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Some results on ϕ-primary submodules

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Abstract

Let R be a commutative ring with identity, M be an unitary R-module, let $\mathcal{S}(M)$ be the set of all submodules of M and $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. A proper submodule N of M is called ϕ-pimary submodule if $rx \in N \setminus \phi(N)$ where $r \in R$ and $x \in M$, implies that $x \in N$ or $r \in \sqrt{(N : M)}$. In this work, ϕ-primary submodules are studied, and some results are obtained.

Keywords: ϕ-prime ideal, ϕ-primary ideal, ϕ-prime submodule, ϕ-primary submodule
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1 Introduction

In this work, all rings are commutative with identity, and modules are unitary. Let M be an R-module, and N be a submodule of M. The ideal $\{r \in R \mid rM \subseteq N\}$ will be denoted by $(N : M)$ and ideal $(0 : M)$ will be denoted by $Ann(M)$. Anderson and Bataineh in [1] introduced various generalizations of prime ideals. Let $\phi' : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all ideals of R. Then a proper ideal I of R is ϕ'-prime if for $a, b \in R, ab \in R \setminus \phi(I)$ implies $a \in R$ or $b \in R$. If $\phi'(I) = \emptyset$ (resp. $\phi'(I) = 0$, $\phi'(I) = I^2$, $\phi'(I) = I^m$ and $\phi'(I) = \bigcap_{m=1}^{\infty} I^m$), then ideal I is called a prime ideal (resp. weakly prime ideal, almost prime ideal, m-almost prime ideal and ω-prime ideal).

Zamni in [10] extended this concept to ϕ-prime submodule. For a function $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$, a proper submodule N of M is called a prime submodule relative to ϕ or ϕ-prime submodule if whenever $r \in R$, $x \in M$, and $rx \in N \setminus \phi(N)$, then $x \in N$ or $r \in (N : M)$. Without loss of generality, throughout this work we will assume $\phi(N) \subseteq N$. If $\phi(N) = \emptyset$ (resp. $\phi(N) = 0$, $\phi(N) = (N : M)N$, $\phi(N) = (N : M)^{m-1}N$ and $\phi(N) = \bigcap_{m=1}^{\infty}(N : M)^m N$), then submodule N is called a prime submodule (resp. weakly prime submodule, almost prime submodule, m-almost prime submodule, and ω-prime submodule). Some properties of generalizations of prime submodules have been studied in [5],[7] and [8].

Now, we extend this concept to a primary relative to ϕ or ϕ-primary submodules, i.e., a proper submodule N of M is ϕ-primary if for every $r \in R$, $x \in M$ and $rx \in N \setminus \phi(N)$ implies $x \in N$ or $r \in \sqrt{(N : M)}$. If $\phi(N) = \emptyset$ (resp. $\phi(N) = 0$, $\phi(N) = (N : M)N$, $\phi(N) = (N : M)^{m-1}N$ and $\phi(N) = \bigcap_{m=1}^{\infty}(N : M)^m N$), then submodule N is called a primary submodule (resp. weakly primary submodule, almost primary submodule, m-almost primary submodule, and ω-primary submodule). We denote the set of ϕ-primary submodules of M by $\mathcal{P}rim_\phi(M)$ and denote the set of ϕ-prime submodules of M by $\mathcal{S}pec_\phi(M)$. Let $\mathcal{S}pec_\phi(R)$ denote the set of ϕ-prime ideals of R and $\mathcal{P}rim_\phi(R)$ denote the set of ϕ-primary ideals of R. We are given some properties of ϕ-primary submodules and we obtain relationships among function ϕ, various modules and the other concepts. Some of the finding in this study have been inspired by the research cited as [10] and [1].

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2 Properties of ϕ-pimary ideal and ϕ-pimary submodules

In this section, we introduce and study several propostions and corollaries of ϕ-pimary submodules.

Proposition 2.1. Let $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all ideals of R such that $\sqrt{\phi(Q)} \subseteq \phi(Q)$. If $Q \in \mathcal{P}rim_\phi(R)$, then $\sqrt{Q} \in \mathcal{S}pec_\phi(R)$.

Proof . Let $a, b \in R$ with $ab \in \sqrt{Q} \setminus \phi(\sqrt{Q})$, so $ab \in \sqrt{Q}$ and $ab \notin \phi(\sqrt{Q})$. Since $\phi(Q) \subseteq \phi(\sqrt{Q})$, $ab \notin \phi(Q)$. Also, since $\sqrt{\phi(Q)} \subseteq \phi(Q)$, $ab \notin \sqrt{\phi(Q)}$. Thus $(ab)^m \notin \phi(Q)$ for each $m \in \mathbb{N}$. On the other hand, $ab \in \sqrt{Q}$ implies $(ab)^n \in Q$, for some $n \in \mathbb{Z}^+$. Therefore $a^n b^n \in Q \setminus \phi(Q)$. Since $Q \in \mathcal{P}rim_\phi(R)$, so $a^n \in Q$ or $(b^n)^k \in Q$, for some $k \in \mathbb{N}$. Thus $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$, i.e., $\sqrt{Q} \in \mathcal{S}pec_\phi(R)$. □

Corollary 2.2. Let $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all ideals of R and $\phi(Q)$ be a radical ideal. If $Q \in \mathcal{P}rim_\phi(R)$, then $\sqrt{Q} \in \mathcal{S}pec_\phi(R)$.

Proof . Since $\sqrt{\phi(Q)} = \phi(Q)$, by Proposition 2.1, the proof is clear. □

Proposition 2.3. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function where $\mathcal{S}(M)$ is the set of all submodules of M such that $(\phi(N) : m) \subseteq \phi((N : M))$ for all $m \in M$. If $N \in \mathcal{P}rim_\phi(M)$, then $(N : M) \in \mathcal{P}rim_\phi(R)$.

Proof . Suppose that $a, b \in R$ with $ab \in (N : M) \setminus \phi((N : M))$ and $a \notin (N : M)$. We show that $b \in \sqrt{(N : M)}$. We have $ab \in (N : M)$ and $ab \notin \phi((N : M))$, so $abm \in N$ for all $m \in M$. Since $(\phi(N) : m) \subseteq \phi((N : M))$, hence $ab \notin (\phi(N) : m)$, so $abm \notin \phi(N)$. Then $abm \in N \setminus \phi(N)$. Now, let $m \notin N$. Since $N \in \mathcal{P}rim_\phi(M)$ and $a \notin (N : M)$, $am \notin N$ and hence $b \in \sqrt{(N : M)}$. Therefore $(N : M) \in \mathcal{P}rim_\phi(R)$. □

Proposition 2.4. Let $f : R \to S$ be a ring homomorphism and $\phi_1 : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$, $\phi_2 : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be two functions such that $f^{-1}(\phi_2(J)) \subseteq \phi_1(f^{-1}(J))$ for all ideal J of R. If $P \in \mathcal{S}pec_{\phi_2}(S)$, then $f^{-1}(P) \in \mathcal{S}pec_{\phi_1}(R)$.

Proof . Assume that $ab \in f^{-1}(P) \setminus \phi_1(f^{-1}(P))$, where $a, b \in R$ and $P \in \mathcal{S}pec_{\phi_2}(S)$, then $ab \in f^{-1}(P)$ and $ab \notin \phi_1(f^{-1}(P))$. Since $f^{-1}(\phi_2(P)) \subseteq \phi_1(f^{-1}(P))$, $ab \notin f^{-1}(\phi_2(P))$, and so $f(ab) \notin \phi_2(P)$. It follows that $f(a)f(b) \in P \setminus \phi_2(P)$. Since $P \in \mathcal{S}pec_{\phi_2}(S)$, $f(a) \in P$ or $f(b) \in P$ and hence $a \in f^{-1}(P)$ or $b \in f^{-1}(P)$. This implies that $f^{-1}(P) \in \mathcal{S}pec_{\phi_1}(R)$. □

Proposition 2.5. Let $f : M \to M'$ be an R-module epimorphism, $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ and $\phi' : \mathcal{S}(M') \to \mathcal{S}(M') \cup \{\emptyset\}$ be two functions with $(\phi'(N') : f(m)) \subseteq (\phi(f^{-1}(N') : m)$ for each $m \in M$ where N' is a submodule M' and $N' \in \mathcal{P}rim_{\phi'}(M')$. Then $f^{-1}(N') \in \mathcal{P}rim_\phi(M)$.

Proof . Let $rm \in f^{-1}(N') \setminus \phi(f^{-1}(N'))$ where $r \in R$, $m \in M$. Then $rm \in f^{-1}(N')$ and $rm \notin \phi(f^{-1}(N'))$, hence $rf(m) \in N'$ and $r \notin (\phi(f^{-1}(N')) : m)$. Since $(\phi'(N') : f(m)) \subseteq (\phi(f^{-1}(N')) : m)$ for each $m \in M$, $r \notin (\phi'(N') : f(m))$. It follows that $rf(m) \in N' \setminus \phi'(N')$. Since $N' \in \mathcal{P}rim_{\phi'}(M')$, $f(m) \in N'$ or $r \in \sqrt{(N' : M')}$. Thus $m \in f^{-1}(N')$ or $r^n \in (N' : M')$ for some $n \in \mathbb{N}$. Assuming $r^n \in (N' : M')$, we have $r^n M' \subseteq N'$ and hence $r^n f(M) \subseteq N'$. Thus $f^{-1}(f(r^n M)) \subseteq f^{-1}(N')$ and so $r^n M \subseteq f^{-1}(N')$ and hence $r^n \in (f^{-1}(N') : M)$. So $r \in \sqrt{(f^{-1}(N') : M)}$. Therefore, $f^{-1}(N') \in \mathcal{P}rim_\phi(M)$. □

The following proposition is stated for ϕ-prime submodules of M (see [10, Theorem 2.12 part (i)]). We assert it for ϕ-primary submodules of M.

Proposition 2.6. Let M be an R-module, K and N be two submodules of M with $K \subseteq N$. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ and $\phi_K : \mathcal{S}(M/K) \to \mathcal{S}(M/K) \cup \{\emptyset\}$ be defined by $\phi_K(N/K) = (\phi(N) + K)/K$ with $K \subseteq \phi(N)$. Then the following statements hold.

(1) If $N \in \mathcal{P}rim_\phi(M)$, then $N/K \in \mathcal{P}rim_{\phi_K}(M/K)$.

(2) If $N/K \in \mathcal{P}rim_{\phi_K}(M/K)$, then $N \in \mathcal{P}rim_\phi(M)$.

Proof . (1) Let $\overline{m} \in M/K$ and $r \in R$ with $r\overline{m} \in N/K \setminus \phi_K(N/K)$, where $\overline{m} = m + K$ for some $m \in M$. So, $rm + K \in N/K$ and $rm + K \notin \phi_K(N/K)$ and hence $rm \in N$. By the definition of ϕ_K, this gives that $rm \notin \phi(N)$.

Therefore $rm \in N \setminus \phi(N)$. Since $N \in \mathcal{P}rim_\phi(M)$, so $m \in N$ or $r \in \sqrt{(N:M)}$. Assuming $m \in N$, we have $\overline{m} = m + K \in N/K$ and $r^n \in (N:M)$ for some $n \in \mathbb{N}$, implies that $r^n(m+K) = r^n m + K \in N/K$. Thus $r^n \in (N/K : M/K)$ and $r \in \sqrt{(N/K : M/K)}$. So $N/K \in \mathcal{P}rim_{\phi_K}(M/K)$.

(2) Let $r \in R$ and $m \in M$ with $rm \in N \setminus \phi(N)$.Then $rm \in N$ and $rm \notin \phi(N)$. Since $K \subseteq \phi(N)$, $rm + K \notin (\phi(N) + K)/K$. Thus $r(m+K) \in N/k \setminus \phi_K(N/K)$. Since $N/K \in \mathcal{P}rim_{\phi_K}(M/K)$, hence $m + K \in N/K$ or $r^n \in (N/K : M/K)$ for some $n \in \mathbb{N}$. It follows that $m \in N$ or $r^n \in (N:M)$. Thus $N \in \mathcal{P}rim_\phi(M)$. □

Corollary 2.7. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function and N be a proper submodule of M. Then N is a ϕ-pimary submodule of M if and only if $N/\phi(N)$ is a weakly primary submodule of $M/\phi(N)$.

Proof . It is straightforward. □

Let S be a multiplicatively closed subset of R and $S^{-1}R$ be a ring of fractions. Let $\phi : \mathcal{S}(R) \to \mathcal{S}(R) \cup \{\emptyset\}$ be a function and define $S^{-1}\phi : \mathcal{S}(S^{-1}R) \to \mathcal{S}(S^{-1}R) \cup \{\emptyset\}$ by $S^{-1}\phi(Q^e) = \phi(Q)^e$ where Q^e is an extension ideal of Q with $Q \cap S = \emptyset$ (i.e., let $\iota : R \to S^{-1}R$ be a canonical ring homomorphism by $\iota(r) = \frac{r}{1}$, the extension Q^e of Q to be the ideal $S^{-1}R\iota(Q)$ generated by $\iota(Q)$ in $S^{-1}R$. Now, we state the following proposition.

Proposition 2.8. Let $\phi : \mathcal{S}(R) \to \mathcal{S}(R) \cup \{\emptyset\}$ be a function and $S^{-1}\phi : \mathcal{S}(S^{-1}R) \to \mathcal{S}(S^{-1}R) \cup \{\emptyset\}$ by $S^{-1}\phi(Q^e) = \phi(Q)^e$. If $Q \in \mathcal{P}rim_\phi(R)$, then $Q^e \in \mathcal{P}rim_{S^{-1}\phi}(S^{-1}R)$.

Proof . Let $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$ with $\frac{a}{s}\frac{b}{t} \in Q^e \setminus S^{-1}\phi(Q^e)$. We have $\frac{ab}{st} \in Q^e$ and $\frac{ab}{st} \notin S^{-1}\phi(Q^e) = \phi(Q)^e$. Then there exists $u \in S$ such that $uab \in Q$ and $uab \notin \phi(Q)$, so $uab \in Q \setminus \phi(Q)$. Since $Q \in \mathcal{P}rim_\phi(R)$, $ua \in Q$ or $b \in \sqrt{Q}$. This shows that $\frac{ua}{us} \in Q^e$ or $\frac{b}{t} \in \sqrt{Q}^e = \sqrt{Q^e}$, as required.

Let S be a multiplicatively closed subset of R and $S^{-1}R$ be a ring of fractions.Then every submodule of $S^{-1}M$ is in the form of $S^{-1}N$ for some submodule N of M (see [9]). Let $N(S) = \{m \in M \ : \ \exists s \in S, sm \in N\}$, it is easy to show that $N(S)$ is a submodule of M containing N and $S^{-1}(N(S)) = S^{-1}(N)$. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function and define $S^{-1}\phi : \mathcal{S}(S^{-1}M) \to \mathcal{S}(S^{-1}M) \cup \{\emptyset\}$ by $S^{-1}\phi(S^{-1}N) = S^{-1}(\phi(N(S))$ if $\phi(N(S)) \neq \emptyset$ and $S^{-1}\phi(S^{-1}N) = \emptyset$ if $\phi(N(S)) = \emptyset$. Since $\phi(N) \subseteq N$, so $S^{-1}\phi(S^{-1}N) \subseteq S^{-1}N$. □

Theorem 2.9. Let M be an R-module and S be a multiplicatively closed subset of R. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function such that $S^{-1}(\phi(N)) \subseteq S^{-1}\phi(S^{-1}N)$. Let N be a ϕ-pimary submodule of M such that $(N:M)$ be a prime ideal of R and $\sqrt{(N:M)} \cap S = \emptyset$. Then $S^{-1}N \in \mathcal{P}rim_{S^{-1}\phi}(S^{-1}M)$.

Proof . Let $\frac{r}{s} \in S^{-1}R$ and $\frac{m}{t} \in S^{-1}M$ with $\frac{r}{s}\frac{m}{t} \in S^{-1}N \setminus S^{-1}\phi(S^{-1}N)$. We have $\frac{rm}{st} \in S^{-1}N$ and $\frac{rm}{st} \notin S^{-1}\phi(S^{-1}N)$. Since $S^{-1}(\phi(N)) \subseteq S^{-1}\phi(S^{-1}N)$, $\frac{rm}{st} \notin S^{-1}(\phi(N))$. Therefore $vrm \notin \phi(N)$ for each $v \in S$. On the other hand, $\frac{rm}{st} \in S^{-1}N$ implies that $urm \in N$ for some $u \in S$ and hence $urm \in N \setminus \phi(N)$. Since N is a ϕ-primary submodule, $m \in N$ or $ur \in \sqrt{(N:M)}$. Since $u \in S$ and $\sqrt{(N:M)} \cap S = \emptyset$, so $u \notin \sqrt{(N:M)}$ and hence $u^n \notin (N:M)$ for every $n \in \mathbb{N}$. It follows that $u^k r^k \in (N:M)$ for some $k \in \mathbb{N}$. Since $u^k \notin (N:M)$ and $(N:M)$ is a prime ideal, $r^k \in (N:M)$ and hence $\frac{r^k}{s^k} \in S^{-1}(N:M)$. Since $S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$, $\frac{r^k}{s^k} \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$ and thus $\frac{r}{s} \in \sqrt{(S^{-1}N :_{S^{-1}R} S^{-1}M)}$. Hence $S^{-1}N$ is a $S^{-1}\phi$-primary submodule of $S^{-1}M$. □

Definition 2.10. A proper submodule N of M is said semiprime, if $r^k m \in N$ for each $r \in R$, $m \in M$ and $k \in \mathbb{N}$, then $rm \in N$.

For more details concerning ϕ-semiprime submodules of an R-module refer to [4].

Proposition 2.11. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function and N be a semiprime submodule of M. If N is a ϕ-primary submodule of M, then N is a ϕ-prime submodule of M.

Proof . Let $r \in R$ and $m \in M$ with $rm \in N \setminus \phi(N)$. Assume that $x \notin N$, since N is a ϕ-primary submodule of M, so $r \in \sqrt{(N:M)}$. It follows that $r^k M \subseteq N$ for some $k \in \mathbb{N}$ and hence $r^k m \in N$ for each $m \in M$. Since N is a semiprime submodule of M, so $rm \in N$ for each $m \in M$. Thus $r \in (N:M)$, therefore $N \in \mathcal{S}pec_\phi(M)$. □

Definition 2.12. Let N be a submodule of M. Then N is called relatively divisible submodule denoted by RD-submodule, if $rN = N \cap rM$ for each $r \in R$. R-module M is said prime module, if $rx = 0$ where $r \in R$ and $x \in M$, then $r \in Ann(M)$ or $x = 0$.

Now, with respect to above definition, we assert the following proposition.

Proposition 2.13. Let M be a prime module, $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function and N be a proper submodule of M. If N is an RD- submodule of M with $Ann(M) \subseteq (\phi(N) : M)$, then $N \in \mathcal{P}rim_{\phi}(M)$.

Proof . Let $r \in R$ and $x \in M$ with $rx \in N \setminus \phi(N)$. Since N is an RD-submodule, $rN = N \cap rM$ and hence $rx \in N \cap rM$. It follows that $rx = rn$ for some $n \in N$, therefore, $r(x - n) = 0$. Since M is prime, hence $r \in Ann(M)$ or $x - n = 0$. But $r \in Ann(M)$, since $Ann(M) \subseteq (\phi(N) : M)$, $r \in (\phi(N) : M)$ and hence $rx \in \phi(N)$ which contradicts with our assumption. Thus $x - n = 0$ and so $x \in N$. Therefore $N \in \mathcal{P}rim_{\phi}(M)$. □

Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ be an R-module where R_i is a commutative ring and M_i is an R_i-module for $i = 1, 2$. Each proper submodule of $M_1 \times M_2$ can be represented in the form of $N_1 \times N_2$ for some proper submodule N_1 of M_1 and N_2 of M_2. Moreover $Q = Q_1 \times Q_2$ is a primary submodule of $M_1 \times M_2$ if and only if $Q = Q_1 \times M_2$ or $Q = M_1 \times Q_2$ for some pimary submodule Q_1 of M_1 and Q_2 of M_2. Now, let $\phi : \mathcal{S}(M_1 \times M_2) \to \mathcal{S}(M_1 \times M_2) \cup \{\emptyset\}$ and $\phi_i : \mathcal{S}(M_i) \to \mathcal{S}(M_i) \cup \{\emptyset\}$ be functions with $\phi(N_1 \times N_2) = \phi_1(N_1) \times \phi_2(N_2)$ for $i = 1, 2$.We state the following proposition that some properties of this concept have been investigated for ϕ-prime submodules of $M_1 \times M_2$ (see [10], Corollary 2.5 and Proposition 2.6).

Proposition 2.14. Let $M = M_1 \times M_2$ be an $R_1 \times R_2$-module, Q_i be a proper submodule of M_i and $\phi : \mathcal{S}(M_1 \times M_2) \to \mathcal{S}(M_1 \times M_2) \cup \{\emptyset\}$ and $\phi_i : \mathcal{S}(M_i) \to \mathcal{S}(M_i) \cup \{\emptyset\}$ be functions with $\phi(Q_1 \times Q_2) = \phi_1(Q_1) \times \phi_2(Q_2)$ and $\phi_i(M_i) = M_i$ for $i = 1, 2$. If $Q_1 \in \mathcal{P}rim_{\phi_1}(M_1)$ $(Q_2 \in \mathcal{P}rim_{\phi_2}(M_2))$, then $Q_1 \times M_2 \in \mathcal{P}rim_{\phi}(M_1 \times M_2)$ $(M_1 \times Q_2 \in \mathcal{P}rim_{\phi}(M_1 \times M_2))$.

Proof . Let $(r_1, r_2) \in R_1 \times R_2$ and $(x_1, x_2) \in M_1 \times M_2$ with $(r_1, r_2)(x_1, x_2) \in Q_1 \times M_2 \setminus \phi(Q_1 \times M_2)$. So $(r_1x_1, r_2x_2) \in Q_1 \times M_2$ and $(r_1x_1, r_2x_2) \notin \phi_1(Q_1) \times \phi_2(M_2) = \phi_1(Q_1) \times M_2$ and hence $r_1x_1 \in Q_1$ and $r_1x_1 \notin \phi_1(Q_1)$. Since Q_1 is a ϕ_1-primary submodule of M_1, therefore $x_1 \in Q_1$ or $r_1 \in \sqrt{(Q_1 :_{R_1} M_1)}$. If $x_1 \in Q_1$, then $(x_1, x_2) \in Q_1 \times M_2$. Thus $Q_1 \times M_2 \in \mathcal{P}rim_{\phi}(M_1 \times M_2)$. On the other hand, if $r_1 \in \sqrt{(Q_1 :_{R_1} M_1)}$, then $r_1^n \in (Q_1 :_{R_1} M_1)$ for some $n \in \mathbb{N}$. It follows that $(r_1^n, r_2^n) \in (Q_1 :_{R_1} M_1) \times (M_2 :_{R_2} M_2)$, so $(r_1^n, r_2^n) \in (Q_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)$ and hence $(r_1, r_2) \in \sqrt{(Q_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)}$. Consequently $Q_1 \times M_2 \in \mathcal{P}rim_{\phi}(M_1 \times M_2)$. □

Corollary 2.15. Let the situation be as described Proposition 2.14. If $Q_1 \times M_2 \in \mathcal{P}rim_{\phi}(M_1 \times M_2)$ $(M_1 \times Q_2 \in \mathcal{P}rim_{\phi}(M_1 \times M_2))$, then $Q_1 \in \mathcal{P}rim_{\phi_1}(M_1)$ $(Q_2 \in \mathcal{P}rim_{\phi_2}(M_2))$.

Firstly, assume that M be a free R-module. We assert the next theorem in connection with ϕ'-primary ideal Q of R and ϕ-primary submodule QM of M.

Theorem 2.16. Let M be a free R-module with a basis $\{x_\alpha\}_{\alpha \in \Omega}$, $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ and $\phi' : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be two functions where $\mathcal{I}(R)$ is the set of all ideals of R. If Q is a ϕ'-primary ideal of R with $\phi'(Q)M \subseteq \phi(QM)$, then QM is a ϕ-primary submodule of M.

Proof . Since Q is proper ideal of R, QM is a proper submodule of M. Assume that $r \in R$ and $x \in M$ with $rx \in QM \setminus \phi(QM)$, so $rx \in QM$ and $rx \notin \phi(QM)$ with $x = \sum_{f.s} r_\alpha x_\alpha$ $(r_\alpha \in R,\ x_\alpha \in \{x_\alpha\}_{\alpha \in \Omega}$). Since M is a free R-module with a basis $\{x_\alpha\}_{\alpha \in \Omega}$, $QM = \{\sum_{f.s} s_i x_i \mid s_i \in Q, x_i \in \{x_\alpha\}_{\alpha \in \Omega}\ \}$. It follows that $rx = \sum_{f.s} r(r_\alpha x_\alpha) = \sum_{f.s} s_i x_i$. Thus $s_\alpha = rr_\alpha \in Q$ for all $\alpha \in \Omega$. But $rr_\alpha \notin \phi'(Q)$ for all $\alpha \in \Omega$, otherwise $rr_\alpha \in \phi'(Q)$, so $rr_\alpha x_\alpha \in \phi'(Q)x_\alpha$ for all $x_\alpha \in \{x_\alpha\}_{\alpha \in \Omega}$. Therefore $r \sum_{f.s} r_\alpha x_\alpha \in \phi'(Q)M$, since $\phi'(Q)M \subseteq \phi(QM)$, $r \sum_{f.s} r_\alpha x_\alpha \in \phi(QM)$. This is a contradiction. We showed that $rr_\alpha \in Q \setminus \phi'(Q)$ for all $\alpha \in \Omega$. Since Q is a ϕ'-primary ideal of R, $r \in Q$ or $r_\alpha \in \sqrt{Q}$. If $r \in Q$, then $rx \in QM$. For let $r_\alpha \in \sqrt{Q}$, we have $r_\alpha^n \in Q$ and hence $r_\alpha^n \in (QM :_R M)$. Thus $r_\alpha \in \sqrt{(QM :_R M)}$. Consequently, $QM \in \mathcal{P}rim_{\phi}(M)$. □

An R-module M is called a multiplication R-module if for every submodule N of M, $N = IM$ for some ideal I of R. It is easily seen that if N is a submodule of M, then $N = (N : M)M$ (see [3, 6]).

Remark 2.17. Let M be a free multiplication R-module and N be a proper submodule of M. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ and $\phi' : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be two functions where $\mathcal{I}(R)$ is the set of all ideals of R. If $(N : M$ is a ϕ'-primary ideal of R with $\phi'((N : M))M \subseteq \phi((N : M)M)$, then N is a ϕ-primary submodule of M.

Theorem 2.18. Let M be a free multiplication R-module and N be a proper submodule of M. If N is a almost primary submodule of M such that $(N : M)^2$ is a prime ideal of R, then N is a primary submodule of M.

Proof . Let $r \in R$ and $x \in M$ with $rx \in N$. If $rx \notin (N : M)N$, then $rx \in N \setminus (N : M)N$. Since N is a almost primary submodule of M, $x \in N$ or $r \in \sqrt{(N : M)}$ and hence N is a primary submodule of M. Now, assume that $rx \in (N : M)N$. Since $N = (N : M)M$, $rx \in (N : M)^2M$. Suppose that $rx \notin \sqrt{(N : M)}$, we prove that $x \in N$. Since M is a free R-module with a basis $\{x_\alpha\}_{\alpha\in\Omega}$, $x = \sum_{f.s} r_\alpha x_\alpha$ where $r_\alpha \in R$ and $x_\alpha \in \{x_\alpha\}_{\alpha\in\Omega}$. Since $rx \in (N : M)^2M$, $rx = \sum_{f.s}(rr_\alpha)x_\alpha = \sum_{f.s} r'_\alpha x_\alpha$ where $r'_\alpha \in (N : M)^2$ for all $\alpha \in \Omega$. It follows that $rr_\alpha = r'_\alpha$ for all $\alpha \in \Omega$ and hence $rr_\alpha \in (N : M)^2$. Since $r \notin \sqrt{(N : M)}$ and $(N : M) \subseteq \sqrt{(N : M)}$, $r \notin (N : M)$ and hence $r \notin (N : M)^2$. Since $(N : M)^2$ is a prime ideal of R, $r_\alpha \in (N : M)^2$ for all $\alpha \in \Omega$. This shows that $x = \sum_{f.s} r_\alpha x_\alpha \in (N : M)^2M$, since $(N : M)^2M \subseteq (N : M)M = N$, $x \in N$ and this completes the proof. □

Corollary 2.19. Let M be a free multiplication R-module and Q be a proper ideal of R such that $(QM : M)^2$ is a prime ideal of R. If QM almost primary submodule of M, then QM is a primary submodule of M.

Proof . Apply Theorem 2.16. □

References

[1] D.D. Anderson and E. Batanieh, *Generalizations of prime ideals*, Commun. Algebra **36** (2008), 686–696.

[2] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[3] A. Barnard, *Multiplication modules*, J. Algebra **71** (1981), 174–178.

[4] M. Ebrahimpour and F. Mirzaee, *On ϕ-semiprime submodules*, J. Korean Math. Soc. **54** (2017), no. 4, 1099–1108.

[5] M. Ebrahimpour and R. Nekooei, *On generalizations of prime submodules*, Bull. Iran. Math. Soc. **39** (2013), no. 5, 919–939.

[6] Z.A. El-Best and P.F. Smith. *Multiplication modules*, Comm. Algebra **16** (1998), no. 4, 755–779.

[7] C.P. Lu, *Prime submodules of modules*, Commun. Math. Univ. Sancti Pauli **33** (1984), 61–69.

[8] R.L. McCasland and M.E. Moore, *Prime submodules*, Commun. Algebra **20** (1992), 1803–1817.

[9] R. Sharp, *Steps in Commutative Algebra*, Cambridge University Press, Cambridge, 2000.

[10] N. Zamani, *ϕ-prime submodules*, Glasgow Math. J. **52** (2010), no. 2, 253–259.