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Maps on Banach $*$-algebras acting at the identity products

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Abstract

Let A be a unital Banach $*$-algebra with unit 1, and X be a Banach $*$-A-bimodule. In this paper, we determining continuous linear maps $\delta : A \longrightarrow X$ that satisfy one of the following conditions:

$$\delta(x \diamond y) = \delta(x) \diamond y,$$

$$\delta(x \diamond y \diamond x) = \delta(x) \diamond y \diamond x,$$

for all $x, y \in A$ with $xy = 1$, where $x \diamond y = x^*y - y^*x$. We also characterize continuous linear maps $\phi : A \longrightarrow B$ which behave like homomorphisms at the identity products.

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1 Introduction and Preliminaries

Let A be an associative algebra over $\mathbb{C}$ and let X be an A-bimodule. A linear map $\delta : A \longrightarrow X$ is called a *left multiplier* (*right multiplier*) if for all $x, y \in A$,

$$\delta(xy) = \delta(x)y, \quad \big(\delta(xy) = x\delta(y)\big),$$

and δ is called a *multiplier* if it is both left and right multiplier. If A is unital, then δ is a left multiplier if and only if δ is of the form $\delta(x) = \delta(1)x$.

A linear map $\delta : A \longrightarrow X$ is called *commuting map* if $[\delta(x), x] = 0$, for every $x \in A$, where $[x, y] = xy - yx$ is the Lie product.

Obviously, each multiplier is a commuting map; however, there exist commuting maps which are not multipliers. Bresar [3] proved that every commuting additive map δ on a prime ring A is of the form

$$\delta(x) = \lambda x + \mu(x), \qquad x \in A,$$

where λ is an element in C, the extended centroid of A, and μ is an additive map from A into C. Recall that a ring A is called *prime* if $aAb = \{0\}$ implies that $a = 0$ or $b = 0$.

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A linear map δ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$, for all $x, y \in A$. Moreover, δ is called a *Jordan derivation* if $\delta(x^2) = \delta(x)x + x\delta(x)$, for every $x \in A$.

We say that a map δ is a *left multiplier* at a given point $w \in A$, if

$$x, y \in A, \quad xy = w \quad \Longrightarrow \quad \delta(xy) = \delta(x)y. \tag{1.1}$$

This type of map has been discussed by several authors. For example, it is shown [16] that if A is a unital C^*-algebra, X is a unital Banach A-bimodule and $\delta : A \longrightarrow X$ is a continuous linear map satisfying (1.1) with $w = 0$, then δ is a left multiplier. The same result was obtained [15] at the identity products. Left multiplier at zero product on a prime ring studied in [4]. For characterization of linear maps, especially, multipliers, derivations and homomorphisms at a given point $w \in A$, see for example [4, 6, 17, 18] and references therein.

Let A be a $*$-algebra. For $x, y \in A$, define $[x, y]_* = xy - yx^*$ and $x \bullet y = xy + yx^*$ for skew Lie product and Jordan $*$-product, respectively. These products are fairly meaningful and important in some research topics, see [1, 8].

A linear map δ from $*$-algebra A into $*$-A-bimodule X is said to be a Jordan $*$-derivation or a skew Lie derivation if δ is self-adjoint, i.e., $\delta(x^*) = \delta(x)^*$ and for all $x, y \in A$,

$$\delta(x \bullet y) = \delta(x) \bullet y + x \bullet \delta(y), \quad \text{or} \quad \delta([x, y]_*) = [\delta(x), y]_* + [x, \delta(y)]_*.$$

Furthermore, δ is called a Jordan triple $*$-derivation or a skew Lie triple derivation if

$$\delta(x \bullet y \bullet z) = \delta(x) \bullet y \bullet z + x \bullet \delta(y) \bullet z + x \bullet y \bullet \delta(z),$$

or

$$\delta(\big[[x, y]_*, z\big]_*) = \big[[\delta(x), y]_*, z\big]_* + \big[[x, \delta(y)]_*, z\big]_* + \big[[x, y]_*, \delta(z)\big]_*, \tag{1.2}$$

for all $x, y, z \in A$. Yu and Zhang [13] proved that every nonlinear skew Lie derivation on factor von Neumann algebras is an additive $*$-derivation. The analogous result was obtained for skew Lie triple derivation [7].

Taghavi et al. [11] proved that every nonlinear Jordan $*$-derivation between factor von Neumann algebras is an additive $*$-derivation. Nonlinear Jordan triple $*$-derivation between von Neumann algebras is discussed in [14].

In [9], the authors introduced the new n-tuple product $x \diamond y = x^*y - y^*x$, and then under certain conditions, characterized maps that preserve the product $a \diamond b$ between factor von Neumann algebras are additive $*$-derivation. The same authors in [10] proved that if A is a unital prime $*$-algebra that possesses a nontrivial projection and $\delta : A \longrightarrow A$ is a nonlinear map which $\delta(\alpha\frac{1}{2})$ is self-adjoint map for all $\alpha \in \{1, i\}$ and satisfies

$$\delta(x \diamond y \diamond z) = \delta(x) \diamond y \diamond z + x \diamond \delta(y) \diamond z + x \diamond y \diamond \delta(z),$$

for all $x, y, z \in A$, then δ is additive $*$-derivation.

Wang and Fei [12] proved that if A is a C^*-subalgebra of B, then every continuous linear map $\delta : A \longrightarrow B$ satisfying (1.2), for all $x, y, z \in A$ with $xy = 1$ and $x = z$ is a $*$-derivation. Moreover, they studied Jordan $*$-homomorphism between two unital C^*-algebras. See also [5], for more results concerning characterization of maps on C^*-algebras.

Motivated by these studies, in this paper, we consider the problem of determining a continuous linear map δ from a Banach $*$-algebra A into a Banach $*$-A-bimodule X satisfying

$$x, y \in A, \quad xy = 1 \quad \Longrightarrow \quad \delta(x \diamond y) = \delta(x) \diamond y,$$

or

$$x, y, z \in A, \quad xy = 1, z = x \quad \Longrightarrow \quad \delta(x \diamond y \diamond z) = \delta(x) \diamond y \diamond z,$$

where $x \diamond y = x^*y - y^*x$ and $x \diamond y \diamond z = (x \diamond y) \diamond z$. We also investigate a continuous linear map $\phi : A \longrightarrow B$ satisfying $\phi(x \diamond y) = \phi(x) \diamond \phi(y)$, for all $x, y \in A$ with $xy = 1$.

Throughout this paper, A is a unital Banach $*$-algebra with unit 1, and X is a unital Banach $*$-A-bimodule. Recall that an element $a \in A$ is self-adjoint if $a = a^*$, and it is unitary if $aa^* = a^*a = 1$. The set of all self-adjoint elements in A will be denoted by A_{sa}.

2 Characterization of commuting maps

In this section, we characterize continuous linear maps on Banach $*$-algebras which behaving like multipliers at the identity products. We commence with the following result.

Theorem 2.1. Let $\delta : A \longrightarrow X$ be a continuous linear map satisfying

$$x, y \in A, \quad xy = 1 \quad \Longrightarrow \quad \delta(x \diamond y) = \delta(x) \diamond y.$$

Then δ is a commuting $*$-map.

Proof . Put any $a \in A_{sa}$, e^{-ita} is a unitary for each $t \in \mathbb{R}$ and $e^{-ita} \diamond e^{ita} = e^{2ita} - e^{-2ita}$. Thus, we have

$$\begin{aligned} \delta(e^{2ita} - e^{-2ita}) &= \delta(e^{-ita} \diamond e^{ita}) \\ &= \delta(e^{-ita}) \diamond e^{ita} \\ &= \delta(e^{-ita})^* e^{ita} - e^{-ita}\delta(e^{-ita}). \end{aligned}$$

By taking derivative of above equation at t, we obtain that

$$\delta(2ae^{2ita} + 2ae^{-2ita}) = \delta(ae^{-ita})^* e^{ita} + \delta(e^{-ita})^* ae^{ita} + ae^{-ita}\delta(e^{-ita}) + e^{-ita}\delta(ae^{-ita}). \tag{2.1}$$

Taking $t = 0$ and $a = 1$ in (2.1), we conclude that $\delta(1) = \delta(1)^*$. Again put $t = 0$ in (2.1) and using $\delta(1) = \delta(1)^*$, we arrive at

$$\delta(a) = \delta(a)^*, \quad a \in A_{sa}.$$

For each $x \in A$, there exist $a, b \in A_{sa}$ such that $x = a + ib$. Hence,

$$\delta(x^*) = \delta(a) - i\delta(b) = \delta(x)^*.$$

Therefore, δ is self-adjoint. Taking derivative of (2.1) in $t = 0$ yields that

$$2\delta(a)a + \delta(1)a^2 = a^2\delta(1) + 2a\delta(a) \quad a \in A_{sa}. \tag{2.2}$$

Replacing a by $a + 1$ in (2.2), we get

$$a\delta(1) = \delta(1)a, \quad a \in A_{sa}. \tag{2.3}$$

From (2.2) and (2.3), we have

$$\delta(a)a = a\delta(a) \quad a \in A_{sa}.$$

Put any $a, b \in A_{sa}$, then

$$\delta(a)b + \delta(b)a = a\delta(b) + b\delta(a).$$

For any $x \in A$, there are $a, b \in A_{sa}$ such that $x = a + ib$. So,

$$\begin{aligned} \delta(x)x &= \delta(a)a - \delta(b)b + i\big(\delta(a)b + \delta(b)a\big) \\ &= a\delta(a) - b\delta(b) + i\big(a\delta(b) + b\delta(a)\big) \\ &= x\delta(x), \end{aligned}$$

for all $x \in A$. Therefore, δ is a commuting $*$-map. $\square$

From Theorem 2.1 and [3, Theorem A] we have the following result.

Corollary 2.2. Let $\delta : A \longrightarrow A$ be a continuous linear map satisfying

$$x, y \in A, \quad xy = 1 \quad \Longrightarrow \quad \delta(x \diamond y) = \delta(x) \diamond y.$$

If A is prime, then

$$\delta(x) = \lambda x + \mu(x), \quad x \in A,$$

where λ is an element in C and μ is a continuous linear map from A into C.

Theorem 2.3. Let $\delta : A \longrightarrow X$ be a continuous linear map satisfying

$$x, y \in A, \quad xy = 1 \quad \Longrightarrow \quad \delta(x \diamond y) = \delta(x) \diamond y - y \diamond \delta(x).$$

Then δ is a commuting $*$-map.

Proof . Let $a \in A_{sa}$, and take $x = e^{-ita}$ and $y = e^{ita}$, for each $t \in \mathbb{R}$. Then we have

$$\begin{aligned} \delta(e^{2ita} - e^{-2ita}) &= \delta(e^{-ita} \diamond e^{ita}) \\ &= \delta(e^{-ita}) \diamond e^{ita} - e^{ita} \diamond \delta(e^{-ita}) \\ &= 2\delta(e^{-ita})^* e^{ita} - 2e^{-ita}\delta(e^{-ita}). \end{aligned} \tag{2.4}$$

It follows from (2.4) with $t = 0$ that $\delta(1) = \delta(1)^*$. By taking derivative of equation (2.4) at t, we obtain

$$\delta(ae^{2ita} + ae^{-2ita}) = \delta(ae^{-ita})^* e^{ita} + \delta(e^{-ita})^* ae^{ita} + ae^{-ita}\delta(e^{-ita}) + e^{-ita}\delta(ae^{-ita}). \tag{2.5}$$

Taking $t = 0$ and $a = 1$ in (2.5), we conclude that $\delta(1) = 0$. Again put $t = 0$ in (2.5) and using $\delta(1) = 0$, we get

$$\delta(a) = \delta(a)^*, \quad a \in A_{sa}.$$

Now one can show that $\delta(x^*) = \delta(x)^*$ for all $x \in A$. By taking derivative of equation (2.5) at t, we obtain

$$\begin{aligned} 2\delta(a^2e^{2ita} - a^2e^{-2ita}) &= \delta(a^2e^{-ita})^* e^{ita} + 2\delta(ae^{-ita})^* ae^{ita} + \delta(e^{-ita})^* a^2 e^{ita} \\ &\quad - a^2e^{-ita}\delta(e^{-ita}) - 2ae^{-ita}\delta(ae^{-ita}) - e^{-ita}\delta(a^2e^{-ita}). \end{aligned} \tag{2.6}$$

Taking $t = 0$ in (2.6), we get

$$\delta(a)a = a\delta(a), \quad a \in A_{sa}.$$

As in the proof of Theorem 2.1, we can see that $\delta(x)x = x\delta(x)$, for all $x \in A$. $\square$

The corollary below follows from Theorem 2.1 and [2, Theorem 2.1].

Corollary 2.4. Let A be a von Neumann algebra and $\delta : A \longrightarrow A$ be a continuous linear map satisfying

$$x, y \in A, \quad xy = 1 \quad \Longrightarrow \quad \delta(x \diamond y) = \delta(x) \diamond y - y \diamond \delta(x).$$

Then

$$\delta(x) = cx + \mu(x), \quad x \in A,$$

where $c \in Z(A)$, the centre of A, and μ is a continuous linear map from A into $Z(A)$.

It should be pointed out that in Corollary 2.4, in fact $c = -\mu(1)$. Indeed,

$$0 = \delta(1) = c + \mu(1),$$

and so $\mu(x) = \delta(x) + \mu(1)x$, for all $x \in A$.

3 Characterization of multipliers

This section devoted to the problem of characterizing continuous linear maps which are necessary $*$-multipliers.

Theorem 3.1. Let $\delta : A \longrightarrow X$ be a continuous linear map satisfying

$$x, y, z \in A, \quad xy = 1, z = x \quad \Longrightarrow \quad \delta(x \diamond y \diamond z) = \delta(x) \diamond y \diamond z.$$

Then

$$2\delta(x) = x\delta(1) + \delta(1)x, \quad x \in A.$$

Moreover, if

$$\delta(1) \in Z(X) = \{x \in X : \ ax = xa \ \text{ for all } a \in A\},$$

then δ is a $*$-multiplier.

Proof . Put any $a \in A_{sa}$, e^{ita} is a unitary for each $t \in \mathbb{R}$ and

$$e^{ita} \diamond e^{-ita} \diamond e^{ita} = e^{3ita} - e^{-ita} - e^{-3ita} + e^{ita}.$$

Therefore,

$$\begin{aligned}\delta(e^{3ita} - e^{-ita} - e^{-3ita} + e^{ita}) &= \delta(e^{ita} \diamond e^{-ita} \diamond e^{ita})\\ &= \delta(e^{ita}) \diamond e^{-ita} \diamond e^{ita}\\ &= e^{ita}\delta(e^{ita})e^{ita} - \delta(e^{ita})^* - e^{-ita}\delta(e^{ita})^* e^{-ita} + \delta(e^{ita}).\end{aligned} \tag{3.1}$$

It follows from (3.1) with $t = 0$ that $\delta(1) = \delta(1)^*$. By taking derivative of equation (3.1) at $t = 0$ and noted that $\delta(1) = \delta(1)^*$, we deduce that

$$3\delta(a) = a\delta(1) + \delta(1)a + \delta(a)^*, \quad a \in A_{sa}. \tag{3.2}$$

Since $\delta(1)$ is self-adjoint, from (3.2) we get

$$\delta(a) = \delta(a)^*, \quad a \in A_{sa}.$$

As in the proof of Theorem 2.1, we can see that $\delta(x^*) = \delta(x)^*$, for all $x \in A$. Now it follows from (3.2) that

$$2\delta(a) = a\delta(1) + \delta(1)a, \quad a \in A_{sa}.$$

One can show that

$$2\delta(x) = x\delta(1) + \delta(1)x, \quad x \in A.$$

If $\delta(1) \in Z(X)$, then $\delta(x) = \delta(1)x = x\delta(1)$, and hence δ is a ∗-multiplier. □

It is clear that A', the dual of A, is a Banach A-bimodule with the following module structures:

$$(f \cdot a)b = f(ab), \quad (a \cdot f)b = f(ba), \quad a, b \in A, f \in A'.$$

Therefore, if A is commutative, then $f \cdot a = a \cdot f$, and so we obtain the next result.

Corollary 3.2. Let $\delta : A \longrightarrow A'$ be a continuous linear map. If A is commutative, then δ is a ∗-multiplier if and only if

$$\delta(x \diamond y \diamond x) = \delta(x) \diamond y \diamond x,$$

for all $x, y \in A$ with $xy = 1$.

Theorem 3.3. Let $\delta : A \longrightarrow X$ be a continuous linear map satisfying

$$\delta(x \diamond y \diamond z) = \delta(x) \diamond y \diamond z - x \diamond \delta(y) \diamond z + x \diamond y \diamond \delta(z),$$

for all $x, y, z \in A$ with $xy = 1, z = x$. Then

$$2\delta(x) = x\delta(1) + \delta(1)x, \quad x \in A.$$

Moreover, if $\delta(1) \in Z(X)$, then δ is a ∗-multiplier.

Proof . Let $a \in A_{sa}$, and take $x = e^{ita}$ and $y = e^{-ita}$. Then for each $t \in \mathbb{R}$, we have

$$\begin{aligned}\delta(e^{3ita} - e^{-3ita} + e^{ita} - e^{-ita}) = {}& e^{ita}\delta(e^{ita})e^{ita} + \delta(e^{ita}) - e^{-ita}\delta(e^{ita})^* e^{-ita} - \delta(e^{ita})^*\\ &- \delta(e^{-ita})^* e^{2ita} + e^{-2ita}\delta(e^{-ita}) - \big(e^{-ita}\delta(e^{-ita})^* + e^{-ita}\delta(e^{-ita})\big)e^{ita}\\ &+ (e^{2ita} - e^{-2ita})\delta(e^{ita}) + \delta(e^{ita})^*(e^{2ita} - e^{-2ita}).\end{aligned} \tag{3.3}$$

Taking $t = 0$ in (3.3), we conclude that $\delta(1) = \delta(1)^*$. By taking derivative of equation (3.3) at $t = 0$ and using $\delta(1) = \delta(1)^*$, we arrive at

$$2\delta(a) = \delta(1)a + a\delta(1).$$

Since $\delta(1)$ is self-adjoint, we get

$$\delta(a) = \delta(a)^*, \quad a \in A_{sa}.$$

The equality above imply that $\delta(x^*) = \delta(x)^*$, for all $x \in A$, and hence δ is self-adjoint. Now one can show that

$$2\delta(x) = x\delta(1) + \delta(1)x, \quad x \in A.$$

If $\delta(1) \in Z(X)$, then $\delta(x) = \delta(1)x = x\delta(1)$, and hence δ is a $*$-multiplier. $\square$

4 Characterization of homomorphisms

In this section, we prove that there is no nonzero continuous linear map $\phi : A \longrightarrow B$ between Banach $*$-algebras with the property that $\phi(x \diamond y) = \phi(x) \diamond \phi(y)$ for all $x, y \in A$ with $xy = 1$.

Theorem 4.1. Let $\phi : A \longrightarrow B$ be a continuous linear map satisfying

$$x, y \in A, \quad xy = 1 \quad \Longrightarrow \quad \phi(x \diamond y) = \phi(x) \diamond \phi(y).$$

Then ϕ is identically zero.

Proof . Put any $a \in A_{sa}$, e^{ita} is a unitary for each $t \in \mathbb{R}$. Therefore,

$$\begin{aligned} \phi(e^{2ita} - e^{-2ita}) &= \phi(e^{-ita} \diamond e^{ita}) \\ &= \phi(e^{-ita})^* \phi(e^{ita}) - \phi(e^{ita})^* \phi(e^{-ita}). \end{aligned} \tag{4.1}$$

By taking derivative of equation (4.1) at t, we obtain

$$2\phi(ae^{2ita} + ae^{-2ita}) = \phi(ae^{-ita})^* \phi(e^{ita}) + \phi(e^{-ita})^* \phi(ae^{ita}) + \phi(ae^{ita})^* \phi(e^{-ita}) + \phi(e^{ita})^* \phi(ae^{ita}). \tag{4.2}$$

Taking $t = 0$ and $a = 1$ in (4.2), we thus get

$$\phi(1) = \phi(1)^* \phi(1).$$

Therefore, $\phi(1) = \phi(1)^*$ and hence $\phi(1)$ is an idempotent in B. Put $t = 0$ in (4.2), we get

$$2\phi(a) = \phi(a)^* \phi(1) + \phi(1)\phi(a). \tag{4.3}$$

From (4.3), we arrive at

$$\phi(a) = \phi(a)^*, \quad a \in A_{sa}.$$

For any $x \in A$, there are $a, b \in A_{sa}$ such that $x = a + ib$. So,

$$\phi(x^*) = \phi(a) - i\phi(b)^* = \phi(x)^*,$$

for all $x \in A$. Thus, ϕ is self-adjoint. Now it follows from (4.3) that

$$2\phi(a) = \phi(a)\phi(1) + \phi(1)\phi(a). \tag{4.4}$$

By taking derivative of equation (4.2) at $t = 0$, we obtain

$$\phi(1)\phi(a^2) = 0, \quad a \in A_{sa}.$$

In particular, $\phi(1) = \phi(1)\phi(1) = 0$, and thus by (4.4),

$$\phi(a) = 0, \quad a \in A_{sa}.$$

Consequently, $\phi(x) = 0$, for all $x \in A$. $\square$

Theorem 4.2. Let $\phi : A \longrightarrow B$ be a continuous linear map satisfying

$$x, y, z \in A, \quad xy = 1, z = x \quad \Longrightarrow \quad \phi(x \diamond y \diamond z) = \phi(x) \diamond \phi(y) \diamond \phi(z).$$

Then

$$4\phi(x) = \phi(x)\phi(1)^2 + 2\phi(1)\phi(x)\phi(1) + \phi(1)^2\phi(x), \quad x \in A.$$

Moreover, if ϕ is surjective and $\phi(1)$ is idempotent, then $\phi(1)$ is the identy of B.

Proof . Let $a \in A_{sa}$, $x = e^{ita}$ and $y = e^{-ita}$ for each $t \in \mathbb{R}$. Then

$$e^{ita} \diamond e^{-ita} \diamond e^{ita} = e^{3ita} - e^{-3ita} + e^{ita} - e^{-ita}.$$

Thus, we get

$$\begin{aligned} \phi(e^{3ita} - e^{-3ita} + e^{ita} - e^{-ita}) &= \phi(e^{ita} \diamond e^{-ita} \diamond e^{ita}) \\ &= \phi(e^{ita}) \diamond \phi(e^{-ita}) \diamond \phi(e^{ita}) \\ &= \phi(e^{-ita})^*\phi(e^{ita})\phi(e^{ita}) - \phi(e^{ita})^*\phi(e^{-ita})\phi(e^{ita}) \\ &\quad - \phi(e^{ita})^*\phi(e^{ita})^*\phi(e^{-ita}) + \phi(e^{ita})^*\phi(e^{-ita})^*\phi(e^{ita}). \end{aligned} \tag{4.5}$$

By taking derivative of equation (4.5) at $t = 0$, we obtain

$$4\phi(a) = \phi(a)^*\phi(1)\phi(1) + \phi(1)^*\phi(a)\phi(1) + \phi(1)^*\phi(a)^*\phi(1) + \phi(1)^*\phi(1)^*\phi(a). \tag{4.6}$$

Put $a = 1$ in (4.6), to get

$$2\phi(1) = \phi(1)^*\phi(1)\phi(1) + \phi(1)^*\phi(1)^*\phi(1).$$

Therefore, $\phi(1) = \phi(1)^*$ and so (4.6) imply that

$$4\phi(a) = \phi(a)^*\phi(1)^2 + \phi(1)\phi(a)\phi(1) + \phi(1)\phi(a)^*\phi(1) + \phi(1)^2\phi(a). \tag{4.7}$$

From (4.7), we deduce that

$$\phi(a) = \phi(a)^*, \quad a \in A_{sa}.$$

Thus, by (4.6) we obtain

$$4\phi(a) = \phi(a)\phi(1)^2 + 2\phi(1)\phi(a)\phi(1) + \phi(1)^2\phi(a).$$

For any $x \in A$, there are $a, b \in A_{sa}$ such that $x = a + ib$. Hence

$$4\phi(x) = \phi(x)\phi(1)^2 + 2\phi(1)\phi(x)\phi(1) + \phi(1)^2\phi(x), \quad x \in A. \tag{4.8}$$

Now suppose that ϕ is surjective and $\phi(1)$ is an idempotent. By multiplying (4.8) on the left and right by $\phi(1)$, respectively, we deduce

$$\begin{aligned} \phi(1)\phi(x) &= \phi(1)\phi(x)\phi(1), \\ \phi(x)\phi(1) &= \phi(1)\phi(x)\phi(1). \end{aligned}$$

Therefore, $\phi(1)$ commutes with $\phi(x)$, for all $x \in A$. From (4.8), we get

$$\phi(x) = \phi(x)\phi(1) = \phi(1)\phi(x), \quad x \in A.$$

The surjectivity of ϕ now implies that $\phi(1)$ is the identy of B. □

References

[1] Z. Bai and S. Du, *Maps preserving product $XY - YX^*$ on von Neumann algebras*, J. Math. Anal. Appl. **386** (2012), no. 1, 103–109.

[2] M. Brešar, *Centralizing mappings on von Neumann algebras*, Proc. Amer. Math. Soc. **111** (1991), no. 2, 501–510.

[3] M. Brešar, *Centralizing mappings and derivations in prime rings*, J. Algebra **156** (1993), 385–394.

[4] M. Brešar, *Characterizing homomorphisms, derivations and multipliers in rings with idempotents*, Proc. R. Soc. Edinb. Sect. A. **137** (2007), 9–21.

[5] A. Essaleh and A. Peralta, *Linear maps on C^*-algebras which are derivations or triple derivations at a point*, Linear Algebra Appl. **538** (2018), 1–21.

[6] H. Ghahramani and W. Jing, *Lie centralizers at zero products on a class of operator algebras*, Ann. Funct. Anal. **12** (2021), 1–12.

[7] C. Li, F. Zhao, and Q. Chen, *Nonlinear skew Lie triple derivations between factors*, Acta Math. Sinica (English Series) **32** (2016) 821–830.

[8] L. Molnar, *A condition for a subspace of $B(H)$ to be an ideal*, Linear Algebra Appl. **235** (1996), 229–234.

[9] M. Shavandi and A. Taghavi, *Maps preserving n-tuple $a^*b - b^*a$ derivations on factor von Neumann algebras*, Publ. Inst. Math. **113** (2023), no. 127, 131–140.

[10] M. Shavandi and A. Taghavi, *Nonlinear triple product $a^*b - b^*a$ derivations on $*$-algebras*, Surv. Math. Appl. **19** (2024), 67–78.

[11] A. Taghavi, H. Rohi, and V. Darvish, *Nonlinear $*$-Jordan derivations on von Neumann algebras*, Linear Multilinear Algebra **64** (2016), 426–439.

[12] Z. Wang and X. Fei, *Maps on C^*-algebras are skew Lie triple derivations or homomorphisms at one point*, Aims Math. **8**(11) (2023), 25564–25571.

[13] W. Yu and J. Zhang, *Nonlinear $*$-Lie derivations on factor von Neumann algebras*, Linear Algebra Appl. **437** (2012) 1979–1991.

[14] F. Zhao and C. Li, *Nonlinear $*$-Jordan triple derivations on von Neumann algebras*, Math. Slovaca **68** (2018), 163—170.

[15] A. Zivari-Kazempour, *Characterization of n-Jordan multipliers*, Vietnam J. Math. **50** (2022), 87–94.

[16] A. Zivari-Kazempour, *Characterizing n-multipliers on Banach algebras through zero products*, Int. J. Nonlinear Anal. Appl. **14** (2023), no. 1, 1071–1078.

[17] A. Zivari-Kazempour, *Characterization of Jordan homomorphisms and Jordan derivations*, Khayyam J. Math. **10** (2024), no. 1, 1–9.

[18] A. Zivari-Kazempour and A. Bodaghi, *Generalized derivations and generalized Jordan derivations on C^*-algebras through zero products*, J. Math. **2022** (2022), Article ID 3386149, 1–6.