Int. J. Nonlinear Anal. Appl. **2** (2011) No.2, 75–85
ISSN: 2008-6822 (electronic)
http://www.ijnaa.com

A CLASS OF NONLINEAR (A, η)-MONOTONE OPERATOR INCLUSION PROBLEMS WITH ITERATIVE ALGORITHM AND FIXED POINT THEORY

M. ALIMOHAMMADY [1] AND M. KOOZEHGAR KALLEGI[2]

Abstract. A new class of nonlinear set-valued variational inclusions involving (A, η)-monotone mappings in a Banach space setting is introduced, and then based on the generalized resolvent operator technique associated with (A, η)-monotonicity, the existence and approximation solvability of solutions using an iterative algorithm and fixed pint theory is investigated.

1. Introduction

Resolvent operator techniques have been applied to studying nonlinear variational inequality/inclusion problems, including problems from model equilibria in economics, optimization and control theory, operations research, transportation network modeling, mathematical programming, and engineering sciences. Recently, Agarwal et al. [5,6] applied the resolvent operator technique to studying sensitivity analysis for qusi-variational inclusions involving strongly monotone mappings, without any differentiability assumptions on solution variables with respect to perturbation parameters. The aim of this paper is to present the sensitivity analysis for the relaxed cocoercive quasi-variational inclusions based on an application of the generalized resolvent operator technique.The framework of the generalized resolvent operator technique heavily relies on A-monotonicityjust recently introduced by the author [8] is more general the existing general class of maximal monotone mappings, so it further empowers the resolvent operator technique. Furthermore, the class of A-monotone mappings generalizes recently introduced and studied notion of the H-monotone mappings by Fang and Huang [14,15,16,19] as well.In recently years, much attention has been given to develop general techniques for the sensitivity analysis of solution set of various classes of variational inequalities (inclusions)point of view, sensitivity properties of various classes of variational inequalities can provide new insight concerning the problem being studied and can stimulate ideas for solving problems. The sensitivity analysis of solution set for variational inequalities have been studied extensively by many authors using quite different techniques. By using the projection technique, Defermos [11], studied the sensitivity analysis of solution

Date: Received:January 2011; Revised: June 2011.
2000 *Mathematics Subject Classification.* Primary 47H05; Secondary 47J20, 49J40.
Key words and phrases. (A, η)-monotonicity, δ-Lipschitz, (H, η)-monotone operator.
.

of some classes of variational inequalities with single-valued mappings.By using resolvent operator technique, Adly [1] and Agarwal et al. [5] studied the sensitivity analysis of solution of some classes of quasi-variational inclusions involving single-valued mappings.

2. Preliminaries

We assume that X is real Banach space equipped with norm $\|.\|$, X^* is the topological dual space of X, $C(X)$ is the family of all nonempty compact subsets of X, $\langle .,. \rangle$ is the dual pair between X and X^* and 2^X is the power set of X, $H(.,.)$ is the Hausdroff metric on $C(X)$, defined by

$$H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} d(x,y), \sup_{y\in B}\inf_{x\in A} d(x,y)\},$$

where $A, B \in C(X)$, assume that $J : X \to 2^{X^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|x\| = \|f\|\},$$

$x \in X$.

Definition 2.1. A mapping $\eta : X \times X \to X$ is said to be

(i) monotone if

$$\langle x - y, \eta(x,y)\rangle \geq 0$$

(ii) strictly monotone if

$$\langle x - y, \eta(x,y)\rangle \geq 0$$

and equality holds if and only if $x = y$.

(iii) δ-strongly monotone if there exists $\delta > 0$ such that

$$\langle x - y, \eta(x,y)\rangle \geq \delta\|x - y\|^2.$$

(iv)Lipschitz continuous if there exists $\tau > 0$ such that

$$\|\eta(x,y)\| \leq \tau\|x - y\|.$$

Definition 2.2. Let $\eta : X \times X \to X^*$ be a single-valued map. Then a mapping $A : X \to X$ is said to be

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0$$

for any $x, y \in X$

(ii) η-monotone if

$$\langle Ax - Ay, \eta(x,y)\rangle \geq 0$$

for any $x, y \in X$

(iii) strictly η-monotone if

$$\langle Ax - Ay, \eta(x,y)\rangle \geq 0$$

for any $x, y \in X$ and equality holds if and only if $x = y$

(iv)δ- strongly η-monotone if there exists $\delta > 0$ such that

$$\langle Ax - Ay, \eta(x,y)\rangle \geq \delta\|x - y\|^2$$

for any $x, y \in X$.

Definition 2.3. [19] Let $T, A : x \to X$ be a single-valued operator. T is said to be

(i) monotone if
$$\langle T(x) - T(y), x - y\rangle \geq 0, \forall x, y \in X;$$

(ii) r-strongly monotone if, there exists a constant $r > 0$ such that
$$\langle T(x) - T(y), x - y\rangle \geq r\|x - y\|^2, \forall x, y \in X;$$

(iii) m-relaxed cocoercive with respect to A, if there exists a constant $m > 0$ such that
$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq (-m)\|T(x) - T(y)\|^2, \forall x, y \in X;$$

(iv) (ϵ, α)-relaxed cocoercive with respect to A, if there exist constants$\epsilon, \alpha > 0$ such
$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq (-\epsilon)\|T(x) - T(y)\|^2 + \alpha\|x - y\|^2, \forall x, y \in X;$$

Definition 2.4. [18]

Let $\eta : X \times X \to X^*$ and $A, H : X \to X$ be single-valued mappings. A set-valued mapping $M : X \to 2^X$ is said to be:

(i) monotone if
$$\langle u - v, x - y\rangle \geq 0$$
for any $x, y \in X$ and $u \in Mx, v \in My$.

(ii) η-monotone if
$$\langle u - v, \eta(x, y)\rangle \geq 0$$
for any $x, y \in X$ and $u \in Mx, v \in My$.

(iii) strictly η-monotone if
$$\langle u - v, \eta(x, y)\rangle \geq 0$$
for any $u \in Mx, v \in My$ except for $x = y$;

(iv) r-strongly η-monotone, if there exists a positive constant r such that if
$$\langle u - v, \eta(x, y)\rangle \geq r\|x - y\|^2$$
for all $x, y \in X$ and $u \in Mx, v \in My$

(v) (m, η)-relaxed monotone, if there is a positive constant m such that
$$\langle u - v, \eta(x, y)\rangle \geq (-m)\|x - y\|^2$$
for all $x, y \in X$ and $u \in Mx, v \in My$

(vi) maximal monotone, if M is monotone and $(I + \lambda M)(X) = X$ for all $\lambda > 0$ where I denotes the identity mapping on X;

(vii) maximal η-monotone, if M is η-monotone and $(I+\lambda M)(X) = X$ for all $\lambda > 0$

(viii) A-monotone, if M is (m)-relaxed monotone and $(A + \lambda M)(X) = X$ for all $\lambda > 0$

(ix) (A, η)-monotone, if M is (m, η)-relaxed monotone and $(A+\lambda M)(X) = X$ for all $\lambda > 0$

(x) H-monotone, if M is monotone and $(H + \lambda M)(X) = X$ for all $\lambda > 0$

(xi) (H, η)-monotone, if M is η-monotone and $(H + \lambda M)(X) = X$ for all $\lambda > 0$

Definition 2.5. Let $\eta : X \times X \to X$ be a single-valued mapping, $A : X \to X$ be a strictly η-monotone mapping and $M : X \to 2^X$ be an (A, η)-monotone mapping. The resolvent operator $J^{A,\eta}_{M,\lambda} : X \to X$ is defined by

$$J^{A,\eta}_{M,\lambda}(z) = (A + \lambda M)^{-1}(z)$$

Theorem 2.6. [11] *Let $A : X \to X$ be an $r-$strongly η-monotone operator and $M : X \to 2^X$ a $A - \eta$-monotone set-valued map. Then*

(a) $\langle u - v, \eta(x, y)\rangle \geq 0$ for any $(v, y) \in Graph(M)$ implies $(u, x) \in Graph(M)$.

(b) the map $(A + \lambda M)^{-1}$ is single-valued for all $\lambda > 0$.

Proposition 2.7. [18] *Let $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping,$A : X \to X$ be an (r, η)-strongly monotone mapping and $M : X \to 2^X$ be an (A, η)-monotone mapping. Then the resolvent operator $J^{A,\eta}_{M,\lambda} : X \to X$ is $\frac{\tau}{r-\lambda m}$ Lipschitz continuous, that is,*

$$\|J^{A,\eta}_{M,\lambda}(x) - J^{A,\eta}_{M,\lambda}(y)\| \leq \frac{\tau}{r - \lambda m}\|x - y\|$$

for all $x, y \in X$.

Proposition 2.8. [12] *Let X be a real uniformly smooth Banach space and let $J : X \to X^*$ be be the normalized duality mapping.Then for all $x, y \in X$ we have*

(a) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle$;

(b) $\langle x - y, Jx - Jy\rangle \leq 2d^2\rho_X(t)(4\|x - y\|/d)$ where , $d = \sqrt{(\|x\|^2 + \|y\|^2)/2}$,

$$\rho_X(t) = \sup\{((\|x\| + \|y\|)/2) - 1 : \|x\| = 1, \|y\| = t\}$$

is called the modulus of smoothness of X.

3. Parametric quasi-variational-like inclusions problem.

In this section we shall introduce a parametric generalized implicit quasi- variational-like inclusions involving (A, η)-monotone operators in Banach spaces. In what follows, unless otherwise specified, we always assume that X is real Banach spaces.

Let Ω be nonempty open subsets of X in which the parameter ω take value,let $A : X \to X, \eta : X \times X \to X, S, T : X \times X \times \Omega \to X$ and $g, m : X \times \Omega \to X$ be single-valued mappings. Suppose that $g \neq 0$ and let $B, C, D, E, F : X \times \Omega \to C(X)$ be set-valued mappings. Suppose that $M : X \times X \times \Omega \to 2^X$ such that $(p, \omega) \in X \times \Omega$, $M(., p, \omega) : X \to 2^X$ is (A, η)-monotone and Rang$(g - m)(x, \omega) \in domain M(., p, \omega)$ for any $(x, \omega) \in X \times \Omega$.
For each $(f, \omega) \in X \times \Omega$ and we consider the parametric quasi-variational-like inclusion problem (PQVLIP for short):
for each $(f, \omega) \in X \times \Omega$ and find (x, y, z, r, u, v) such that $y \in B(x, \omega)$, $z \in C(x, \omega), r \in D(x, \omega), u \in E(x, \omega), v \in F(x, \omega)$

$$f \in S(y, z, \omega) - T(r, u, \lambda) + M((g - m)(x, \omega), v, \omega)$$

Some special cases of PQVLIP:

I. If $X = H$ is a Hilbert space, $A = I$ the identity map, and $\eta(x, t) = x - t$ for any $t \in H$, then PQVLIP reduces to the problem of finding (x, y, z, r, u, v) such that $y \in B(x, \omega), z \in C(x, \omega), r \in D(x, \omega)$, $u \in E(x, \omega), v \in F(x, \omega)$ and

$$f \in S(y, z, \omega) - T(r, u, \lambda) + M((g - m)(x, \omega), v, \omega)$$

which has been studied by Liu [15].

II. If $f = 0$, $u = T(u, v, \lambda), B = C = D = E = F = g = m$, then PQVLIP reduces to the problem of finding (x, y, z, r, u, v) such that $y \in B(x, \omega), z \in C(x, \omega), r \in D(x, \omega), u \in E(x, \omega), v \in F(x, \omega)$ and

$$0 \in S(y, z, \omega) - u + M((g - m)(x, \omega), v, \omega)$$

which has been studied by Heng-you Lan [12].

III. If $X = H$ a Hilbert space, $B = I$ is an identity and $\eta(x, t) = x - t$ for any $t \in H$, $T_1 = 0$ zero map ,$f = 0$ then PQVLIP reduces to the problem of finding (x, y, z, r, s) such that $y \in B(x, \omega), z \in C(x, \omega), v \in F(x, \omega)$ such that $(g - m)(x) \in domain M(.)$ and

$$0 \in S(y, z) + M(v)$$

which has been studied by verma [18].

IV. If $X = H, B = I, f = 0$ and $T = C = D = E = F = m = 0$ and $B(x, \omega) = x, S(y, z, \omega) = S(y, \omega), M(x, y, \omega) = W(x, \omega)$, where $S : H \times \Omega \to H, W : H \times \Omega \to 2^H$ then PQVLIP reduces to the problem of finding $x = x(\omega) \in H$ such that $g(x, \omega) \in domain M(., \omega)$ and

$$0 \in S(x, \omega) + W(g(x, \omega), \omega)$$

which has been studied by Adly [1].

Definition 3.1. A set-valued mapping $B : X \times \Omega \to C(X)$ is said to be s-H-Lipschitz continuous, if there exists constant $s > 0$ such that

$$H(B(x_1, \omega), B(x_2, \omega)) \leq s\|x_1 - x_2\|$$

4. Existence and convergence

In this section we will prove existence for solution of problem. We have the following characterization of solution of problem.

Lemma 4.1. *Let X be a real Banach space then the following conclusions are equivalent to each other:*
(i) (x, y, z, r, u, v) is solution of variational inclusion problem.
(ii) $x = (g-m)(x, \omega) + J_\rho^M(., v, \omega)[A(x) - \rho S(y, z, \omega) + \rho T(r, u, \omega) + \rho f]$.
(iii) The map of $G : X \times \Omega \to 2^X$ defined by

$$G(t, \omega) = f - S(y, z, \omega) + T(r, u, \omega) - M((g-m)(x, \omega), v, \omega) + t$$

such that $y \in B(x, \omega), z \in C(x, \omega), r \in D(x, \omega), u \in E(x, \omega), v \in F(x, \omega)$ and $(t, \omega) \in X \times \Omega$ then $x \in X$ is a fixed point of G.

Proof. $(i \leftrightarrow ii)$ Suppose that $(x, y, z, r, u, v,)$ is a solution of problem then

$$f \in S(y, z, \omega) - T(r, u, \omega) + M((g-m)(x, \omega), v, \omega) \Leftrightarrow$$
$$\rho f \in \rho S(y, z, \omega) - \rho T(r, u, \omega) + \rho M((g-m)(x, \omega), v, \omega) \Leftrightarrow$$
$$A(x) + \rho f - \rho S(y, z, \omega) + \rho T(r, u, \omega) \in A(x) - \rho\varphi((g-m)(x, \omega), v, \omega) \Leftrightarrow$$
$$x - (g-m)(x, \omega) = J_\rho^{M(., v, \omega)}[A(x) - \rho S(y, z, \omega) + \rho T(r, u, \omega) + \rho f] \Leftrightarrow$$
$$x = (g-m)(x, \omega) + J_\rho^{M(., v, \omega)}[A(x) - \rho S(y, z, \omega) + \rho T(r, u, \omega) + \rho f].$$

$(i \leftrightarrow iii)$ If (x, y, z, r, u, v) is a solution of the problem then

$$f \in S(y, z, \omega) - T(r, u, \omega) + M((g-m)(x, \omega), v, \omega) \Leftrightarrow$$
$$0 \in f - S(y, z, \omega) - T(r, u, \omega) - M((g-m)(x, \omega), v, \omega) \Leftrightarrow$$
$$x \in f - S(y, z, \omega) + T(r, u, \omega) - M((g-m)(x, \omega), v\omega) + x \Leftrightarrow$$
$$x \in G(x, \omega)$$

therefore $x \in X$ is a fixed point of G. so this completes the proof.

Lemma 4.2. [10] *Let $\{c_n\}$ and $\{k_n\}$ be real sequences of nonnegative number such that satisfy the following conditions:*
(i) $0 \leq k_n < 1, n = 0, 1, 2, ...$ and $\limsup_n k_n < 1$
(ii) $c_{n+1} \leq k_n c_n, n = 0, 1, 2, ...$
Then $c_n \to 0$ as $n \to \infty$

Algorithm (I)

For any given $(x_0, y_0, z_0, r_0, u_0, v_0)$ such that $x_0 \in X$ and $y_0 \in A(x_0, \omega), z_0 \in B(x_0, \omega), r_0 \in C(x_0, \omega), u_0 \in D(x_0, \omega), v_0 \in F(x_0, \omega)$ define iterative sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{r_n\}, \{u_n\}, \{v_n\}$

$$x_{n+1} = \alpha_n(x_n - (g-m)(x_n, \omega)) + (1-\alpha_n)J_\rho{}^{M(., v_n, \omega)}(x_n - \rho S(y_n, z_n, \omega) + \rho T(r_n, u_n, \omega) + \rho f)$$

and

$$y_n \in B(x_n, \omega), \|y_{n+1} - y_n\| \leq \alpha_n H(B(x_{n+1}, \omega), B(x_n, \omega)),$$
$$z_n \in C(x_n, \omega), \|z_{n+1} - z_n\| \leq \alpha_n H(C(x_{n+1}, \omega), C(x_n, \omega)),$$
$$r_n \in D(x_n, \omega), \|r_{n+1} - r_n\| \leq \alpha_n H(D(x_{n+1}, \omega), D(x_n, \omega)),$$

$$u_n \in E(x_n,\omega), \|u_{n+1}-u_n\| \leq \alpha_n H(E(x_{n+1},\omega), E(x_n,\omega)),$$
$$v_n \in F(x_n,\omega), \|v_{n+1}-v_n\| \leq \alpha_n H(F(x_{n+1},\omega), F(x_n,\omega)),$$

$0 \leq \alpha_n < 1$ and $\limsup_n \alpha_n < 1$

Algorithm (II)

Define multifunction $G : X \times \Omega \to 2^X$ by

$$G(t,\omega) = f - S(y,z,\omega) + T(r,u,\omega) - M((g-m)(x,\omega), v, \omega) + t$$

for all $(t,\omega) \in X \times \Omega$. For any given $(x_0, y_0, z_0, r_0, u_0, v_0)$ we consider iterative sequences as following

$$x_{n+1} \in G(x_n,\omega)$$

and for a sequence $0 < a_n < 1$ for all $n = 1,2,...$ and

$$y_n \in B(x_{n+1},\omega), y_{n+1} = a_n y_n + (1-a_n)S(y_n, z_0, \omega)$$
$$z_n \in C(x_{n+1},\omega), z_{n+1} = a_n z_n + (1-a_n)S(y_0, z_n, \omega)$$
$$r_n \in D(x_{n+1},\omega), r_{n+1} = a_n r_n + (1-a_n)T(r_n, u_0, \omega)$$
$$u_n \in E(x_{n+1},\omega), u_{n+1} = a_n u_n + (1-a_n)T(y_0, u_0, \omega)$$
$$v_n \in F(x_{n+1},\omega), v_{n+1} = a_n v_n + (1-a_n)M((g-m)(x_0,\omega), v_n, \omega)$$

Theorem 4.3. *Let $\eta : X \times \Omega \to X$ be a Lipschitz continuous operator with constant τ. Let $A : X \times \Omega \to X$ be a strongly η-monotone, Lipschitz continuous operator with constants γ, σ.Let $g, m : X \times \Omega \to X$ such that $g-m$ is s-relaxed cocoercive in the first argument.Let $B, C, D, E, F : X \times \Omega \to C(X)$ be (H)-Lipschitz continuous in the first argument with constants $L_B, L_C, L_D, L_E, L_F > 0$ respectively. Let $M : X \times \Omega \to 2^X$ be a (A,η)-monotone and let $S : X \times X \times \Omega \to X$ be an operator such that for any given $(x,y,\omega) \in X \times \Omega$, S is δ-relaxed cocoercive and Lipschitz continuous with respect to first and second variables with constants $\alpha_1 > 0, \beta_1 > 0$ respectively and (s,H)Lipschitz continuous $T : X \times X \times \Omega \to X$ be an operator such that for any given $(x,y,\omega) \in X \times \Omega$, T is Lipschitz continuous with respect to first and second variables with constants $\alpha_2 > 0, \beta_2 > 0$ respectively and (s,H)Lipschitz continuous.If $(x_n, y_n, z_n, r_n, u_n, v_n)$ be sequences that generated by Algorithm(I) and hold the following conditions*

$$\begin{cases} \lambda > 0, \\ \frac{\gamma}{\lambda} > m, \\ 0 < s < \frac{1}{2}, \rho > 0, \delta > 0, 2(\alpha_1 L_B + \beta_1 L_C) < \frac{1}{\rho\delta} \\ 0 < (\frac{1}{\sqrt{1-2s}})(\frac{\tau}{\gamma - \lambda m})\sqrt{(1 - 2\rho\delta s(\alpha_1 L_B + \beta_1 L_C))} < 1 \end{cases}$$

then theses sequences converge strongly to the unique solution (x,y,z,r,u,v) of problem and there exists $d \in [0,1)$ such that

$$\|x_n - x\| \leq d^n \|x_0 - x\|$$

for all $n > 0$.

Proof.

Let (x, y, z, r, u, v) be a unique solution of problem.It follows from lemma12 that

$$\|x_{n+1}-x\| = \|\alpha_n(x_n-(g-m)(x_n,\omega))+(1-\alpha_n)J_\rho^M(x_n-\rho S(y_n,z_n,\omega)+\rho T(r_n,u_n,\omega)+\rho f)-$$
$$[\alpha_n(x-(g-m)(x,\omega))+(1-\alpha_n)J_\rho^M(x-\rho S(y,z,\omega)+\rho T(r,u,\omega)+\rho f)]\| \leq$$
$$\alpha_n\|x_n-x-((g-m)(x_n,\omega)-(g-m)(x,\omega))\|+(1-\alpha_n)\|J_\rho^M(x_n-\rho S(y_n,z_n,\omega)+\rho T(r_n,u_n,\omega)+\rho f)-$$
$$J_\rho^M(x-\rho S(y,z,\omega)+\rho T(r,u,\omega)+\rho f)\| \leq \alpha_n\|x_n-x-((g-m)(x_n,\omega)-(g-m)(x,\omega))\|+$$
$$(1-\alpha_n)(\frac{\tau}{\gamma-\lambda m})\|x_n-\rho S(y_n,z_n,\omega)+\rho T(r_n,u_n,\omega)+\rho f-[x-\rho S(y,z,\omega)+\rho T(r,u,\omega)+\rho f]\| \Rightarrow$$
$$\alpha_n\|x_n-x-((g-m)(x_n,\omega)-(g-m)(x,\omega))\|+(1-\alpha_n)(\frac{\tau}{\gamma-\lambda m})\|x_n-\rho S(y_n,z_n,\omega)+\rho T(r_n,u_n,\omega)$$
$$-x+\rho S(y,z,\omega)-\rho T(r,u,\omega)\| \leq$$
$$\alpha_n\|x_n-x-((g-m)(x_n,\omega)-(g-m)(x,\omega))\|+(1-\alpha_n)(\frac{\tau}{\gamma-\lambda m})\|x_n-x-$$
$$[\rho S(y_n,z_n,\omega)-\rho S(y_n,z_n,\omega)]+\rho T(r_n,u_n,\omega)-\rho T(r,u,\omega)\| \leq$$
$$\alpha_n\|x_n-x-((g-m)(x_n,\omega)-(g-m)(x,\omega))\|+(1-\alpha_n)(\frac{\tau}{\gamma-\lambda m})\{\|x_n-x-$$
$$(\rho S(y_n,z_n,\omega)-\rho S(y_n,z_n,\omega))\|+\|\rho T(r_n,u_n,\omega)-\rho T(r,u,\omega)\|\}$$

on the other hand

$$\|-((g-m)(x_n,\omega)-(g-m)(x,\omega))\|^2 \leq$$
$$\|x_n-x\|^2-2\langle(g-m)(x_n,\omega)-(g-m)(x,\omega),J(x_n-x)\rangle \leq$$
$$(1-2s)\|x_n-x\|^2$$

and also

$$\|x_n-x-(\rho S(y_n,z_n,\omega)-\rho S(y_n,z_n,\omega))\|^2 \leq$$
$$\|x_n-x\|^2-2\rho\langle S(y_n,z_n,\omega)-S(y_n,z_n,\omega),J(x_n-x)\rangle \leq$$
$$(1-2\rho\delta(\alpha_1\|y_n-y_{n-1}\|+\beta_1\|z_n-z_{n-1}\|))\|x_n-x\| \leq$$
$$(1-2\rho\delta(\alpha_1 H(B(x_{n+1},\omega),B(x_n,\omega))(+\beta_1 H(C(x_{n+1},\omega),C(x_n,\omega))))\|x_n-x\| \leq$$
$$(1-2\rho\delta(\alpha_1 L_B+\beta_1 L_C))\|x_n-x\|^2$$

and

$$\|\rho T(r_n,u_n,\omega)-\rho T(r,u,\omega)\| \leq (\alpha_2 L_D+\beta_2 L_E)\|x_n-x_{n-1}\|$$

therefore

$$\|x_{n+1}-x\| \leq \alpha_n\sqrt{(1-2s)}\|x_n-x\|+(1-\alpha_n)(\frac{\tau}{\gamma-\lambda m})\{\|x_n-x-(\rho S(y_n,z_n,\omega)-\rho S(y_n,z_n,\omega))\|+$$
$$\|\rho T(r_n,u_n,\omega)-\rho T(r,u,\omega)\|\} \leq$$
$$\{\alpha_n\sqrt{(1-2s)}+(1-\alpha_n)(\frac{\tau}{\gamma-\lambda m})\{\sqrt{(1-2\rho\delta s(\alpha_1 L_B+\beta_1 L_C))}+(\alpha_2 L_D+\beta_2 L_E)\}\|x_n-x\|$$

It follows from conclusions above that

$$\|x_{n+1}-x\| \leq \alpha_n\sqrt{(1-2s)}\|x_n-x\|+(1-\alpha_n)k\|x_n-x\| = (k+(1-k)\alpha_n\sqrt{1-2s})\|x_n-x\|(*)$$

where $0 \leq k < 1$ is defined by

$$k := (\frac{1}{\sqrt{1-2s}})(\frac{\tau}{\gamma-\lambda m})\sqrt{(1-2\rho\delta s(\alpha_1 L_B+\beta_1 L_C))}(\alpha_2 L_D+\beta_2 L_E)$$

Let $C_n := \|x_n - x\|$ and $k_n := k + (1-k)\alpha_n\sqrt{1-2s}$. Then by $(*)$ can be rewritten as

$$c_{n+1} \le k_n c_n$$

for all $n = 0, 1, 2, 3, ...$ so by lemma11 we know that $\limsup_n k_n < 1$. It follows from lemma11 that $0 \le k_n \le d < 1$ and

$$\|x_n - x\| \le d^n \|x_0 - x\|$$

for all $n \ge 0$.

Theorem 4.4. *Let X be real Banach space $\eta : X \times X \to X$ be a Lipschitz continuous operator with constant τ. Let $A : X \times \Omega \to X$ be a strongly η-monotone, Lipschitz continuous operator with constants γ, σ. Let $g, m : X \times \Omega \to X$ such that $g - m$ is s-relaxed monotone in the first argument. Let $B, C, D, E, F : X \times \Omega \to C(X)$ be (H)-Lipschitz continuous in the first argument with constants $L_B, L_C, L_D, L_E, L_F > 0$ respectively. Let $S : X \times X \times \Omega \to X$ be an operator such that for any given $(x, y, \omega) \in X \times \Omega$, S is δ-relaxed monotone and Lipschitz continuous with respect to first and second variables with constants $\alpha_1 > 0, \beta_1 > 0$ respectively and (s, H)Lipschitz continuous $T : X \times X \times \Omega \to X$ be an operator such that for any given $(x, y, \omega) \in X \times \Omega$, T is δ_2-strongly monotone and Lipschitz continuous with respect to first and second variables with constants $\alpha_2 > 0, \beta_2 > 0$ respectively and (s, H)Lipschitz continuous and let $M : X \times \Omega \to 2^X$ be a (A, η)-monotone and (ν)-relaxed monotone with respect to T.If $(x_n, y_n, z_n, r_n, u_n, v_n)$ be sequences that generated by Algorithm(II) hold the following conditions*

$$\begin{cases} \delta > 0, 2(\alpha_1 L_B + \beta_1 L_C) \le \frac{1}{\delta} \\ 0 < \nu < \frac{1}{2}, \\ \\ \sqrt{(1 - 2\delta(\alpha_1 L_B + \beta_1 L_C)} + \sqrt{(1-2\nu)(\alpha_2 L_D + \beta_2 L_E)} < 1 \end{cases}$$

then theses sequences strongly converge to the unique solution (x, y, z, r, u, v) of problem.

Proof.

Let $(x_n, y_n, z_n, r_n, u_n, v_n)$ be sequences that generated by Algorithm(II) so

$$\|x_{n+1} - x_n\| = \|G(x_n, \omega) - G(x_{n-1}, \omega)\| = \|f - S(y_n, z_n, \omega) + T(r_n, u_n, \omega) -$$
$$M((g-m)(x_n, \omega), v_n, \omega) + x_n - (f - S(y_{n-1}, z_{n-1}, \omega) + T(r_{n-1}, u_{n-1}, \omega) -$$
$$M((g-m)(x_{n-1}, \omega), v_{n-1}, \omega) + x_{n-1}\| \le$$
$$\|S(y_{n-1}, z_{n-1}, \omega) - S(y_n, z_n, \omega) + x_n - x_{n-1}\| + \|T(r_n, u_n, \omega) - T(r_{n-1}, u_{n-1}, \omega) -$$
$$(M((g-m)(x_n, \omega), v_n, \omega) - M((g-m)(x_{n-1}, \omega), v_{n-1}, \omega)\| \Rightarrow$$

$$\|S(y_{n-1}, z_{n-1}, \omega) - S(y_n, z_n, \omega) + x_n - x_{n-1}\|^2 \le \|x_n - x_{n-1}\|^2 -$$
$$2\langle S(y_{n-1}, z_{n-1}, \omega) - S(y_n, z_n, \omega), J(x_n - x_{n-1})\rangle \le \|x_n - x_{n-1}\|^2 -$$
$$2\delta(\alpha_1\|y_n - y_{n-1}\| + \beta_1\|z_n - z_{n-1}\|).\|x_n - x_{n-1}\| \le$$

$$(1-2\delta(\alpha_1 L_B+\beta_1 L_C))\|x_n-x_{n-1}\|^2$$

On the other hand

$$\|T(r_n,u_n,\omega)-T(r_{n-1},u_{n-1},\omega)-(M((g-m)(x_n,\omega),v_n,\omega)-M((g-m)(x_{n-1},\omega),v_{n-1},\omega)\|^2$$
$$\leq \|T(r_n,u_n,\omega)-T(r_{n-1},u_{n-1},\omega)\|^2-2\langle(M((g-m)(x_n,\omega),v_n,\omega)-M((g-m)(x_{n-1},\omega),$$
$$T(r_n,u_n,\omega)-T(r_{n-1},u_{n-1},\omega)\rangle \leq (\alpha_2\|r_n-r_{n-1}\|^2+\beta_2\|u_n-u_{n-1}\|^2)-$$
$$2\nu\|T(r_n,u_n,\omega)-T(r_{n-1},u_{n-1},\omega)\|^2 \leq [(\alpha_2 L_D+\beta_2 L_E)-2\nu(\alpha_2 L_D+\beta_2 L_E)]\|x_n-x_{n-1}\|^2,$$

therefore

$$\|x_{n+1}-x_n\| \leq \sqrt{(1-2\delta(\alpha_1 L_B+\beta_1 L_C)}+\sqrt{(1-2\nu)(\alpha_2 L_D+\beta_2 L_E)}\|x_n-x_{n-1}\|$$

so there exists $x \in X$ such that $x_n \to x$ and hence for $(x_0,y_0,z_0,r_0,u_0,v_0)$ we have

$$\|y_{n+1}-y_n\| = \|a_k y_n+(1-a_k)S(y_n,z_0,\omega)-(a_k y_{n-1}+(1-a_k)S(y_{n-1},z_0,\omega))\| =$$
$$\|a_k(y_n-y_{n-1})+(1-a_k)(S(y_n,z_0,\omega)-S(y_{n-1},z_0,\omega))\| \leq a_k\|y_n-y_{n-1}\|+$$
$$(1-a_k)\|(S(y_n,z_0,\omega)-S(y_{n-1},z_0,\omega)) \leq (a_k+(1-a_k)\alpha_1)\|y_n-y_{n-1}\| \leq$$
$$(a_k+(1-a_k)\alpha_1)H(B(x_{n+1},\omega),B(x_n,\omega)) \leq (a_k+(1-a_k)\alpha_1)L_B\|x_n-x_{n-1}\|.$$

Then there exists $y \in B(x,\omega)$ such that $y_n \to y$, by similar to way we have $z_n \to z, r_n \to r, u_n \to u, v_n \to v$ thus (x,y,z,r,u,v) is solution of problem.

References

1. S. Adly, Perturbed algorithms and sensitivity analysis for a general class of variational inclusions, *J. Math. Anal. Appl*, **201(3)**(1996), 609–630. 3
2. R.P. Agarwal, N.-J. Huang and Y.-J. Cho, Generalized nonlinear mixed implicit quasi-variational inclusions with set-valued mappings, *J. Inequal. Appl*, **7(6)**(2002), 807–828.
3. R.P. Agarwal, Y.-J. Cho and N.-J. Huang, Sensitivity analysis for strongly nonlinear quasi-variational inclusions, *Appl. Math. Lett*, **13**(2002), 19–24.
4. M. Alimohammady, M. Ramzannezhad, On ϕ-fixed point for maps on uniform spaces, (H,η)-monotone operators, *J. Nonlinear Sci.Appl.* **1(4)** (2008), 241–243.
5. M. Alimohammady, M. Roohi, Implicit variational-like inclusions involving general (H,η)-monotone operators, *J. Nonlinear Sci.Appl.* **1(3)**(2008), 145-154.
6. C.E. Chidume, K.R. Kazmi and H. Zegeye, Iterative approximation of a solution of a general variational-like inclusion in Banach spaces, *International J. Math. Math. Sci*, **22**(2004), 1159–1168.
7. S. Dafermos, Sensitivity analysis in variational inequalities, *Math. Oper. Res*,**13**(1998), 421–434.
8. X.-P. Ding, Parametric completely generalized mixed implicit quasivariational inclusions involving h-maximal monotone mappings, *J. Comput. Appl. Math*, **182(2)**(2005), 252–269.
9. Y.-P. Fang and N.-J. Huang, H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces, *Appl. Math. Lett*, **17**(2004), 647–653.
10. Y.P.Fang and N.J.Huang, A new system of variational inclusion with (H,η)-monotone operators in Hilbert spaces, *Comput. Math. Appl*, **(49)**(2005), 365–374. 4.2
11. F. Giannessi., and A. Maugeri., *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York, 1995. 2.6
12. Heng-you Lan, A Class of Nonlinear (A,η)-Monotone Operator Inclusion Problems with Relaxed Cocoercive Mappings,*Advances in Nonlinear Variational Inequalities*,**9(2)**(2006), 1-11. 2.8, 3
13. N.-J. Huang, Nonlinear implicit quasi-variational inclusions involving generalized m-accretive mappings, *Archives of Inequalities and Applications* , **2**(2004), 413–426.
14. N.-J. Huang, P.Fang and Y.J. Cho, Generalized m-accretive mappings and variational inclusions in Banach spaces, *Journal of Concretive and Applicable Mathematics*, **3**(2005), 31–40.

15. Z. Liu, L. Debnath, S.M. Kang and J.S. Ume, Sensitivity analysis for parametric completely generalized nonlinear implicit quasi-variational inclusions, *J. Math. Anal. Appl*, **277(1)**(2003), 142–154. 3
16. J.Y.Park and J.U. Jeong, Parametric generalized mixed variational inequalities, *Appl. Math. Lett*, **17**(2004), 43–48.
17. S.M. Robinson, Sensitivity analysis for variational inequalities by normal-map techniques, in: F. Giannessi, A. Maugeri (Eds.), Variational Inequalities and Network Equilibrium Problems, *Plenum Press, New York*, 1995.
18. R.U.Verma, Approximation solvability of a class of nonlinear set-valued varoiational inclusion involving (A, η)-monotone mappings,*J.Math.Anal.Appl*, **337**(2008), 969–975. 2.4, 2.7, 3
19. R.U. Verma, Generalized (A, η)- resolvent operator technique and senstivity analysis for relaxed coercive variational inclusion, *Journal of Inequalities in Pure and Applied Mathematics*, **7(3)** article 83 (2006). 2.3
20. R.U. Verma, Sensitivity analysis for generalized strongly monotone variational inclusions based on the (A, η)-resolvent operator technique, *Appl. Math. Lett*, **19(12)**(2006),1409–1413.
21. R. U.Verma, A-monotonicity and applications to nonlinear variational inclusion problems, *J. Appl. Math. Stochastic Anal*,17(2)(2004), 193–195.

[1,2] Department of Mathematics, University of Mazandaran, Babolsar, Iran.
E-mail address: amohsen@umz.ac.ir, m.kallegi@gmail.com