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On the zeros of polynomials and their generalized derivative

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Abstract

In this paper, we present findings on the placement of zeros of generalized derivative of polynomials, drawing parallels to those observed in the ordinary derivative of polynomials. Mathematicians have broadened the scope of the Gauss-Lucas Theorem, a classic principle that deals with zero location in polynomials and their derivatives. The new work expands it to cover convex linear combinations of incomplete polynomials.

Keywords: Polynomials, Half-plane, Specht Theorem, Gauss-Lucas Theorem, Generalized Derivative, Grace-Heawood Theorem
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1 Introduction and Preliminaries

Polynomials play a significant role in almost all branches of mathematics. Virtually every branch of mathematics, from algebraic number theory and algebraic geometry to Fourier analysis and computer science, has a corpus of theory arising from the study of polynomials. Historically, polynomials have given rise to some of the most important problems. The subject has now grown too large to attempt encyclopedic coverage. One of the most fascinating problems in algebra is to find the location of the zeros of a polynomial. However, as the degree of a polynomial increases, it becomes more and more difficult to find the exact location of its zeros. This makes the identification of regions containing the zeros of a polynomial a significant problem. In 1829, Cauchy [2] provided a very simple expression for the zero-bound in terms of the coefficients of a polynomial.

Let $A(z)$ be a polynomial of degree n with zeros at $z_1, z_2, \ldots, z_n$. The zeros of the derivative $A'(z)$ are also called critical points of $A(z)$. A critical point of $A(z)$ is said to be non-trivial if it is not a zero of $A(z)$. The classical Gauss–Lucas theorem states that all the zeros of $A'(z)$ lie in the convex hull $H(z_1, z_2, \ldots, z_n)$ of $z_1, z_2, \ldots, z_n$. It is also obvious from the proof of Gauss–Lucas theorem that if the zeros of $A(z)$ do not lie on a straight line, then all its non-trivial critical points lie inside $H(z_1, z_2, \ldots, z_n)$.

Definition 1.1 (Generalized Derivative). Given a polynomial $A(z) = c(z-z_1)(z-z_2)\cdots(z-z_n)$ of degree n and an n-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of non-negative real numbers (not all zero), Sz-Nagy (see [6]) introduced a generalized derivative of $A(z)$ defined by:

$$A^{\gamma}(z) = A(z)\sum_{j=1}^{n}\frac{\gamma_j}{z-z_j} = c\sum_{j=1}^{n}\gamma_j A_j(z),$$

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where $A_j(z) = c\prod_{\substack{i=1\\i\neq j}}^{n}(z-z_i)$ for $1 \leq j \leq n$. Note that the ordinary derivative $A'(z)$ of $A(z)$ can be obtained from $A^{\gamma}(z)$ by setting $\gamma_j = 1$ for $j = 1, 2, \ldots, n$, that is,

$$A^{\gamma}(z) = A'(z), \text{ for } \gamma = \underbrace{(1, 1, \ldots, 1)}_{n\text{-times}}.$$

Diaz-Barrero et al. [3] extended the Gauss-Lucas theorem to generalized derivative of polynomials. In fact they proved the following result.

Theorem 1.2. Let $z_1, z_2, \ldots, z_n$ denote n, not necessarily distinct, complex numbers. Then the polynomial

$$A_n^{\gamma}(z) = \sum_{k=1}^{n}\gamma_k g_k(z), \text{ where } g_k(z) = \prod_{\substack{j=1\\j\neq k}}^{n}(z-z_j)$$

has all its roots inside or on the convex hull $H(z_1, z_2, \ldots, z_n)$.

The following result, proved by Specht [7], is an improvement of the Gauss-Lucas theorem.

Theorem 1.3 (Specht). Let $f(z)$ be a polynomial of degree n with zeros $z_1, \ldots, z_n$. Then the convex hull $K^*(f)$ of the $n^2 - n$ points

$$w_{v\mu} = \frac{z_v + (n-1)z_\mu}{n}, \quad \text{where} \quad v, \mu \in \{1, ..., n\},\ v \neq \mu$$

contains all the critical points of $f(z)$.

The following result, proved by Mir et al. [5], is an improvement of a result in [1], and it gives a relationship between the zeros and critical points of a polynomial.

Theorem 1.4. Let $A(z)$ be a polynomial of degree n. If $A'(\omega) = 0$, then for every given real or complex number α, $A(z)$ has at least one zero in each of the regions

$$\left|\omega - \frac{\alpha+z}{2}\right| \leq \left|\frac{\alpha-z}{2}\right| \quad \text{and} \quad \left|\omega - \frac{\alpha+z}{2}\right| \geq \left|\frac{\alpha-z}{2}\right|. \tag{1.1}$$

The following well-known result gives us the region which contains at least one critical point of the polynomial $f(z)$

Theorem 1.5 (Grace-Heawood). Let $f(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree $n \geq 2$. If $z_1, z_2 \in \mathbb{C}$ are any two distinct points at which f takes the same value, then the disc

$$D(z_1, z_2, n) = \left\{z \in \mathbb{C} : \left|z - \frac{z_1+z_2}{2}\right| \leq \left|\frac{z_1-z_2}{2}\right| \cot\frac{\pi}{n}\right\}$$

contains at least one zero of $f'(z)$.

2 Lemmas

In this part, we first prove the following lemma, which will be employed to establish our main results in the next section.

Lemma 2.1. Let $z_1, z_2, \ldots, z_n$ be n, not necessarily distinct points, but z_1 comes only once. Let

$$\gamma_v = \frac{\lambda_1 z_v + (n-\lambda_1)z_1}{n}, \quad v = 2, 3, \ldots, n.$$

where, $0 \leq \lambda_1 \leq 1$. Let M be an open disk or a half-plane containing z_1 but none of the points γ_v, $v = 2, 3, \ldots, n$. Then M cannot contain any zero of $A_n^{\gamma}(z)$.

Proof . Let $A(z) = \prod_{v=1}^{n}(z - z_v)$. We have $\gamma_v \notin M, \ v = 2, 3, \ldots, n$ and $z_1 \in M$. Let ξ be a zero of $A_n^{\gamma}(z)$ but not of $A(z)$. Assume to the contrary that $\xi \in M$. Consider the mobius transformation

$$\phi(z) \ = \ \frac{\lambda_1}{\lambda_1\xi + (n - \lambda_1)z_1 - nz}. \tag{2.1}$$

The denominator of $\phi(z)$ vanishes at $z = \frac{\lambda_1}{n}\xi + (1 - \frac{\lambda_1}{n})z_1 \in M$. Hence, $\phi(M^c)$ is a disk D and $\phi(\gamma_v) \in D$. Thus their convex linear combination given by $\frac{\sum_{v=2}^{n}\lambda_v\phi(\gamma_v)}{\sum_{v=2}^{n}\lambda_v} \in D$, where $\lambda_v \geq 0$ and $\sum_{v=1}^{n} \ \lambda_v = 1$. Hence there exits some $\hat{\gamma} \in M^c$, such that

$$\phi(\hat{\gamma}) \ = \ \frac{\sum_{v=2}^{n}\lambda_v\phi(\gamma_v)}{\sum_{v=2}^{n}\lambda_v}.$$

Using equation (2.1) this gives us

$$\frac{\lambda_1\left(\sum_{v=2}^{n}\lambda_v\right)}{\lambda_1\xi + (x - \lambda_1)z_1 - n\hat{\gamma}} = \sum_{v=2}^{n}\frac{\lambda_1\lambda_v}{\lambda_1\xi + (x - \lambda_1)z_1 - n\gamma_v}$$

$$= \sum_{v=2}^{n}\frac{\lambda_v}{\xi - z_v}.$$

This can be write as

$$\frac{\lambda_1(1 - \lambda_1)}{\lambda_1\xi + (x - \lambda_1)z_1 - n\hat{\gamma}} \ = \ \sum_{v=2}^{n}\frac{\lambda_v}{\xi - z_v}. \tag{2.2}$$

Now, we can write

$$\begin{aligned} 0 \ &= \ \frac{A_n^{\gamma}(\xi)}{A_n(\xi)} \\ &= \ \sum_{v=1}^{n}\frac{\lambda_v}{\xi - z_v} \\ &= \ \frac{\lambda_1}{\xi - z_1} + \frac{\lambda_1(1 - \lambda_1)}{\lambda_1\xi + (x - \lambda_1)z_1} - n\hat{\gamma}. \end{aligned}$$

Simplify the above equation, we obtain $\hat{\gamma} = (1 - \frac{1}{n})z_1 + \frac{\xi}{n}$. But M is a convex set, hence $\hat{\gamma} \in M$. This is a contradiction, to the fact that $\hat{\gamma} \notin M$. This proves the Lemma. □

3 Main Results

In this paper, we first prove the following result which provides a simple proof of Theorem 1.2 besides giving some insight about the geometrical relationship between the points $z_1, z_2, \ldots, z_n$ and the roots of $A_n^{\gamma}(z)$ which is lacking in the original proof as given in [3].

Theorem 3.1. Let $z_1, z_2, \ldots, z_n$ denote n, not necessarily distinct, complex numbers. Then the polynomial

$$A_n^{\gamma}(z) = \sum_{k=1}^{n}\gamma_k g_k(z), \text{ where } g_k(z) = \prod_{\substack{j=1\\ j\neq k}}^{n}(z - z_j)$$

has all its roots inside or on the convex hull $H(z_1, z_2, \ldots, z_n)$.

Remark 3.2. It follows from the proof of Theorem 3.1 that if ξ is a zero of $A_n^{\gamma}(z)$ and not equal to any z_v, then ξ lies inside the convex hull $H(z_1, z_2, \ldots, z_n)$. In other words any line passing through ξ separate the points $z_v, \ v = 1, 2, \ldots, n$.

Our next result is an improvement of [4, Theorem 1.3], which in fact gives a smaller region than prescribed in Theorem 1.3 containing all the zeros of $A_n^\gamma(z)$.

Theorem 3.3. Let $z_1, z_2, \ldots, z_n$ be n distinct complex numbers. Then the polynomial $A_n^\gamma(z) = \sum_{k=1}^n \lambda_k g_k(z)$ has all its zeros in the convex hull of the points

$$\beta_{v\mu} = \frac{\lambda_\mu z_v + (n-\lambda_\mu)z_\mu}{n}, \quad \text{where} \quad v, \mu \in \{1, 2, \ldots, n, \quad v \neq \mu\}. \tag{3.1}$$

We next prove the following result which extends [5, Theorem 1.4] to generalized derivative.

Theorem 3.4. Let $A(z) = \prod_{v=1}^n (z - z_v)$ be a polynomial of degree n and $A_n^\gamma(\xi) = 0$. Then for every given real or complex number α, $A(z)$ has at least one zero in each of the regions given by

$$\left|\xi - \frac{\alpha+z}{2}\right| \leq \left|\frac{\alpha - z}{2}\right| \quad \text{and} \quad \left|\xi - \frac{\alpha+z}{2}\right| \geq \left|\frac{\alpha - z}{2}\right|. \tag{3.2}$$

Finally, we prove the following result, which provides a circular region which contains at least one zero of $f'(z)$ under conditions similar to those of the Grace-Heawood Theorem [7].

Theorem 3.5. Let $f(z)$ be a polynomial such that $f(z_1) = f(z_2)$. Then $f'(z)$ has a zero in the circular region

$$|z - z_1| \geq \frac{|z_1 - z_2|}{2}.$$

4 Proof of Theorems

Proof .[Proof of Theorem 3.1] Let $A(z) = \prod_{v=1}^n (z - z_v)$. We can write

$$A_n^\gamma(z) \ = \ A(z)\frac{A_n^\gamma(z)}{A(z)} \quad \text{for} \quad A(z) \ \neq \ 0$$

That is;

$$A_n^\gamma(z) = A(z)B_n(z) \tag{4.1}$$

where

$$\begin{aligned} B_n(z) &= \frac{A_n^\gamma(z)}{A(z)} \\ &= \frac{1}{A(z)}\sum_{k=1}^n \gamma_k g_k(z) \\ &= \sum_{k=1}^n \frac{\gamma_k}{z - z_k}. \end{aligned} \tag{4.2}$$

The theorem is proved if we show that every half plane H containing all the zeros of $A(z)$ also contains all the zeros of $A_n^\gamma(z)$. Assume that $A_n^\gamma(\xi) = 0$ and $\xi \notin H$, where H is any half plane given by $H = \{z \in \mathbb{C} \ / \ Re(e^{i\alpha}z) \ \leq \ b\}$. Since $A(\xi) \ \neq \ 0$, we have $B_n(\xi) = 0$. Thus, from equation (4.2), we get

$$B_n(\xi) = \sum_{k=1}^n \frac{\gamma_k}{\xi - z_k}.$$

Therefore, we can write

$$\begin{aligned} Re\left(e^{-i\alpha}B_n(\xi)\right) &= Re\left(\overline{e^{-i\alpha}B_n(\xi)}\right) \\ &= \sum_{k=1}^{n}\gamma_k Re\left(\frac{e^{i\alpha}}{\overline{\overline{\xi - z_k}}}\right) \\ &= \sum_{k=1}^{n}\gamma_k Re\left(\frac{e^{i\alpha}(\xi - z_k)}{|\xi - z_k|^2}\right) \\ &= \sum_{k=1}^{n}\gamma_k Re\left(\frac{(e^{i\alpha}\xi - b) - (e^{i\alpha}z_k - b)}{|\xi - z_k|^2}\right). \end{aligned}$$

This gives $Re\left(e^{-i\alpha}B_n(\xi)\right) > 0$. That is $B_n(\xi) \neq 0$, for all $\xi \notin H$. Hence from equation (4.1), it follows that all the zeros of $A_n^{\gamma}(z)$ lie inside H unless all the zeros of $A(z)$ lie on a line. □

Proof .[Proof of Theorem 3.3] We need to show that every closed half plane H containing all the points $\beta_{v\mu}$ must contain all the zeros of $A_n^{\gamma}(z)$. If H contains all the points $z_1, z_2, \ldots, z_n$, then by Theorem 1.2, H also contains all the zeros of $A_n^{\gamma}(z)$. So we may assume some zero, say z_1, is not contained in H. Then z_1 can not coincide with any other z_v, $v = 2, 3, \ldots, n$, because otherwise z_1 coincides with some $\beta_{v\mu}$ and hence cannot lie outside H. For $\mu = 1$, the points

$$\beta_{v1} = \frac{\lambda_1 z_v + (n - \lambda_1)z_1}{n}, \quad v \;=\; 2, 3, \cdots, n$$

are all in H^c and $z_1 \in H^c$. Hence by the Lemma (2.1), H^c contains no zero of $A_n^{\gamma}(z)$. That is, all zeros of $A_n^{\gamma}(z)$ lie in H. This proves the theorem. □

Proof .[Proof of Theorem 3.4] We observe that the circular regions

$$\left|\xi - \frac{\alpha + z}{2}\right| \;\leq\; \left|\frac{\alpha - z}{2}\right| \quad \text{and} \quad \left|\xi - \frac{\alpha + z}{2}\right| \;\geq\; \left|\frac{\alpha - z}{2}\right|$$

are respectively equivalent to the circular regions

$$|z - (2\xi - \alpha)| \;\leq\; |z - \alpha| \quad \text{and} \quad |z - (2\xi - \alpha)| \;\geq\; |z - \alpha|.$$

These two regions denote the right and left half-plane respectively formed by a line passing through ξ. The direction of line depends on the points α. Since ξ is a zero of $A_n^{\gamma}(z)$. Therefore by Remark 3.2 there are zeros of $A(z)$ in both the half-planes. This proves the result. □

Proof .[Proof of Theorem 3.5] Let $f(z)$ be a polynomial of degree $n \geq 1$. We first assume that $f(0) = f(1)$. Let $f'(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}$. Then we can write

$$\begin{aligned} 0 &= f(1) - f(0) \\ &= \int_0^1 f'(z)dz \\ &= \left[a_0 z + \frac{a_1 z^2}{2} + \cdots + \frac{a_{n-1}z^n}{n}\right]_0^1 \\ &= a_0 + \frac{a_1}{2} + \cdots + \frac{a_{n-1}}{n}. \end{aligned} \tag{4.3}$$

Consider the polynomial

$$g(z) \;=\; z^{n-1} - \binom{n-1}{1}\frac{z^{n-2}}{2} + \binom{n-1}{2}\frac{z^{n-3}}{3} - \cdots + (-1)^{n-1}\frac{1}{n}. \tag{4.4}$$

Then from (4.3), it follows that the polynomial $g(z)$ is apolar to $f'(z)$. Hence by Grace's theorem every circular domain containg all the zeros of $g(z)$ also contains at least one zero of $f'(z)$. We first assume that n is odd. It is easy to verify that

$$\binom{m}{k}\frac{1}{k+1} = \frac{1}{1+m}\binom{m+1}{k+1}.$$

We can write $g(z)$ in the form given by

$$\begin{aligned} g(z) &= \frac{1}{n}\left[z^{n-1} - \binom{n}{2} z^{n-2} + \binom{n}{3} z^{n-3} - \cdots + \binom{n}{n}\right] \\ &= -\frac{1}{n}\left[z^{n} - \binom{n}{1} z^{n-1} + \binom{n}{2} z^{n-2} - \cdots \binom{n}{n} - z^{n}\right] \\ &= -\frac{1}{n}\left[(1-z)^{n} - z^{n}\right]. \end{aligned}$$

The zeros of $g(z)$ are given by the points

$$z = \frac{1}{2}, \ \frac{1}{1+\omega}, \ \frac{1}{1+\omega^2}, \ \ldots, \ \frac{1}{1+\ \omega^{n-1}}$$

where $1,\ \omega, \omega^2, \cdots, \omega^{n-1}$ are the n-th roots of unity. Since $|\omega^v| = 1$ *for all* $v = 0, 1, 2, \cdots, n-1$, it is easy to verify that all the zeros of $g(z)$ lie in $|z| \ \geq \ \frac{1}{2}$. The same holds if we assume n is even. Hence, $f'(z)$ has at least one zero in $|z| \geq \frac{1}{2}$. In general, if $f(z_1) = f(z_2)$ for any two distinct complex numbers z_1 and z_2, we consider the polynomial

$$g(z) \ = \ f\left[(z_2 - z_1)z + z_1\right].$$

Then $g(1) = g(0)$. Hence, the derivative

$$g'(z) \ = \ (z_2 - z_1)\, f'\left[(z_2 - z_1)z + z_1\right]$$

has at least one zero in $|z| \geq \frac{1}{2}$. This implies that $f'(z)$ has at least one zero in the region

$$|z - z_1| \geq \frac{|z_1 - z_2|}{2}.$$

This proves the theorem. □

5 Conclusion

In this paper, we present findings on the placement of zeros of generalized derivative of polynomials, drawing parallels to those observed in the ordinary derivative of polynomials. Mathematicians have broadened the Gauss-Lucas Theorem, a classic principle that deals with zero location of polynomials and their derivatives. The new work deals with the location of zeros of incomplete polynomials and generalized derivatives, and also throws light upon the location of zeros of a polynomial in half planes.

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References

[1] A. Aziz, *On the zeros of a polynomials and its derivative,* Bull. Aust. Math. Soc. **31** (1985), no. 4, 245–255.

[2] A.L. Cauchy, *Exercises de mathématique*, Oeuvres **9** (1829), 122. 2000.

[3] J.L. Díaz-Barrero and J.J. Egozcue, *A generalization of the Gauss-Lucas theorem*, Czech. Math. J. **58** (2008), no. 2, 481–486.

[4] M. Marden, *The Geometry of the Zeros of a Polynomial in a Complex Variable*, American Mathematical Society, Rhode Island, 1966.

[5] M.I. Mir, I.A. Wani, and I. Nazir, *On the zeros and critical points of a polynomial*, Math. Anal. Contemp. Appl. **4** (2022), no. 1, 25–28.

[6] J. Sz-Nagy, *Verallgemeinerung der derivierten in der geometric der polynome*, Acta Univ. Szeged. Sect. Sci. Math. **13** (1950), 169–178.

[7] Q.I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials,* Oxford University Press Inc., New York, 2002.