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Global existence and blow-up results for a nonlinear viscoelastic higher-order $p(x)$-Laplacian equation

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Abstract

This study aims for the global existence and blow-up of solutions for a class of nonlinear viscoelastic higher-order $p(x)$-Laplacian equations. First, we prove the global existence of solutions in the appropriate range of the variable exponents and next, by using different methods, we prove the blow-up of solutions with positive and negative initial energy. Our results are new, and it is the first time that taken into consideration, extending and improving the earlier results in the literature, such as (Bol. Soc. Mat. Mex.., 2023, https://doi.org/10.1007/s40590-023-00551-x).

Keywords: global existence, blow-up, higher-order, viscoelastic, $p(x)$-Laplacian
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1 Introduction

In this work, we consider the following viscoelastic higher-order $p(x)$-Laplacian equation with variable-exponent logarithmic source term in $\Omega \times (0, \infty)$

$$u_{tt} + (-\Delta)^\alpha u + (-\Delta)^\alpha_{p(x)} u - \int_0^t g(t-s)(-\Delta)^\alpha u(s)ds + u_t = |u|^{q(x)-2} u \ln |u|, \tag{1.1}$$

with the following initial-boundary conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega \tag{1.2}$$

$$\frac{\partial^i u}{\partial \nu^i} = 0, \qquad i = 0, 1, 2, \cdots, \alpha - 1, \; x \in \partial\Omega, t > 0 \tag{1.3}$$

where $\alpha \geq 1$ is a natural number, Ω is a bounded domain of $R^n (n \geq 1)$ with smooth boundary $\partial\Omega$ so that the divergence theorem could be applied ν is unit outward normal vector on $\partial\Omega$ and $\frac{\partial^i u}{\partial \nu^i}$ denotes the i-order normal derivation of u. Here, $(-\Delta)^\alpha_{p(x)} u$ is higher-order $p(x)$-Laplacian operator defined as

$$(-\Delta)^\alpha_{p(x)} u = D^\alpha (|D^\alpha u|^{p(x)-2} D^\alpha u)$$

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where D denotes the gradient operator, that is $D. = \nabla. = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n})$. Moreover, $D^\alpha. = \Delta^j.$ if $\alpha = 2j$ and $D^\alpha. = D\Delta^j.$ if $\alpha = 2j+1$. The following additional properties are assumed on the variable exponents and the kernel of the memory:

(A1) The functions $p(\cdot), m(\cdot)$ and $q(\cdot)$ are measurable on $\overline{\Omega}$, such that:

$$\begin{aligned} &2 \leq p_1 \leq p(x) \leq p_2 < \frac{2n}{n-2}, \quad n \geq 3 \\ &2 \leq m_1 \leq m(x) \leq m_2 < \frac{2n}{n-2}, \quad n \geq 3 \\ &2 \leq q_1 \leq q(x) \leq q_2 < \frac{2n}{n-2}, \quad n \geq 3 \end{aligned}$$

with

$$\begin{aligned} &p_1 := \operatorname*{ess\,inf}_{x\in\overline{\Omega}} p(x), \quad p_2 := \operatorname*{ess\,sup}_{x\in\overline{\Omega}} p(x), \\ &m_1 := \operatorname*{ess\,inf}_{x\in\overline{\Omega}} m(x), \quad m_2 := \operatorname*{ess\,sup}_{x\in\overline{\Omega}} m(x), \\ &q_1 := \operatorname*{ess\,inf}_{x\in\overline{\Omega}} q(x), \quad q_2 := \operatorname*{ess\,sup}_{x\in\overline{\Omega}} q(x). \end{aligned}$$

(A2) Kernel of the memory g is a non-increasing and non-negative function satisfying

$$g(t) \geq 0, \quad g'(t) \leq -g(t), \quad 1 - \int_0^\infty g(t)dt = \ell > 0.$$

Study of the behavior of solutions for p-Laplacian type equations attracted great deal of attention in the last decade. Regarding the analysis of this type of equations, we mention the work of Benaissa and Mokeddem [3], where studied decay properties of solutions for the following nonlinear wave equation of p-Laplacian type with a weak nonlinear dissipation

$$\begin{cases} u'' - div(|\nabla_x u|^{p-2}\nabla_x u) - \sigma(t) div(|\nabla_x u'|^{p-2}\nabla_x u') = 0 & \text{in } \Omega \times [0,+\infty[\\ u = 0, & \text{on } \Gamma \times [0,+\infty[\\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) & \text{on } \Omega \end{cases}$$

where Ω is a bounded domain of R^n with a smooth boundary $\Gamma = \partial\Omega$ and σ is a positive function. In another study, Pei et. al [17] investigated the following quasilinear wave equation with Kelvin-Voigt damping

$$u_{tt} - \Delta_p u - \Delta u_t = f(u),$$

in a bounded domain $\Omega \subset R^3$ where $\Delta_p u$ is the nonlinear p-Laplacian operator, $p \geq 2$ in which

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u).$$

They proved the existence of local weak solutions and extended the local weak solutions to global solutions when the damping term dominated the source in an appropriate sense. Moreover, a blow-up result has been proved for solutions with negative initial total energy.

Raposo et. al [24] studied the global solution, uniqueness and asymptotic behavior of the following nonlinear equation

$$u_{tt} - \Delta_p u = \Delta u - g * \Delta u,$$

such that $g * \Delta u$ is a memory damping term that is

$$g * \Delta u = \int_0^t g(t-\tau)\Delta u(\tau)d\tau.$$

They constructed the global solution by means of the Faedo-Galerkin approximations whereas the initial data is in appropriated set of stability created from the Nehari manifold and the asymptotic behavior has been obtained by using a result of P. Martinez based on new inequality that generalizes the results of Haraux and Nakao. For more information about the p-Laplacian equations we refer to [14, 15, 27, 32, 36] and references therein. Recently, Zu et. al [37] considered the following quasilinear wave equation of p-Laplacian type with $2 < p < 3$

$$u_{tt} - \Delta_p u - \Delta u_t = |u|^{r-1}u,$$

they obtained an energy estimate for the solutions and proved a blow-up result for the solutions with arbitrarily positive initial energy. Moreover, estimate of the lifespan of the solutions has been showed.

It is known that the logarithmic nonlinearity arises in a lot of different areas of sciences. This type of nonlinearity was introduced in the nonrelativistic wave equations describing spinning particles moving in an external electromagnetic field and also in the relativistic wave equation for spinless particles. With all those specific underlying meaning in physics,the global-in-time well-posedness of solution to the problem of evolution equation with such logarithmic type nonlinearity captures lots of attention. Piskin and Irkiil [20] proved the local existence of the solutions for the following p-Laplacian equation with logarithmic nonlinearity

$$u_{tt} - div(|\nabla u|^{p-2}\nabla u) - \Delta u + u_t = ku \ln |u|,$$

where $p > 2$ is a constant number and k is the smallest positive constant. Next, the same authors in [21] investigated the following equation

$$u_{tt} - \Delta u - div(|\nabla u|^{p-2}\nabla u) - \Delta u_t + |u_t|^{k-2}u_t = |u|^{p-2}u \ln |u|,$$

and proved the finite time blow up of solutions with negative initial energy when initial data satisfy some suitable conditions. Ferreira et. al [10] considered a Petrovsky type viscoelastic equation

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds + |u_t|^{m-2}u_t = |u|^{p-2}u \ln |u|,$$

and proved that any solution with initial data blows up in finite time provided that $E(0) < E_1$. Recently, Pereira et. al [18] investigated the existence, uniqueness, exponential decay, and blow-up behavior of the viscoelastic beam equation involving the p-Laplacian operator, strong damping, and a logarithmic source term, given by

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{r-2}u \ln |u|,$$

and by using the Faedo–Galerkin approximation, they established the existence and uniqueness result for the global solutions, taking into account that the initial data must belong to an appropriate stability set created from the Nehari manifold. Also, the exponential decay of solutions has been proved based on Nakao's method and they proved the blow-up of solutions by using the concavity argument. For more information regarding the equations with logarithmic nonlinearity see the selected works [2, 6, 8, 19, 33].

Numerous researchers have investigated equations characterized by nonstandard growth conditions, specifically those involving variable exponents in nonlinearities, due to their significant theoretical and practical implications. For example, Boudjeriou [4] considered the following class of heat equation involving $p(x)$-Laplacian with variable-exponent logarithmic nonlinearity

$$u_t - \Delta_{p(x)}u = |u|^{s(x)-2}u \log(|u|), \tag{1.4}$$

by using the concavity method he proved that the local solutions blow-up in finite time under suitable conditions. Also, he applied the potential well theory combined with the Pohozaev manifold to prove the global existence result. In another study, Zeng et. al [35] considered (1.4) in the presence of strong damping term $-\Delta u_t$ and proved the existence of the global solution by using the potential well method and the logarithmic inequality. In addition, the sufficient conditions of the blow-up have been obtained by concavity method. Recently, Bu et. al [7] studied the existence of solutions for the following Kirchhoff-type equations driven by the $p(x)$-Laplacian:

$$(-\Delta)^{M_{p(x)}}_{p(x)}\eta = \lambda|\eta|^{p(x)-2}\eta \ln |\eta| + g(x, \eta, \nabla u),$$

where $\Delta^{M_{p(x)}}_{p(x)}$ denotes the $p(x)$-Kirchhoff-type operator expressed as

$$(-\Delta)^{M_{p(x)}}_{p(x)}\eta = -M_{p(x)}\Delta_{p(x)}\eta, \quad M_{p(x)} = \int_\Omega |\nabla\eta|^{p(x)}dx.$$

Using a topological approach based on the Galerkin method together with fixed point theorem, they obtained the existence of the finite-dimensional approximate solutions, generalized solutions, and strong generalized solutions.

Kafini and Noor [13] considered the following delayed nonlinear wave equation with logarithmic variable-exponent nonlinearity

$$u_{tt} - \Delta u + \mu_1 u_t(x,t)|u_t|^{m(x)-2}(x,t) + \mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau) = u|u|^{p(x)-2}\ln|u|^k,$$

and proved a global existence result under sufficient conditions on the initial data only without imposing the Sobolev Logarithmic Inequality. After that, they established global results of exponential and polynomial types according to the range values of the exponents. At the end, a numerical study that supports their theoretical results has been given. In another study, Pan et. al [16] investigated the following a pseudoparabolic equation problem with variable exponents and logarithmic nonlinear term

$$u_t - \Delta u_t - div(|\nabla u|^{p(x)-2}\nabla u) = |u|^{q(x)-2}u\ln|u|.$$

By using the energy functional and the classical potential well, they obtained the global existence and blow-up results of weak solutions with variable exponents.
Boughamsa and Ouaoua [5] considered a boundary value problem related to the following nonlinear higher-order wave equation with variable-exponent nonlinearity

$$\eta_{tt} + (-\Delta)^m\eta - \int_0^t g(t-s)(-\Delta)^m\eta(s)ds + \eta_t = |\eta|^{p(x)-2}\eta.$$

They proved the existence and uniqueness of the local solution under suitable conditions for the relaxation function g and viable-exponent $p(\cdot)$, using a method, which is a mixture of the Faedo-Galarkin and Banach fixed point theorem.Also the blow up of solutions in finite time has been obtained and given a two-dimensional numerical example to illustrate the blow-up result. Recently, Shahrouzi [29] considered the following viscoelastic higher-order $p(x)$-Laplacian equation with variable-exponent logarithmic source term

$$u_{tt} + (-\Delta)^\alpha u + (-\Delta)^\alpha_{p(x)}u - (g\odot(-\Delta)^\alpha u)(t) + |u_t|^{m(x)-2}u_t = |u|^{q(x)-2}u\ln|u|.$$

He proved the global existence of solutions in the appropriate range of the variable exponents and next, by using Martinez's approach, the asymptotic stability of solutions has been established.
In several mathematical models we face higher-order partial differential equations. For example it can be found in Fluid Dynamics, Mechanics, Biology, Electromagnetism, image processing, where three-dimensional problems are represented on surfaces, for instance in the case of thin geometries, modeled as membranes, plates or shells, depending on the structure of the original domain. This leads to defining surface partial differential equations which often involve high-order differential operators. Ye [34] studied the following initial–boundary value problem of higher-order nonlinear viscoelastic wave equation

$$\begin{aligned}
&u_{tt} + (-\Delta)^m u - \int_0^t g(t-s)(-\Delta)^m u(s)ds = |u|^{p-2}u, && (x,t)\in\Omega\times R^+,\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), && x\in\Omega,\\
&\frac{\partial^i u}{\partial\nu^i} = 0, \quad i = 0,1,2,\dots,m-1, && (x,t)\in\partial\Omega\times R^+,
\end{aligned}$$

and proved the existence of global weak solutions for this problem by using the Galerkin method. Meanwhile, under suitable conditions on relaxation function $g(\cdot)$ and the positive initial energy as well as non-positive initial energy, he proved that the solution blows up in the finite time and obtained the lifespan estimates of solutions. Next, Pişkin and Irkıil [22] investigated the following nonlinear higher-order wave equation

$$u_{tt} + [Pu_{tt} + Pu_t] + Pu + u - \int_0^t g(t-s)Puds + u_t = u\ln|u|^k,$$

where $P = (-\Delta)^m$, ($m \geq 1$ and $m \in N$). By using Faedo-Galerkin method and a logarithmic Sobolev inequality, they proved local existence general decay of solutions. see also [23, 25].

Regarding equations with variable-exponent nonlinearity, Shahrouzi [26] studied the solution behavior of the following viscoelastic equation involving the $m(x)$-Laplacian operator

$$u_{tt} - \Delta u - div(|\nabla u|^{m(x)}\nabla u) + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + h(x,t,u,\nabla u) + \beta u_t = |u|^{p(x)}u,$$

with nonlinear boundary conditions. Under appropriate conditions, He proved a general decay result associated to solution energy. It is also shown that regarding arbitrary positive initial energy, solutions blow-up in a finite time. For more information on the problems with variable-exponent nonlinearities we refer to [28, 30, 31].

Inspired by the previous studies, to the best of our knowledge, the present paper is the first to study the global existence and blow-up of solutions to the initial-boundary value problem (1.1)-(1.3) which involves the higher-order viscoelastic $p(x)$-Laplacian equation and variable-exponent logarithmic source term.

The rest of the paper is organized as follows. In section 2, some definitions and Lemmas about the variable-exponent Lebesgue space, $L^{p(\cdot)}(\Omega)$, the Sobolev space, $W^{1,p(\cdot)}(\Omega)$ are presented and used, for the main results. Section 3 proves the global existence of solutions for the problem (1.1)-(1.3) and next, in section 4, the blow-up of solutions with positive and negative initial energy are proved.

2 Preliminaries

To prove our results for the problem (1.1)-(1.3), we need to present some theories about the function spaces with variable-exponents as Lebesuge and Sobolev (See [1, 9]). Suppose that Ω is a subset of R^n and the function $p:\Omega\to[1,\infty]$ is measurable. The variable exponent Lebesgue space is defined by:

$$L^{p(x)}(\Omega)=\Big\{u\big|\text{ u is measurable in }\Omega\text{ and }\int_\Omega|\lambda u(x)|^{p(x)}dx<\infty\text{ for some }\lambda>0\Big\}.$$

The Lebesgue space, $L^{p(\cdot)}(\Omega)$, is equipped with the below Luxembourg-type norm:

$$\|u\|_{p(x)}:=\inf\Big\{\lambda>0\Big|\int_\Omega|\frac{u(x)}{\lambda}|^{p(x)}dx\le 1\Big\}.$$

Lemma 2.1. [9] Let Ω be a bounded domain in R^n

(i) the space $(L^{p(\cdot)}(\Omega),\|.\|_{p(\cdot)})$ is a Banach space, and its conjugate space is $L^{q(\cdot)}(\Omega)$, where $\frac{1}{q(x)}+\frac{1}{p(x)}=1$.

(ii) For any $f\in L^{p(\cdot)}(\Omega)$ and $g\in L^{q(\cdot)}(\Omega)$, the generalized Hölder inequality holds

$$\left|\int_\Omega fgdx\right|\le\left(\frac{1}{p_1}+\frac{1}{q_1}\right)\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}\le 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

The following formula is used to determine the relationship between the modular $\int_\Omega|f|^{p(x)}dx$ and the norm

$$\min(\|f\|^{p_1}_{p(\cdot)},\|f\|^{p_2}_{p(\cdot)})\le\int_\Omega|f|^{p(x)}dx\le\max(\|f\|^{p_1}_{p(\cdot)},\|f\|^{p_2}_{p(\cdot)}).$$

The variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega)=\{u\in L^{p(\cdot)}(\Omega):\nabla_x u\text{ exists and }|\nabla_x u|\in L^{p(\cdot)}(\Omega)\}.$$

This space is a Banach space with respect to the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)}=\|u\|_{p(\cdot)}+\|\nabla_x u\|_{p(\cdot)}.$$

Furthermore, let $W_0^{1,p(\cdot)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$. For $u\in W_0^{1,p(\cdot)}(\Omega)$, an equivalent norm is defined as

$$\|u\|_{1,p(\cdot)}=\|\nabla_x u\|_{p(\cdot)}.$$

Let the log-Hölder continuity condition be satisfied by the variable component $p(.)$

$$|p(x)-p(y)|\le\frac{-A}{\log|x-y|},\text{ for all }x,y\in\Omega\text{ with }|x-y|<\delta,$$

where $A>0$ and $0<\delta<1$.

Lemma 2.2. (Sobolev-Poincaré inequality) Suppose that Ω is a bounded domain of R^n and the log-Hölder condition is satisfied by $p(\cdot)$. Then we have $H_0^\alpha(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ and

$$\|u\|_{p(\cdot)} \le c_*\|D^\alpha u\|_{p(\cdot)}, \text{ for all } u \in H_0^\alpha(\Omega), \tag{2.1}$$

where $c_* = c(p_1, p_2, |\Omega|) > 0$.

For completeness, the local existence result for the problem (1.1)-(1.3) is stated as follows. This theorem could be proved by the Faedo-Galerkin approximation method that has been used [11, 12].

Theorem 2.3. (Local existence) Let $(u_0, u_1) \in H_0^\alpha(\Omega) \times L^2(\Omega)$ be given. Assume that (A1) and (A2) are satisfied; then the problem (1.1)-(1.3) has at least one weak solution such that

$$\begin{aligned} &u \in C\big((0,T), H_0^\alpha(\Omega)\big) \cap L^{p(\cdot)}(\Omega) \cap L^{q(\cdot)}(\Omega),\\ &u_t \in C\big((0,T), H_0^\alpha(\Omega)\big) \cap L^2(\Omega). \end{aligned}$$

The energy of the system is defined by

$$\begin{aligned} E(t) =& \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|D^\alpha u\|^2 + \int_\Omega \frac{1}{p(x)}|D^\alpha u|^{p(x)}dx \\ &+ \frac{1}{2}(g * D^\alpha u)(t) + \int_\Omega \frac{1}{q^2(x)}|u|^{q(x)}dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)} \ln|u|dx, \end{aligned} \tag{2.2}$$

where

$$(g * D^\alpha u)(t) = \int_0^t g(t-s)\|D^\alpha u(t) - D^\alpha u(s)\|^2 ds.$$

Lemma 2.4. (Monotonicity of energy) Assume that $u(x,t)$ be a local solution of (1.1)-(1.3). Then, along the solution, $E(t)$ is a nonincreasing functional.

Proof . Multiplying equation (1.1) by u_t and integrating it over Ω, we easily get

$$E'(t) = -\|u_t\|^2 - g(t)\|D^\alpha u\|^2 + \frac{1}{2}(g' * D^\alpha u)(t),$$

and by using hypotheses (A2), desired result could be obtained for any weak solution. □

3 Global existence

To prove the global existence of solutions for the problem (1.1)-(1.3), we define:

$$I(t) = \left(1 - \int_0^t g(s)ds\right)\|D^\alpha u\|^2 + (g * D^\alpha u)(t) + \int_\Omega |D^\alpha u|^{p(x)}dx - \int_\Omega |u|^{q(x)}\ln|u|dx, \tag{3.1}$$

and

$$\begin{aligned} J(t) =& \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|D^\alpha u\|^2 + \frac{1}{2}(g * D^\alpha u)(t) + \int_\Omega \frac{1}{p(x)}|D^\alpha u|^{p(x)}dx \\ &+ \int_\Omega \frac{1}{q^2(x)}|u|^{q(x)}dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\ln|u|dx. \end{aligned} \tag{3.2}$$

From the definitions (3.1) and (3.2), we have $E(t) = \frac{1}{2}\|u_t\|^2 + J(t)$.

Lemma 3.1. Under the assumptions of Theorem 2.3, we assume that $I(0) > 0$, $p_2 < q_1$ and moreover, for some $\theta > 0$

$$\gamma_1 := \frac{C_{*,\theta}^\theta}{e\theta}\left(\frac{E(0)}{\gamma_0\ell}\right)^{\frac{\theta-2}{2}} \max\left\{C_{*,\theta}^{q_1}\left(\frac{E(0)}{\gamma_0\ell}\right)^{\frac{q_1}{2}},\ C_{*,\theta}^{q_2}\left(\frac{E(0)}{\gamma_0\ell}\right)^{\frac{q_2}{2}}\right\} < \ell, \tag{3.3}$$

where $C_{*,\theta}$ is the best constant of the embedding $H^{\alpha}(\Omega) \hookrightarrow L^{q(\cdot)+\theta}(\Omega)$ and

$$\gamma_0 := \min\left\{\frac{1}{q_2^2},\ \frac{q_1-2}{2q_1},\ \frac{q_1-p_2}{p_2q_1}\right\} < \frac{1}{2}.$$

Then, we have

$$I(t) > 0, \qquad \text{for all } t \in [0,T].$$

Proof . By using the continuity of $u(t)$ and since $I(0) > 0$, thus there exists a time $T^* < T$ such that

$$I(t) \geq 0, \qquad \text{for all } t \in [0,T^*].$$

Now, under the condition (A1) and from the definition of $J(t)$ in (3.2), we deduce that

$$\begin{aligned} J(t) \geq & \frac{1}{2}(1-\int_0^t g(s)ds)\|D^{\alpha}u\|^2 + \frac{1}{2}(g * D^{\alpha}u)(t) + \frac{1}{p_2}\int_{\Omega}|D^{\alpha}u|^{p(x)}dx \\ & + \frac{1}{q_2^2}\int_{\Omega}|u|^{q(x)}dx - \frac{1}{q_1}\int_{\Omega}|u|^{q(x)}\ln|u|dx. \end{aligned}$$

From the definition of $I(t)$, we obtain

$$\begin{aligned} J(t) \geq & \frac{q_1-2}{2q_1}(1-\int_0^t g(s)ds)\|D^{\alpha}u\|^2 + \frac{q_1-2}{2q_1}(g * D^{\alpha}u)(t) \\ & + \frac{q_1-p_2}{p_2q_1}\int_{\Omega}|D^{\alpha}u|^{p(x)}dx + \frac{1}{q_2^2}\int_{\Omega}|u|^{q(x)}dx + \frac{1}{q_1}I(t), \forall t \in [0,T^*]. \end{aligned}$$

Thanks to the assumptions of Lemma 3.1 on the variable-exponents and γ_0, we get

$$J(t) \geq \gamma_0\Big((1-\int_0^t g(s)ds)\|D^{\alpha}u\|^2 + (g * D^{\alpha}u)(t) + \int_{\Omega}|D^{\alpha}u|^{p(x)}dx + \int_{\Omega}|u|^{q(x)}dx\Big). \tag{3.4}$$

Inequality (3.4) yields

$$E(0) \geq E(t) \geq J(t) \geq \gamma_0(1-\int_0^t g(s)ds)\|D^{\alpha}u\|^2 \geq \gamma_0\ell\|D^{\alpha}u\|^2,$$

and therefore, we have

$$\|D^{\alpha}u\|^2 \leq \frac{E(0)}{\gamma_0\ell}. \tag{3.5}$$

Suppose that

$$\Omega_1 = \{u | u \in W_0^{1,q(x)}(\Omega), 0 < u < 1\}, \quad \text{and} \quad \Omega_2 = \{u | u \in W_0^{1,q(x)}(\Omega), u \geq 1\},$$

and thus, it is easy to see that for any $u \in \Omega_2$ and $\theta > 0$, we have $0 \leq u^{-\theta}\ln u \leq \frac{1}{e\theta}$. Consequently, we deduce

$$\begin{aligned} \int_{\Omega}|u|^{q(x)}\ln|u|dx = & \int_{\Omega_1}|u|^{q(x)}\ln|u|dx + \int_{\Omega_2}|u|^{q(x)}\ln|u|dx \\ \leq & \int_{\Omega_2}|u|^{q(x)}\ln|u|dx \\ \leq & \frac{1}{e\theta}\int_{\Omega_2}|u|^{q(x)+\theta}dx \leq \frac{1}{e\theta}\int_{\Omega}|u|^{q(x)+\theta}dx \\ \leq & \frac{1}{e\theta}\max\{\left(\int_{\Omega}|u|^{q(x)+\theta}dx\right)^{q_1+\theta}, \left(\int_{\Omega}|u|^{q(x)+\theta}dx\right)^{q_2+\theta}\} \\ \leq & \frac{1}{e\theta}\max\{C_{*,\theta}^{q_1+\theta}\|D^{\alpha}u\|^{q_1+\theta}, C_{*,\theta}^{q_2+\theta}\|D^{\alpha}u\|^{q_2+\theta}\} \\ = & \frac{1}{e\theta}\max\{C_{*,\theta}^{q_1+\theta}\|D^{\alpha}u\|^{q_1+\theta-2}, C_{*,\theta}^{q_2+\theta}\|D^{\alpha}u\|^{q_2+\theta-2}\}\|D^{\alpha}u\|^2 \\ \leq & \frac{1}{e\theta}\max\{C_{*,\theta}^{q_1+\theta}\left(\frac{E(0)}{\gamma_0\ell}\right)^{\frac{q_1+\theta-2}{2}}, C_{*,\theta}^{q_2+\theta}\left(\frac{E(0)}{\gamma_0\ell}\right)^{\frac{q_2+\theta-2}{2}}\}\|D^{\alpha}u\|^2. \end{aligned} \tag{3.6}$$

where $C_{*,\theta}$ is the best constant of embedding $H^{\alpha}(\Omega) \hookrightarrow L^{q(x)+\theta}(\Omega)$ and (3.5) has been used. Utilizing (3.3) into (3.6), we obtain

$$\int_{\Omega} |u|^{q(x)} \ln |u| dx \leq \gamma_1 \|D^{\alpha}u\|^2. \tag{3.7}$$

At this point, we shall prove that $I(t) > 0,\ \forall t \in [0, T^*]$. For this goal, we have

$$\begin{aligned} I(t) =& \left(1 - \int_0^t g(s)ds\right) \|D^{\alpha}u\|^2 + (g * D^{\alpha}u)(t) + \int_{\Omega} |D^{\alpha}u|^{p(x)} dx - \int_{\Omega} |u|^{q(x)} \ln |u| dx \\ \geq& (\ell - \gamma_1)\|D^{\alpha}u\|^2 + (g * D^{\alpha}u)(t) + \int_{\Omega} |D^{\alpha}u|^{p(x)} dx, \end{aligned} \tag{3.8}$$

where $(A2)$ and (3.7) have been used. Therefore, since $\gamma_1 < \ell$ we deduce

$$I(t) > 0, \quad \forall t \in [0, T^*].$$

By repeating this procedure, T^* extended to T. $\square$ At this point, we state and prove the global existence result as follows:

Theorem 3.2. Let $u(x,t)$ be the local solution of (1.1)-(1.3). Under the assumption of Lemma 3.1, $u(x,t)$ is global.

Proof . By virtue of (3.2) and(3.4), we obtain

$$\begin{aligned} E(t) =& \frac{1}{2}\|u_t\|^2 + J(t) \\ \geq& \frac{1}{2}\|u_t\|^2 + \gamma_0 \left(\ell\|D^{\alpha}u\|^2 + (g * D^{\alpha}u)(t) + \int_{\Omega} |D^{\alpha}u|^{p(x)} dx + \int_{\Omega} |u|^{q(x)} dx\right) \\ \geq& \gamma_0 \ell \left(\|u_t\|^2 + \|D^{\alpha}u\|^2 + (g * D^{\alpha}u)(t) + \int_{\Omega} |D^{\alpha}u|^{p(x)} dx + \int_{\Omega} |u|^{q(x)} dx\right). \end{aligned}$$

Thus, we get

$$\|u_t\|^2 + \|D^{\alpha}u\|^2 + (g * D^{\alpha}u)(t) + \int_{\Omega} |D^{\alpha}u|^{p(x)} dx + \int_{\Omega} |u|^{q(x)} dx \leq \frac{E(t)}{\gamma_0 \ell}.$$

Hence, considering the nonincreasingness of $E(t)$, we get

$$\|u_t\|^2 + \|D^{\alpha}u\|^2 + (g * D^{\alpha}u)(t) + \int_{\Omega} |D^{\alpha}u|^{p(x)} dx + \int_{\Omega} |u|^{q(x)} dx \leq \frac{E(0)}{\gamma_0 \ell}. \tag{3.9}$$

and this shows that the local solution $u(x,t)$ of (1.1)-(1.3) is global and bounded. $\square$

4 Blow-up results

This section aims at proving the blow-up of solutions for the problem (1.1)-(1.3). Firstly, we show that the solutions with positive initial energy blow-up at infinite time when $q_1 > \max\{p_2, 2 + (\varepsilon+1)^2 c_*^2\}$ and $\int_0^{\infty} g(s)ds \leq \min\{\frac{q_1(q_1-2)}{q_1^2-2q_1+2}, \frac{q_1-2-(\varepsilon+1)^2c_*^2}{q_1-2}\}$, where ε is a small enough positive number and c_* is the best constant in Sobolev-Poincaré inequality (Lemma 2.2). Next, we shall prove that the solutions with negative initial energy blow-up at a finite time when $q_1 > \max\{p_2, 2\varepsilon + \frac{c_*^2}{2}\}$ and $\int_0^{\infty} g(s)ds \leq \frac{q_1 - 2\varepsilon - \frac{c_*^2}{2}}{q_1}$, where ε is a positive number satisfying $1 < \varepsilon < \frac{q_1}{p_2}$.

4.1 blow-up at infinite time with $E(0) > 0$

In this part we are going to prove the blow up of solutions for the problem (1.1)-(1.3) with positive initial energy. To prove this result, we assume that:

(B1) For sufficiently small $\varepsilon > 0$

$$q_1 > \max\{p_2, 2 + (\varepsilon + 1)^2 c_*^2\},$$
$$\int_0^\infty g(s)ds \le \min\{\frac{q_1(q_1 - 2)}{q_1^2 - 2q_1 + 2}, \frac{q_1 - 2 - (\varepsilon + 1)^2 c_*^2}{q_1 - 2}\},$$

where c_* is the best constant in Sobolev-Poincaré inequality (Lemma 2.2). Our main result with positive initial energy reads in the following theorem:

Theorem 4.1. Suppose that the assumptions of Theorem 2.3 and (B1) hold. Moreover, $E(0) > 0$ is a given initial energy level. If we choose initial data u_0, u_1 satisfying

$$\int_\Omega u_0(x)u_1(x)dx \ge \frac{q_1}{\varepsilon} E(0), \tag{4.1}$$

then the solution of (1.1)-(1.3) blows up at infinite time, i.e.

$$\lim_{t\to+\infty} \left(\|D^\alpha u\|^2 + \|u_t\|^2\right) = +\infty.$$

Proof . Define

$$H(t) = \int_\Omega u u_t dx - \frac{q_1}{\varepsilon} E(t), \tag{4.2}$$

where $E(t)$ satisfy (2.2) and ε is a small enough positive constant. Differentiating $H(t)$, we obtain

$$\begin{aligned} H'(t) =& \|u_t\|^2 + \int_\Omega u u_{tt} dx - \frac{q_1}{\varepsilon} E'(t) \\ \ge& \|u_t\|^2 + \int_\Omega u u_{tt} dx, \end{aligned} \tag{4.3}$$

where Lemma 2.4 has been used. By using (A1) and for any $\varepsilon > 0$, (4.3) is rewritten as:

$$\begin{aligned} H'(t) \ge& \varepsilon H(t) + (1 + \frac{q_1}{2})\|u_t\|^2 + (\frac{q_1}{2} - 1)(1 - \int_0^t g(s)ds)\|D^\alpha u\|^2 \\ &+ (\frac{q_1}{p_2} - 1)\int_\Omega |D^\alpha u|^{p(x)}dx + \frac{q_1}{2}(g * D^\alpha u)(t) + \frac{q_1}{q_2^2}\int_\Omega |u|^{q(x)}dx \\ &- (\varepsilon + 1)\int_\Omega u u_t dx + \int_0^t g(t - s)\int_\Omega D^\alpha u(D^\alpha u(s) - D^\alpha u)dxds. \end{aligned} \tag{4.4}$$

At this point, taking into account Young and Sobolev-Poincaré inequalities to get the following estimates:

$$\begin{aligned} \left|\int_\Omega u u_t dx\right| \le& \frac{\varepsilon + 1}{4}\|u\|^2 + \frac{1}{\varepsilon + 1}\|u_t\|^2 \\ \le& \frac{c_*^2(\varepsilon + 1)}{4}\|D^\alpha u\|^2 + \frac{1}{\varepsilon + 1}\|u_t\|^2. \end{aligned} \tag{4.5}$$

$$\left|\int_0^t g(t - s)\int_\Omega D^\alpha u(D^\alpha u(s) - D^\alpha u)dxds\right| \le \frac{(q_1 - 2)\ell}{4}\|D^\alpha u\|^2 + \frac{1 - \ell}{(q_1 - 2)\ell}(g * D^\alpha u)(t). \tag{4.6}$$

Applying (4.5) and (4.6) into (4.4) to obtain

$$\begin{aligned} H'(t) \ge& \varepsilon H(t) + q_1\|u_t\|^2 + \left[\frac{(q_1 - 2)\ell}{4} - \frac{(\varepsilon + 1)^2 c_*^2}{4}\right]\|D^\alpha u\|^2 \\ &+ (\frac{q_1}{p_2} - 1)\int_\Omega |D^\alpha u|^{p(x)}dx + \left(\frac{q_1}{2} - \frac{1 - \ell}{(q_1 - 2)\ell}\right)(g * D^\alpha u)(t) + \frac{q_1}{q_2^2}\int_\Omega |u|^{q(x)}dx, \end{aligned} \tag{4.7}$$

where (A2) has been used. Now, we apply assumptions of Theorem 4.1, since we have $H(0) > 0$, therefore, inequality (4.7) get to us

$$H'(t) \geq \varepsilon H(t), \tag{4.8}$$

and hence we conclude that

$$H(t) \geq e^{\varepsilon t} H(0), \quad \forall t \geq 0.$$

This shows that the functional $H(t)$ exponentially growth when time goes to infinity.
It is easy to see that

$$e^{\varepsilon t} H(0) \leq H(t) = \int_\Omega u u_t dx - \frac{q_1}{\varepsilon} E(t) \leq \int_\Omega u u_t dx.$$

Thanks to the estimation (4.5), thus there exists a constant C such that

$$e^{\varepsilon t} H(0) \leq H(t) \leq C \left(\|D^\alpha u\|^2 + \|u_t\|^2 \right). \tag{4.9}$$

Thus inequality (4.9) shows that

$$\lim_{t \to +\infty} \left(\|D^\alpha u\|^2 + \|u_t\|^2 \right) = +\infty,$$

and proof of Theorem 4.1 is completed. □

4.2 blow-up at finite time with $E(0) < 0$

In the last part we shall prove the blow up of solutions for the problem (1.1)-(1.3) with negative initial energy. For this goal, we assume that:
(B2) For any $1 < \varepsilon < \frac{q_1}{p_2}$

$$q_1 > \max\{p_2, 2\varepsilon + \frac{c_*^2}{2}\},$$
$$\int_0^\infty g(s) ds \leq \frac{q_1 - 2\varepsilon - \frac{c_*^2}{2}}{q_1},$$

where c_* is the best constant in Sobolev-Poincaré inequality (Lemma 2.2). Now we are in a position to state and prove our blow-up result as follows:

Theorem 4.2. Let the conditions of Theorem 2.3 and (B2), are satisfied. Assume that $E(0) < 0$. Then the solutions of the problem (1.1)-(1.3) blows up in finite time T^*, and

$$T^* \leq \frac{1 - \sigma}{\eta \sigma \Gamma^{\frac{\sigma}{1-\sigma}}(0)},$$

where $0 < \sigma < 1$ and $\Gamma(t)$ is given in (4.12).

Proof . Define $\psi(t) = -E(t)$ and thus by using Monotonicity of energy Lemma 2.4, we arrive at

$$\psi'(t) = -E'(t) \geq \|u_t\|^2, \tag{4.10}$$

then negative initial energy and (4.10) gives $\psi(t) \geq \psi(0) > 0$. Also, by definition $\psi(t)$, it is easy to see that

$$\psi(t) \leq \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \ln |u| dx \leq \frac{1}{q_1} \int_\Omega |u|^{q(x)} \ln |u| dx. \tag{4.11}$$

Define for $0 < \sigma < 1$

$$\Gamma(t) = \psi^{1-\sigma}(t) + \varepsilon \int_\Omega u u_t dx, \tag{4.12}$$

where ε is a positive constant satisfying (B2). By taking a derivative of (4.12) and using (1.1), we get

$$\begin{aligned}\Gamma'(t) =&(1-\sigma)\psi'(t)\psi^{-\sigma}(t)+\varepsilon\|u_t\|^2+\varepsilon\int_\Omega uu_{tt}dx\\ =&(1-\sigma)\psi'(t)\psi^{-\sigma}(t)+\varepsilon\|u_t\|^2-\varepsilon(1-\int_0^t g(s)ds)\|D^\alpha u\|^2\\ &-\varepsilon\int_\Omega |D^\alpha u|^{p(x)}dx+\varepsilon\int_\Omega |u|^{q(x)}\ln|u|dx-\underbrace{\varepsilon\int_\Omega uu_tdx}_{I_1}\\ &+\underbrace{\varepsilon\int_0^t g(t-s)\int_\Omega D^\alpha u(D^\alpha u(s)-D^\alpha u)dxds}_{I_2}.\end{aligned} \tag{4.13}$$

By using (A2), Young and Sobolev-Poincaré inequalities, we could estimate I_1 and I_2 as

$$|I_1| \leq \varepsilon\|u_t\|^2+\frac{c_*^2}{4\varepsilon}\|D^\alpha u\|^2, \tag{4.14}$$

and

$$\begin{aligned}|I_2| \leq&\varepsilon(1-\ell)\|D^\alpha u\|^2+\frac{\varepsilon}{4(1-\ell)}\int_\Omega\left(\int_0^t g(t-s)\left(D^\alpha u(s)-D^\alpha u\right)ds\right)^2dx\\ =&\varepsilon(1-\ell)\|D^\alpha u\|^2+\frac{\varepsilon}{4(1-\ell)}\int_\Omega\left(\int_0^t \sqrt{g(t-s)}\sqrt{g(t-s)}\left(D^\alpha u(s)-D^\alpha u\right)ds\right)^2dx\\ \leq&\varepsilon(1-\ell)\|D^\alpha u\|^2+\frac{\varepsilon}{4(1-\ell)}\int_\Omega\left(\int_0^t g(t-s)ds\right)\left(\int_0^t g(t-s)ds\left(D^\alpha u(s)-D^\alpha u\right)^2ds\right)dx\\ \leq&\varepsilon(1-\ell)\|D^\alpha u\|^2+\frac{\varepsilon}{4}(g*D^\alpha u)(t).\end{aligned} \tag{4.15}$$

Combination of (4.14) and (4.15) into (4.13), yields

$$\begin{aligned}\Gamma'(t) \geq&(1-\sigma)\psi'(t)\psi^{-\sigma}(t)-\varepsilon\int_\Omega |D^\alpha u|^{p(x)}dx-\varepsilon\big[(1-\int_0^t g(s)ds)+1-\ell+\frac{c_*^2}{4\varepsilon}\big]\|D^\alpha u\|^2\\ &-\frac{\varepsilon}{4}(g*D^\alpha u)(t)+\varepsilon\int_\Omega |u|^{q(x)}\ln|u|dx.\end{aligned}$$

Taking into account the definition of the $\psi(t)$, conditions (A1) and (A2), it follows that

$$\begin{aligned}\Gamma'(t) \geq& q_1\psi(t)+(1-\sigma)\psi'(t)\psi^{-\sigma}(t)+\frac{q_1}{2}\|u_t\|^2+\left[(\frac{q_1}{2}-\varepsilon)(1-\int_0^t g(s)ds)-\varepsilon(1-\ell)-\frac{c_*^2}{4}\right]\|D^\alpha u\|^2\\ &+\frac{2q_1-\varepsilon}{4}(g*D^\alpha u)(t)+(\frac{q_1}{p_2}-\varepsilon)\int_\Omega |D^\alpha u|^{p(x)}dx+(\varepsilon-1)\int_\Omega |u|^{q(x)}\ln|u|dx.\end{aligned} \tag{4.16}$$

On the other hand, by using (4.11) we have

$$(\varepsilon-1)\int_\Omega |u|^{q(x)}\ln|u|dx \geq q_1(\varepsilon-1)\psi(t). \tag{4.17}$$

Therefore, we get

$$\begin{aligned}\Gamma'(t) \geq& q_1\varepsilon\psi(t)+(1-\sigma)\psi'(t)\psi^{-\sigma}(t)+\frac{q_1}{2}\|u_t\|^2\\ &+\big[(\frac{q_1}{2}-\varepsilon)(1-\int_0^t g(s)ds)-\varepsilon(1-\ell)-\frac{c_*^2}{4}\big]\|D^\alpha u\|^2\\ &+\frac{2q_1-\varepsilon}{4}(g*D^\alpha u)(t)+(\frac{q_1}{p_2}-\varepsilon)\int_\Omega |D^\alpha u|^{p(x)}dx.\end{aligned} \tag{4.18}$$

From the assumption of Theorem 4.2, we know that $q_1 > 2\varepsilon$ and therefore we have

$$\begin{aligned}(\frac{q_1}{2}-\varepsilon)(1-\int_0^t g(s)ds)-\varepsilon(1-\ell)-\frac{c_*^2}{4} \geq&(\frac{q_1}{2}-\varepsilon)\ell-\varepsilon(1-\ell)-\frac{c_*^2}{4}\\ =&\frac{q_1\ell}{2}-\varepsilon-\frac{c_*^2}{4}\end{aligned} \tag{4.19}$$

Using (4.19) into (4.18), we deduce

$$\begin{aligned}\Gamma'(t) \geq& q_1\varepsilon\psi(t)+(1-\sigma)\psi'(t)\psi^{-\sigma}(t)+\frac{q_1}{2}\|u_t\|^2+\left(\frac{q_1\ell}{2}-\varepsilon-\frac{c_*^2}{4}\right)\|D^\alpha u\|^2\\ &+\frac{2q_1-\varepsilon}{4}(g*D^\alpha u)(t)+(\frac{q_1}{p_2}-\varepsilon)\int_\Omega |D^\alpha u|^{p(x)}dx.\end{aligned}$$

At this point, assumption (B2) guaranties that there exists positive constant c_1 such that

$$\Gamma'(t) \geq q_1\varepsilon\psi(t)+c_1(\|u_t\|^2+\|D^\alpha u\|^2). \tag{4.20}$$

Therefore we deduce that $\Gamma(t) \geq \Gamma(0) > 0$, for all $t \geq 0$. Now, by using the definition of $\Gamma(t)$, Hölder and Young inequalities, we have

$$\begin{aligned}\Gamma^{\frac{1}{1-\sigma}}(t) =&\left[\psi^{1-\sigma}(t)+\varepsilon\int_\Omega uu_tdx\right]^{\frac{1}{1-\sigma}}\\ \leq& 2^{\frac{1-\sigma}{\sigma}}\left(\psi(t)+\varepsilon^{\frac{1}{1-\sigma}}|\int_\Omega uu_tdx|^{\frac{1}{1-\sigma}}\right)\\ \leq& 2^{\frac{1-\sigma}{\sigma}}\left(\psi(t)+c_2\|D^\alpha u\|^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}}\right)\\ \leq& 2^{\frac{1-\sigma}{\sigma}}\left[\psi(t)+c_3(\|u_t\|^2+\|D^\alpha u\|^2)\right]\\ \leq& \eta^{-1}\Gamma'(t),\end{aligned}$$

where (4.20) has been used and

$$\eta := \frac{\min\{q_1\varepsilon,\ c_1\}}{2^{\frac{1-\sigma}{\sigma}}\max\{1,c_3\}}.$$

Therefore

$$\Gamma'(t) \geq \eta\Gamma^{\frac{1}{1-\sigma}}(t). \tag{4.21}$$

Integrating (4.21) from 0 to t, we deduce

$$\Gamma^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Gamma^{-\frac{\sigma}{1-\sigma}}(0)-\frac{\eta\sigma t}{1-\sigma}}.$$

This shows that solutions blow up in finite time $T^* = \frac{1-\sigma}{\eta\sigma\Gamma^{\frac{\sigma}{1-\sigma}}(0)}$, and proof of Theorem 4.2 has been completed. □

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