}^{f} R_0 v_j \oplus \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l$$

$$(Z(R))^2 = p^2 R_0 \oplus p \sum_{i=1}^{e} R_0 u_i \oplus \sum_{j=1}^{f} R_0 v_j \oplus \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l$$

$$(Z(R))^3 = p^3 R_0 \oplus p^2 \sum_{i=1}^{e} R_0 u_i \oplus p \sum_{j=1}^{f} R_0 v_j \oplus \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l$$

$$(Z(R))^4 = p^4 R_0 \oplus p^3 \sum_{i=1}^{e} R_0 u_i \oplus p^2 \sum_{j=1}^{f} R_0 v_j \oplus p \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l$$

$$(Z(R))^5 = (0)$$

3 Main Results

Proposition 3.1. Suppose that R is a ring construct according to the previous section, and of characteristic p^n, $1 \leq n \leq 5$ with $p^{\xi} u_i = 0$, $pv_j = 0$, $pw_k = 0$, $py_l = 0$, $\xi = 1$ for $n = 1$, $1 \leq \xi \leq 2$ for $n = 2$, $1 \leq \xi \leq 3$ for $n = 3$, and $1 \leq \xi \leq 4$ for $n = 4, 5$. Then $\Gamma(R)$ satisfies the following:

(i) $|V(\Gamma(R))| = p^{((n-1)+\xi e+f+g+h)r} - 1$.
(ii) $\Gamma(R)$ is incomplete.
(iii) $\Gamma(R)$ is connected.
(iv) $diam(\Gamma(R)) = 2$.
(v) $gr(\Gamma(R)) = 3$.

Proof .

(i) To show (i), we consider two cases: $n = 1$ & $\xi = 1$ and $2 \leq n \leq 5$. When $n = 1$, and $\xi = 1$, $Z(R) = \sum_{i=1}^{e} R_0 u_i \oplus \sum_{j=1}^{f} R_0 v_j \oplus \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l$. Thus, we have $|Z(R)| = p^{(e+f+g+h)r}$ and since the set $Z(R)^* = Z(R) - \{(0,0,0,0,0)\}$, it follows that $|Z(R)^*| = |V(\Gamma(R))| = p^{(e+f+g+h)r} - 1$.
When $2 \leq n \leq 5$, $Z(R) = pR_0 \oplus \sum_{i=1}^{e} R_0 u_i \oplus \sum_{j=1}^{f} R_0 v_j \oplus \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l$, thus $|Z(R)| = p^{((n-1)+\xi e+f+g+h)r}$ which clearly follows that $|V(\Gamma(R))| = p^{((n-1)+\xi e+f+g+h)r} - 1$.

(ii) Let $\left(\sum_{i=1}^{e} r_i u_i + \sum_{j=1}^{f} s_j v_j + \sum_{k=1}^{g} t_k w_k + \sum_{l=1}^{h} z_l y_l\right)$ and $\left(\sum_{i=1}^{e} r'_i u_i + \sum_{j=1}^{f} s'_j v_j + \sum_{k=1}^{g} t'_k w_k + \sum_{l=1}^{h} z'_l y_l\right)$ be any two vertices in the graph set $V(\Gamma(R))$. Clearly their product is a nonzero element in $V(\Gamma(R))$. Since $(Z(R))^2 \neq (0)$, it is evident that not all the vertices in the graph set $V(\Gamma(R))$ are adjacent. This shows that $\Gamma(R)$ is incomplete.

(iii) The $ann(Z(R)^*) = (Z(R))^4 = Y$ i.e. $ann(Z(R)^*) \neq \emptyset$. At least there is some vertex $\sum_{l=1}^{h} z_l y_l \in ann(Z(R)^*)$ which is adjacent to every other vertex in the graph set $V(\Gamma(R))$. Thus $\Gamma(R)$ is connected.

(iv) Since $(Z(R))^2 \neq (0)$, there exists non-adjacent vertices in the graph set $V(\Gamma(R))$ each of whose product with $ann(Z(R)^*)$ is zero. Thus $diam(\Gamma(R)) = 2$.

(v) Since $(Z(R))^2 \neq (0)$, the $gr(\Gamma(R)) \not< 3$. Consider the two cases where the two vertices in $V(\Gamma(R))$ are adjacent.
Case(i): $Z(R)^*$ is adjacent to $(Z(R)^4)^*$. Since $(Z(R)^4)^* = ann(Z(R)^*)$, this cycle is of length 1.
Case (ii): $(Z(R)^2)^*$ is adjacent to both $(Z(R)^3)^*$ and $ann(Z(R)^*)$. Also $(Z(R)^3)^*$ is adjacent to $ann(Z(R)^*)$. Clearly the girth of the graph, $gr(\Gamma(R)) \not> 3$ since $ann(Z(R)^*) = (Z(R)^4)^* = Y$. Thus $gr(\Gamma(R)) = 3$.

□

Proposition 3.2. Suppose R is a ring constructed in section 2, and of characteristic p^n, $1 \leq n \leq 5$ with $p^\xi u_i = 0$, $pv_j = 0$, $pw_k = 0$, $py_l = 0$, $\xi = 1$ for $n = 1$, $1 \leq \xi \leq 2$ for $n = 2$, $1 \leq \xi \leq 3$ for $n = 3$, and $1 \leq \xi \leq 4$ for $n = 4, 5$. Then $\Gamma(R)$ satisfies the following:

(i) $$b(\Gamma(R)) = \begin{cases} \frac{p^{((n-1)+g+h)r}-1}{p^{((n-1)+g+h)r}\left(p^{(e+f)r}-1\right)}, & \xi = 1 \\ \frac{p^{((n-1)+e+g+h)r}-1}{p^{((n-1)+e+g+h)r}\left(p^{(e+f)r}-1\right)}, & \xi = 2 \\ \frac{p^{((n-1)+2e+g+h)r}-1}{p^{((n-1)+2e+g+h)r}\left(p^{(e+f)r}-1\right)}, & \xi = 3 \\ \frac{p^{((n-1)+3e+g+h)r}-1}{p^{((n-1)+3e+g+h)r}\left(p^{(e+f)r}-1\right)}, & \xi = 4 \end{cases}$$

(ii) $$\Gamma(R) = \begin{cases} p^{((n-1)+g+h)r} - partite, & \xi = 1 \\ p^{((n-1)+e+g+h)r} - partite, & \xi = 2 \\ p^{((n-1)+2e+g+h)r} - partite, & \xi = 3 \\ p^{((n-1)+3e+g+h)r} - partite, & \xi = 4 \end{cases}$$

Proof . Let $\varepsilon_1, \ldots, \varepsilon_r \in R_0$ with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. We consider two cases: When $n = 1$, $\xi = 1$ and $2 \leq n \leq 5$, separately.

When $n = 1$ and $\xi = 1$, we let $H_{\mu,k,l} = \left\{ \sum_{k=1}^{g} \sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h} \sum_{\mu=1}^{r} \varepsilon_\mu y_l \right\}$. Then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\left\{ V_{\sum_{k=1}^{g} \sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h} \sum_{\mu=1}^{r} \varepsilon_\mu y_l} \right\} = H_{\mu,k,l} - \{(0,0,0,0,0)\}$$

and

$$V_1 = Z(R)^* - \bigcup_{\mu,k,l} \left\{ V_{\sum_{k=1}^{g} \sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h} \sum_{\mu=1}^{r} \varepsilon_\mu y_l} \right\}.$$

From the definition of V_1, we obtain the set of neighbours of V_1 as

$$N(V_1) = \bigcup_{\mu,k,l} \left\{ V_{\sum_{k=1}^{g} \sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h} \sum_{\mu=1}^{r} \varepsilon_\mu y_l} \right\}.$$

Clearly, $|N(V_1)| = \left| \bigcup_{\mu,k,l} \left\{ V_{\sum_{k=1}^{g} \sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h} \sum_{\mu=1}^{r} \varepsilon_\mu y_l} \right\} \right| = p^{(g+h)r} - 1$. Therefore

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| \\ &= p^{(e+f+g+h)r} - 1 - \left(p^{(g+h)r} - 1\right) \\ &= p^{(g+h)r}\left(p^{(e+f)r} - 1\right) \end{aligned}$$

Thus, the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{(g+h)r} - 1}{p^{(g+h)r}\left(p^{(e+f)r} - 1\right)}.$$

Since $|N(V_1)| = p^{(g+h)r} - 1$, it follows that $\Gamma(R)$ is $p^{(g+h)r}$ partite. When $2 \leq n \leq 5$, we consider cases A and B where the set $Z(R)^*$ is partitioned for $\xi = 1, 2, 3, 4$ followed by determination of the binding number of $\Gamma(R)$.

Case A : When n is even. For $\xi = 1$, let

$$H_{\mu,k,l} = \left\{ \sum_{\mu=1}^{r} m_\mu \varepsilon_\mu + \sum_{k=1}^{g}\sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r} \varepsilon_\mu y_l, m_\mu \in \{0, \lambda(p)\} : 1 \leq \lambda \leq p^{(n-1)} - 1 \right\},$$

then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\left\{ V_{\sum_{\mu=1}^{r} m_\mu \varepsilon_\mu + \sum_{k=1}^{g}\sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r} \varepsilon_\mu y_l} \right\} = H_{\mu,k,l} - \{(0,0,0,0,0)\}$$

and

$$V_1 = Z(R)^* - \bigcup_{\mu,k,l} \left\{ V_{\sum_{\mu=1}^{r} m_\mu \varepsilon_\mu + \sum_{k=1}^{g}\sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r} \varepsilon_\mu y_l} \right\}.$$

From the definition of V_1, we have

$$N(V_1) = \bigcup_{\mu,k,l} \left\{ V_{\sum_{\mu=1}^{r} m_\mu \varepsilon_\mu + \sum_{k=1}^{g}\sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r} \varepsilon_\mu y_l} \right\}.$$

Clearly, $|N(V_1)| = \left| \bigcup_{\mu,k,l} \left\{ V_{\sum_{\mu=1}^{r} m_\mu \varepsilon_\mu + \sum_{k=1}^{g}\sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r} \varepsilon_\mu y_l} \right\} \right| = p^{((n-1)+g+h)r} - 1$. Therefore,

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| \\ &= p^{((n-1)+e+f+g+h)r} - 1 - \left(p^{((n-1)+g+h)r} - 1\right) \\ &= p^{((n-1)+g+h)r}\left(p^{(e+f)r} - 1\right). \end{aligned}$$

Thus, the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{((n-1)+g+h)r} - 1}{p^{((n-1)+g+h)r}\left(p^{(e+f)r} - 1\right)}.$$

Since $|(N(V_1))| = p^{((n-1)+g+h)r} - 1$, it follows that $\Gamma(R)$ is $p^{((n-1)+g+h)r}$ partite. For $\xi = 2$, let

$$H_{\mu,i,k,l} = \left\{ \sum_{\mu=1}^{r} m_\mu \varepsilon_\mu + p\sum_{i=1}^{e}\sum_{\mu=1}^{r} \varepsilon_\mu u_i + \sum_{k=1}^{g}\sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r} \varepsilon_\mu y_l, m_\mu \in \{0, \lambda(p)\} : 1 \leq \lambda \leq p^{(n-1)} - 1 \right\}$$

then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\left\{ V_{\sum_{\mu=1}^{r} m_\mu \varepsilon_\mu + p\sum_{i=1}^{e}\sum_{\mu=1}^{r} \varepsilon_\mu u_i + \sum_{k=1}^{g}\sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r} \varepsilon_\mu y_l} \right\} = H_{\mu,i,k,l} - \{(0,0,0,0,0)\}$$

and

$$V_1 = Z(R)^* - \bigcup_{\mu,i,k,l} \left\{ V_{\sum\limits_{\mu=1}^{r} m_\mu \varepsilon_\mu + p \sum\limits_{i=1}^{e} \sum\limits_{\mu=1}^{r} \varepsilon_\mu u_i + \sum\limits_{k=1}^{g} \sum\limits_{\mu=1}^{r} \varepsilon_\mu w_k + \sum\limits_{l=1}^{h} \sum\limits_{\mu=1}^{r} \varepsilon_\mu y_l} \right\}.$$

From the definition of V_1, we have

$$N(V_1) = \bigcup_{\mu,i,k,l} \left\{ V_{\sum\limits_{\mu=1}^{r} m_\mu \varepsilon_\mu + p \sum\limits_{i=1}^{e} \sum\limits_{\mu=1}^{r} \varepsilon_\mu u_i + \sum\limits_{k=1}^{g} \sum\limits_{\mu=1}^{r} \varepsilon_\mu w_k + \sum\limits_{l=1}^{h} \sum\limits_{\mu=1}^{r} \varepsilon_\mu y_l} \right\}.$$

Clearly,

$$|N(V_1)| = \left| \bigcup_{\mu,i,k,l} \left\{ V_{\sum\limits_{\mu=1}^{r} m_\mu \varepsilon_\mu + p \sum\limits_{i=1}^{e} \sum\limits_{\mu=1}^{r} \varepsilon_\mu u_i + \sum\limits_{k=1}^{g} \sum\limits_{\mu=1}^{r} \varepsilon_\mu w_k + \sum\limits_{l=1}^{h} \sum\limits_{\mu=1}^{r} \varepsilon_\mu y_l} \right\} \right| = p^{((n-1)+e+g+h)r} - 1.$$

Therefore,

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| \\ &= p^{((n-1)+2e+f+g+h)r} - 1 - \left(p^{((n-1)+e+g+h)r} - 1\right) \\ &= p^{((n-1)+e+g+h)r} \left(p^{(e+f)r} - 1\right). \end{aligned}$$

Thus, the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{((n-1)+e+g+h)r} - 1}{p^{((n-1)+e+g+h)r} \left(p^{(e+f)r} - 1\right)}.$$

Since $|(N(V_1))| = p^{((n-1)+e+g+h)r} - 1$, it follows that $\Gamma(R)$ is $p^{((n-1)+e+g+h)r}$ partite. For $\xi = 3$, $Z(R)^*$ is partitioned as in the case when $\xi = 2$ with

$$|N(V_1)| = \left| \bigcup_{\mu,i,k,l} \left\{ V_{\sum\limits_{\mu=1}^{r} m_\mu \varepsilon_\mu + p \sum\limits_{i=1}^{e} \sum\limits_{\mu=1}^{r} \varepsilon_\mu u_i + \sum\limits_{k=1}^{g} \sum\limits_{\mu=1}^{r} \varepsilon_\mu w_k + \sum\limits_{l=1}^{h} \sum\limits_{\mu=1}^{r} \varepsilon_\mu y_l} \right\} \right| = p^{((n-1)+2e+g+h)r} - 1$$

and

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| \\ &= p^{((n-1)+3e+f+g+h)r} - 1 - \left(p^{((n-1)+2e+g+h)r} - 1\right) \\ &= p^{((n-1)+2e+g+h)r} \left(p^{(e+f)r} - 1\right). \end{aligned}$$

Thus, the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{((n-1)+2e+g+h)r} - 1}{p^{((n-1)+2e+g+h)r} \left(p^{(e+f)r} - 1\right)}.$$

Since $|(N(V_1))| = p^{((n-1)+2e+g+h)r} - 1$, it follows that $\Gamma(R)$ is $p^{((n-1)+2e+g+h)r}$ partite. For $\xi = 4$, $Z(R)^*$ is partitioned as in the case when $\xi = 2$ with

$$|N(V_1)| = \left| \bigcup_{\mu,i,k,l} \left\{ V_{\sum\limits_{\mu=1}^{r} m_\mu \varepsilon_\mu + p \sum\limits_{i=1}^{e} \sum\limits_{\mu=1}^{r} \varepsilon_\mu u_i + \sum\limits_{k=1}^{g} \sum\limits_{\mu=1}^{r} \varepsilon_\mu w_k + \sum\limits_{l=1}^{h} \sum\limits_{\mu=1}^{r} \varepsilon_\mu y_l} \right\} \right| = p^{((n-1)+3e+g+h)r} - 1,$$

and

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| \\ &= p^{((n-1)+4e+f+g+h)r} - 1 - \left(p^{((n-1)+3e+g+h)r} - 1\right) \\ &= p^{((n-1)+3e+g+h)r}\left(p^{(e+f)r} - 1\right). \end{aligned}$$

Thus, the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{((n-1)+3e+g+h)r} - 1}{p^{((n-1)+3e+g+h)r}\left(p^{(e+f)r} - 1\right)}.$$

Since $|(N(V_1))| = p^{((n-1)+3e+g+h)r} - 1$, it follows that $\Gamma(R)$ is $p^{((n-1)+3e+g+h)r}$ partite.

Case B : When n is odd. For $\xi = 1$, let

$$H_{\mu,k,l} = \left\{\sum_{\mu=1}^{r} m_\mu \varepsilon_\mu + \sum_{k=1}^{g}\sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r} \varepsilon_\mu y_l, m_\mu \in \{0, (\lambda-1)(p)\} : 2 \leq \lambda \leq p^{(n-1)}\right\},$$

then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets:

$$\left\{V_{\sum_{\mu=1}^{r} m_\mu\varepsilon_\mu + \sum_{k=1}^{g}\sum_{\mu=1}^{r}\varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r}\varepsilon_\mu y_l}\right\} = H_{\mu,k,l} - \{(0,0,0,0,0)\}$$

and

$$V_1 = Z(R)^* - \bigcup_{\mu,k,l}\left\{V_{\sum_{\mu=1}^{r} m_\mu\varepsilon_\mu + \sum_{k=1}^{g}\sum_{\mu=1}^{r}\varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r}\varepsilon_\mu y_l}\right\}.$$

From the definition of V_1, we have

$$N(V_1) = \bigcup_{\mu,k,l}\left\{V_{\sum_{\mu=1}^{r} m_\mu\varepsilon_\mu + \sum_{k=1}^{g}\sum_{\mu=1}^{r}\varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r}\varepsilon_\mu y_l}\right\}.$$

Clearly,

$$|N(V_1)| = \left|\bigcup_{\mu,k,l}\left\{V_{\sum_{\mu=1}^{r} m_\mu\varepsilon_\mu + \sum_{k=1}^{g}\sum_{\mu=1}^{r}\varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r}\varepsilon_\mu y_l}\right\}\right| = p^{((n-1)+g+h)r} - 1.$$

Therefore,

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| \\ &= p^{((n-1)+e+f+g+h)r} - 1 - \left(p^{((n-1)+g+h)r} - 1\right) \\ &= p^{((n-1)+g+h)r}\left(p^{(e+f)r} - 1\right). \end{aligned}$$

Thus, the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{((n-1)+g+h)r} - 1}{p^{((n-1)+g+h)r}\left(p^{(e+f)r} - 1\right)}.$$

Since $|(N(V_1))| = p^{((n-1)+g+h)r} - 1$, it follows that $\Gamma(R)$ is $p^{((n-1)+g+h)r}$ partite. For $\xi = 2$, let

$$H_{\mu,i,k,l} = \left\{\sum_{\mu=1}^{r} m_\mu \varepsilon_\mu + p\sum_{i=1}^{e}\sum_{\mu=1}^{r} \varepsilon_\mu u_i + \sum_{k=1}^{g}\sum_{\mu=1}^{r} \varepsilon_\mu w_k + \sum_{l=1}^{h}\sum_{\mu=1}^{r} \varepsilon_\mu y_l, m_\mu \in \{0, (\lambda-1)(p)\} : 2 \leq \lambda \leq p^{(n-1)}\right\}$$

then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets:

$$\left\{V_{\sum\limits_{\mu=1}^{r} m_\mu\varepsilon_\mu + p\sum\limits_{i=1}^{e}\sum\limits_{\mu=1}^{r}\varepsilon_\mu u_i + \sum\limits_{k=1}^{g}\sum\limits_{\mu=1}^{r}\varepsilon_\mu w_k + \sum\limits_{l=1}^{h}\sum\limits_{\mu=1}^{r}\varepsilon_\mu y_l}\right\} = H_{\mu,i,k,l} - \{(0,0,0,0,0)\}$$

and

$$V_1 = Z(R)^* - \bigcup_{\mu,i,k,l}\left\{V_{\sum\limits_{\mu=1}^{r} m_\mu\varepsilon_\mu + p\sum\limits_{i=1}^{e}\sum\limits_{\mu=1}^{r}\varepsilon_\mu u_i + \sum\limits_{k=1}^{g}\sum\limits_{\mu=1}^{r}\varepsilon_\mu w_k + \sum\limits_{l=1}^{h}\sum\limits_{\mu=1}^{r}\varepsilon_\mu y_l}\right\}.$$

From the definition of V_1, we have

$$N(V_1) = \bigcup_{\mu,i,k,l}\left\{V_{\sum\limits_{\mu=1}^{r} m_\mu\varepsilon_\mu + p\sum\limits_{i=1}^{e}\sum\limits_{\mu=1}^{r}\varepsilon_\mu u_i + \sum\limits_{k=1}^{g}\sum\limits_{\mu=1}^{r}\varepsilon_\mu w_k + \sum\limits_{l=1}^{h}\sum\limits_{\mu=1}^{r}\varepsilon_\mu y_l}\right\}.$$

Clearly,

$$|N(V_1)| = \left|\bigcup_{\mu,i,k,l}\left\{V_{\sum\limits_{\mu=1}^{r} m_\mu\varepsilon_\mu + p\sum\limits_{i=1}^{e}\sum\limits_{\mu=1}^{r}\varepsilon_\mu u_i + \sum\limits_{k=1}^{g}\sum\limits_{\mu=1}^{r}\varepsilon_\mu w_k + \sum\limits_{l=1}^{h}\sum\limits_{\mu=1}^{r}\varepsilon_\mu y_l}\right\}\right| = p^{((n-1)+e+g+h)r} - 1.$$

Therefore,

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| \\ &= p^{((n-1)+2e+f+g+h)r} - 1 - \left(p^{((n-1)+e+g+h)r} - 1\right) \\ &= p^{((n-1)+e+g+h)r}\left(p^{(e+f)r} - 1\right). \end{aligned}$$

Thus, the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{((n-1)+e+g+h)r} - 1}{p^{((n-1)+e+g+h)r}\left(p^{(e+f)r} - 1\right)}.$$

Since $|(N(V_1))| = p^{((n-1)+e+g+h)r} - 1$, it follows that $\Gamma(R)$ is $p^{((n-1)+e+g+h)r}$ partite. For $\xi = 3$, $Z(R)^*$ is partitioned as in the case when $\xi = 2$ with

$$|N(V_1)| = \left|\bigcup_{\mu,i,k,l}\left\{V_{\sum\limits_{\mu=1}^{r} m_\mu\varepsilon_\mu + p\sum\limits_{i=1}^{e}\sum\limits_{\mu=1}^{r}\varepsilon_\mu u_i + \sum\limits_{k=1}^{g}\sum\limits_{\mu=1}^{r}\varepsilon_\mu w_k + \sum\limits_{l=1}^{h}\sum\limits_{\mu=1}^{r}\varepsilon_\mu y_l}\right\}\right| = p^{((n-1)+2e+g+h)r} - 1,$$

and

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| \\ &= p^{((n-1)+3e+f+g+h)r} - 1 - \left(p^{((n-1)+2e+g+h)r} - 1\right) \\ &= p^{((n-1)+2e+g+h)r}\left(p^{(e+f)r} - 1\right). \end{aligned}$$

Thus, the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{((n-1)+2e+g+h)r} - 1}{p^{((n-1)+2e+g+h)r}\left(p^{(e+f)r} - 1\right)}.$$

Since $|(N(V_1))| = p^{((n-1)+2e+g+h)r} - 1$, it follows that $\Gamma(R)$ is $p^{((n-1)+2e+g+h)r}$ partite. For $\xi = 4$, $Z(R)^*$ is partitioned as in the case when $\xi = 2$ with

$$|N(V_1)| = \left|\bigcup_{\mu,i,k,l}\left\{V_{\sum\limits_{\mu=1}^{r} m_\mu\varepsilon_\mu + p\sum\limits_{i=1}^{e}\sum\limits_{\mu=1}^{r}\varepsilon_\mu u_i + \sum\limits_{k=1}^{g}\sum\limits_{\mu=1}^{r}\varepsilon_\mu w_k + \sum\limits_{l=1}^{h}\sum\limits_{\mu=1}^{r}\varepsilon_\mu y_l}\right\}\right| = p^{((n-1)+3e+g+h)r} - 1,$$

and

$$\begin{aligned}|V_1| &= |Z(R)^*| - |N(V_1)| \\ &= p^{((n-1)+4e+f+g+h)r} - 1 - \left(p^{((n-1)+3e+g+h)r} - 1\right) \\ &= p^{((n-1)+3e+g+h)r}\left(p^{(e+f)r} - 1\right).\end{aligned}$$

Thus, the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{((n-1)+3e+g+h)r} - 1}{p^{((n-1)+3e+g+h)r}\left(p^{(e+f)r} - 1\right)}.$$

Since $|(N(V_1))| = p^{((n-1)+3e+g+h)r} - 1$, it follows that $\Gamma(R)$ is $p^{((n-1)+3e+g+h)r}$ partite. □

4 Conclusion

This work has evidently laid bare the interplay between ring theory and graph theory. The algebraic theory of zero-divisor graphs for certain classes of five-radical zero, completely primary, finite rings has been considered. It has been noted that for such classes of rings, $\Gamma(R)$ is incomplete, connected with a diameter of 2 and a girth of 3. These graphical properties have been realized to be invariant for all the characteristics of the ring R while the binding number, $b(\Gamma(R))$ and $\Gamma(R)$ partite have been noted to be directly proportional to the characteristic of the ring. Since zero-divisor graphs consist of vertices and edges between them, urban planners may find it useful as a model, where vertices represent cities and edges represent roads between them. From it, the shortest route between any two cities through another city and cities that are well bounded with fairly distributed roads between them may be developed. For such classes of rings, the spectral theory of the zero divisor graphs and compressed zero divisor graphs has been left open for further exploration.

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