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Convergence of trajectories in infinite horizon optimization

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Abstract

In this paper, we investigate the convergence of a sequence of minimizing trajectories in infinite horizon optimization problems. The convergence is considered in the sense of ideals and their particular case called the statistical convergence. The optimality is defined as a total cost over the infinite horizon.

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1. Introduction

Many important planning problems such as capacity expansion, equipment replacement and production planning involve sequences of related decisions over an infinite time horizon. The mathematical formulation of such problems lead to infinite horizon optimization which is the problem of selecting an infinite sequence of decisions such that the associated cost over an unbounded horizon is minimum [1, 2, 7, 10, 14, 16, 15, 17, 23].

In many studies an optimal solution/trajectory to infinite horizon problem is approximated by a sequence of finite horizon optimal solutions [2, 18, 19, 23]. In [20], a general method for approximating optimal solution via the solutions to a simpler approximating problems is presented.

The uniqueness of optimal solution is a common assumption used in many studies [1, 2, 10]. For example in [2], under this assumption, an algorithm is developed for finding optimal solution and the results are applied to undiscounted Markov decision processes. Among the studies that do not use the uniqueness assumption we mention [17, 18, 23]. For example, in [18] a tie-breaking algorithm

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is presented based on selection of a nearest-point efficient solutions that converges to an optimal solution and the results are applied to the scheduling production problem to meet demand over an infinite horizon.

Since the cost over an unbounded horizon may be infinite or diverge, a discounting factor is applied in the definition of the total cost. It is clear that even in the presence of discounting, the total cost may still be infinite. In this case, different optimality criteria apart from minimal total cost are required [5, 12, 19, 21, 22]; the average cost [3, 8, 26], overtaking optimality[5, 13, 27] and 1-optimality [4, 25] are some examples of such optimality criteria.

In this paper, we consider systems described by the decision network as in [2]. These systems generate trajectories of decisions and there is a cost associated to each decision that could be used to define the functional - the total cost for a trajectory. The aim of this paper is to investigate the convergence of a sequence of trajectories under the assumption that the functional values (total costs) converge to the optimal value (i.e. the minimal total cost). The convergence is considered in the sense of ideals and their particular case called the statistical convergence.

The paper is organized as follows. Notations and the problem statement are presented in the next section. Some preliminary results about convergence of the sequence of trajectories are established in Section 3. The I-convergence and the statistical convergence of a sequences of trajectories are considered in Section 4.

2. Notations and problem statement

We begin with the decision network, (Σ, A, C), where Σ is the set of states (nodes), A is the set of decisions (arcs) and C is a real-valued cost function $C : A \to R$. We assume that the decision network satisfies the following conditions [2]:

- there is a node called single root with the following properties
  - there is no incoming arcs to this node,
  - every other node can be reached from the single root,
- the set of decisions available at any node is nonempty and finite,
- the set of incoming decisions to any node is also finite.

Under these assumptions, it has been proved that [24, Theorem 1] the set of nodes can be numbered as $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, ...\}$ such that if $(\sigma_i, \sigma_j) \in A$ where $\sigma_i, \sigma_j \in N$, then $i < j$.

An infinite trajectory $\mathbf{s}$ is an infinite sequence of states $(s_1, s_2, s_3, ...)$ where s_1 is a given fixed root, $s_i \in \Sigma$ and $(s_i, s_{i+1}) \in A$ for all $i = 1, 2, ...$. The cost $C(s_i, s_{i+1})$ associates with the decision (s_i, s_{i+1}). The set of all trajectories $\mathbf{s}$ is denoted by $\prod$.

Now we introduce the metric in the set of trajectories. Consider two trajectories $\mathbf{s} = (s_1, s_2, s_3, ...)$ and $\mathbf{s}' = (s'_1, s'_2, s'_3, ...)$. In [2], the metric ρ, on $\prod$ is constructed as follows:

$$\rho(\mathbf{s}, \mathbf{s}') = \sum_{i=1}^{\infty} \phi_i(\mathbf{s}, \mathbf{s}')2^{-i}, \tag{2.1}$$

where

$$\phi_i(\mathbf{s},\mathbf{s}') = \begin{cases} 0 & \text{if} \quad s_i = s'_i \\ 1 & \text{otherwise} \end{cases} .$$

In [2] (Lemma 1), it is proved that the set $\prod$ is complete and hence compact in the sense of this metric.

Under this metric, the closeness of trajectories depends on the number of initial nodes over which they agree. For example, given any $i \in \mathbf{N}$ it can easily be verified that the following hold:

$$\rho(\mathbf{s},\mathbf{s}') > \frac{1}{2^i} \quad \Rightarrow \quad s'_r \neq s_r,\ \exists r \in \{1,2,\cdots,i\}. \tag{2.2}$$

Functional - the total cost $f(\mathbf{s})$ of trajectory $\mathbf{s}$ is defined as in [2] given by

$$f(\mathbf{s}) = \sum_{i=1}^{\infty} C(s_i, s_{i+1}). \tag{2.3}$$

We will assume that f is uniformly convergent over $\prod$; that is, for any $\varepsilon > 0$ there exists n_ε such that for all trajectories $\mathbf{s}$ the relation $\sum_{i=n}^{\infty} C(s_i, s_{i+1}) < \varepsilon$ holds for all $n \geq n_\varepsilon$. In this case f is continuous on $\prod$. Note that this is not a restrictive assumption; it holds if the cost function $C(s_i, s_{i+1})$ is uniformly bounded and also is discounted, for example, by $(1/2)^i$ (see Assumption 1 and Lemma 2 in [2]).

We consider the following optimization problem

$$\text{Minimize} \quad f(\mathbf{s}), \qquad \text{subject to} \quad \mathbf{s} \in \prod. \tag{2.4}$$

Since f is continuous and $\prod$ is compact, an optimal solution s^* to problem (2.4) exists. We call $\mathbf{s}^n$ a minimizing sequence if $f(\mathbf{s}^n)$ converges to the minimal value $f(\mathbf{s}^*)$ of the objective function in this problem. The aim of this paper is to investigate the convergence of minimizing sequence $\mathbf{s}^n$ to $\mathbf{s}^*$ by considering different types of convergence.

3. Preliminary results

In this section, we consider the convergence of a sequence of trajectories $\{\mathbf{s}^n\}_{n\in\mathbf{N}}$ to the trajectory $\mathbf{s}$ in the sense of ideals as well as their particular case called the statistical convergence. We recall that the initial point of all sequences is the same; that is, $s_1 = s_1^n$ for all n. We will use the notation $\{\{\mathbf{s}\}\} := \{s_1, s_2, s_3, ...\}$ to denote the set of states for trajectory $\mathbf{s}$.

First we give the definition of ideal and I-convergence.

Definition 3.1. A family $I \subset 2^{\mathbf{X}}$ of subsets of a nonempty set $\mathbf{X}$ is said to be an ideal in X if

- $A, B \in I$ implies $A \cup B \in I$,
- $B \subset A,\ A \in I$ implies $B \in I$,

while an admissible ideal I of $\mathbf{X}$ further satisfy $\{x\} \in I$ for each $x \in \mathbf{X}$.

Clearly, an ideal admissible contains all finite sets in $\mathbf{X}$. In the remainder of this section, we assume that any ideal is admissible and I is an ideal in $\mathbf{N}$.

Definition 3.2. [6, 11] A sequence $\mathbf{s}^n$ in a metric space $(\mathbf{X}, \rho)$ is said to be I-convergent to $\mathbf{s} \in \mathbf{X}$ (in short $\mathbf{s} = \mathrm{I} - \lim_{n\to\infty} \mathbf{s}^n$) if $K(\epsilon) \in I$ for each $\epsilon > 0$, where $K(\epsilon) = \{n \in \mathbf{N} : \ \rho(\mathbf{s}^n, \mathbf{s}) \geq \epsilon\}$.

Below we consider two special cases of ideals.

1. Classical convergence. In this case the ideal is the set of all finite subsets of $\mathbf{N}$; that is

$$I = I_{fin} \doteq \{A \subset \mathbf{N} : \ |A| < \infty\}.$$

Clearly, both of the conditions in Definition 3.1 are satisfied.

2. Statistical convergence. First we give the definition of the statistical convergence in terms of the notion of density. Assume K is a subset of the positive integers $\mathbf{N}$. $K_n = \{k \in K : \ k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n. The natural density of K is given by $\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$. It may not exist; in this case the upper and lower asymptotic densities for the set K are defined as follows:

$$\bar{\delta}(K) = \limsup_{n\to\infty} \frac{|K_n|}{n} \text{ and } \underline{\delta}(K) = \liminf_{n\to\infty} \frac{|K_n|}{n}.$$

Note that $\underline{\delta}(K) \leq \delta(K) \leq \bar{\delta}(K)$.

Definition 3.3. [9] A sequence $\{\mathbf{s}^n\}_{n\in\mathbf{N}}$ is statistically convergent to $\mathbf{s}$ provided that for every $\epsilon > 0$, the set $K(\epsilon) = \{n \in \mathbf{N} : \ \rho(\mathbf{s}^n, \mathbf{s}) \geq \epsilon\}$ has natural density zero.

It is not difficult to observe that both of the conditions in Definition 3.1 are satisfied if we define the ideal as the set of subsets of $\mathbf{N}$ with density zero. Thus in this case we set

$$I = I_d \doteq \{A \subset \mathbf{N} : \ \delta(A) = 0\}.$$

In the next lemma, the convergence of the sequence $\mathbf{s}^n$ to the trajectory $\mathbf{s}$ is considered.

Lemma 3.4. *Assume that* $\delta(\{n \in \mathbf{N} : \ s_i^n \neq s_i\}) = 0$ *for all* $i \in \mathbf{N}$. *Then sequence* $\mathbf{s}^n$ *statistically converges to* $\mathbf{s}$.

Proof: Take an arbitrary $\epsilon > 0$ and denote $A_\epsilon := \{n \in \mathbf{N} : \ \rho(\mathbf{s}^n, \mathbf{s}) > \epsilon\}$. We show that $\delta(A_\epsilon) = 0$.

Let $r_\epsilon \in \mathbf{N}$ such that $\frac{1}{2^{r_\epsilon}} < \epsilon$. Consider the sets

$$A_{r_\epsilon} := \{n \in \mathbf{N} : \ \rho(\mathbf{s}^n, \mathbf{s}) > \frac{1}{2^{r_\epsilon}}\}$$

and

$$B_{r_\epsilon} := \{n \in \mathbf{N} : \ s_i^n \neq s_i^* \text{ for some } i \in \{1, 2, ..., r_\epsilon\}\}.$$

From (2.2) we have $A_\epsilon \subset A_{r_\epsilon} \subset B_{r_\epsilon}$. On the other hand, B_{r_ϵ} can be represented in the form

$$B_{r_\epsilon} = \cup_{i=1}^{r_\epsilon} \{n \in \mathbf{N} : \ s_i^n \neq s_i\}.$$

By the assumption of the lemma, $\delta(\{n \in \mathbf{N} : \; s_i^n \neq s_i\}) = 0$ for all $i = 1, ..., r_\epsilon$ and therefore $\delta(B_{r_\epsilon}) = 0$. Thus, since $A_\epsilon \subset A_{r_\epsilon} \subset B_{r_\epsilon}$ we have $\delta(A_\epsilon) = 0$. Lemma is proved. □

It is clear that under the conditions of lemma 3.4, the classical convergence may not be true. Indeed, for example, if $\mathbf{s}^n = \mathbf{s}$ for all $n \in \mathbf{N} \setminus \{3^j\}_{j\in\mathbf{N}}$ and $\mathbf{s}^n = (s_1, \bar{s}_2, s_3, s_4, s_5, ...)$ for $n \in \{3^j\}_{j\in\mathbf{N}}$ where $\bar{s_2} \neq s_2$ then it is not difficult to show that $\mathbf{s}^n$ is statistically convergent to $\mathbf{s}$ while the classical convergence is not true.

4. Convergence of a sequence of minimizing trajectories

In this section, we investigate the convergence of the minimizing sequence $\mathbf{s}^n$ to the optimal trajectory $\mathbf{s}^*$ of the problem (2.4); that is, under the assumption that $f(\mathbf{s}^n) \to f(\mathbf{s}^*)$ we investigate the convergence $\mathbf{s}^n \to s^*$ as $n \to \infty$. We do not assume the uniqueness of $\mathbf{s}^*$; however, we will consider a fixed optimal trajectory $\mathbf{s}^*$ and will formulate the main assumptions by using this trajectory. Note that $\mathbf{s}^n$ may not converge to $\mathbf{s}^*$, in this case the $I-$convergence and statistical convergence will be considered.

Given sequence $\mathbf{s}^n$ and set $K \subset \mathbf{N}$ we define

$$H(K) = \{j \in \mathbf{N} : \; s_j^* \in \{\{\mathbf{s}^n\}\}, \; \forall \, n \in K\}. \tag{4.1}$$

In the case $K = \mathbf{N}$ for the sake of simplicity we denote

$$H = H(\mathbf{N}) = \{j \in \mathbf{N} : \; s_j^* \in \{\{\mathbf{s}^n\}\}, \; \forall \, n \in \mathbf{N}\}. \tag{4.2}$$

For trajectory $\mathbf{s}$, we denote the section connecting two nodes a and b by

$$P(\mathbf{s} : a, b) = \{s_{n_1}, s_{n_1+1}, ..., s_{n_2-1}, s_{n_2}\};$$

where $s_{n_1} = a$ and $s_{n_2} = b$.

The corresponding cost is

$$f(P(\mathbf{s} : a, b)) = C(s_{n_1}, s_{n_1+1}) + ... + C(s_{n_2-1}, s_{n_2}).$$

Let $\mathbf{s}^* = (s_1^*, s_2^*, s_3^*, s_4^*, ...)$.

Condition A: For any $s_i^*, s_j^* \in \mathbf{s}^*$ with $i < j$ and any trajectory $\mathbf{s}$ connecting points s_i^*, s_j^* (that is, $s_{n_i} = s_i^*, s_{n_j} = s_j^*$) the following inequality holds

$$f(P(\mathbf{s} : s_{n_i}, s_{n_j})) - f(P(\mathbf{s}^* : s_i^*, s_j^*)) > 0.$$

A trajectory is said to be efficient if it reaches each of the states through which it passes at minimum cost. Efficient solutions are shown in [19] to be average-cost optimal under a state reachability property. Clearly, Condition A implies that $\mathbf{s}^*$ is an efficient trajectory.

Condition A means any finite section $(s_i^*, s_{i+1}^*, ..., s_j^*)$ in terms of an "optimality" connecting nodes s_i^* and s_j^* is unique. Note also that as $s_i^*, s_j^* \in \Sigma$ and there is a finite number of nodes between s_i^* and s_j^*; that is, the number of trajectories connecting these two nodes is finite.

In the following example, we show that Condition A does not mean the uniqueness of optimal trajectory.

Example 4.1. Assume that the set of states (nodes) in a decision network consists of nodes $\{s_k\}_{k\in\mathbf{N}}$ and the cost function is given by

$$C(s_1,s_2)=\frac{1}{2},\ C(s_{2k-1},s_{2k+1})=\frac{1}{2^k},\ C(s_{2k},s_{2k+1})=C(s_{2k},s_{2k+2})=\frac{1}{2^{k+1}},\quad \forall k\geq 1. \tag{4.3}$$

Clearly, $\mathbf{s}^*=(s_1,s_3,s_5,s_7,\cdots)$ is an optimal trajectory. We construct a sequence of trajectories $\mathbf{s}^n$ in the following form:

$$\mathbf{s}^n=(s_1,s_2,s_{2+2},\cdots,s_{2n},s_{2n+1},s_{2n+3},\cdots),\quad n\in\mathbf{N}. \tag{4.4}$$

We show that the condition A holds. For this aim, it is enough to show that condition A holds for $s_1,s_{2j+1}\in\mathbf{s}^*,\ j\geq 1$. Taking into account the definition of the cost function in (4.3), we have

$$C(s_1,s_2)+(\sum_{k=1}^{j-1}C(s_{2k},s_{2k+2}))+C(s_{2j},s_{2j+1})+\sum_{k=j}^{\infty}C(s_{2k-1},s_{2k+1})=\frac{1}{2^{j+1}}>0;$$

that is, condition A holds.

On the other hand, for the trajectory $\bar{\mathbf{s}}=(s_1,s_2,s_4,...,s_{2n},s_{2n+2},...)$ the relation $f(\bar{\mathbf{s}})=f(\mathbf{s}^*)$ holds which means optimal trajectory $\mathbf{s}^*$ is not unique. □

In the next theorem, we establish the I-convergence of the sequence $\mathbf{s}^n$ to the optimal trajectory $\mathbf{s}^*$ when $f(\mathbf{s}^n)\to f(\mathbf{s}^*)$.

Theorem 4.2. *Assume that optimal trajectory $\mathbf{s}^*$ satisfies Condition A, $\mathbf{s}^n$ is a minimizing sequence and there exists $K\subset\mathbf{N}$ such that $|H(K)|=\infty$ and $|K\cap A|=\infty$ for all $A\notin I$. Then $\mathbf{s}^n$ is I-convergent to $\mathbf{s}^*$ as $n\to\infty$.*

Proof: On the contrary assume there exists $\epsilon>0$ such that $\{n\in\mathbf{N}:\ \rho(\mathbf{s}^n,\mathbf{s}^*)>\epsilon\}\notin I$. Take any $r_\varepsilon\in\mathbf{N}$ satisfying $\frac{1}{2^{r_\varepsilon}}<\epsilon$ and denote

$$A_{r_\varepsilon}=\{n\in\mathbf{N}:\ \rho(\mathbf{s}^n,\mathbf{s}^*)>\frac{1}{2^{r_\varepsilon}}\}. \tag{4.5}$$

Clearly $\{n\in\mathbf{N}:\ \rho(\mathbf{s}^n,\mathbf{s}^*)>\epsilon\}\subset A_{r_\varepsilon}$ and therefore $A_{r_\varepsilon}\notin I$. From (2.2) it follows that

$$A_{r_\varepsilon}\subset\{n\in\mathbf{N}:\ s_i^n\neq s_i^*,\ \ \exists\, i\in\{2,3,...,i\}\}.$$

By the assumption of the lemma we have $|K\cap A_{r_\varepsilon}|=\infty$. Consider the set $H(K)$ defined in (4.1) and denote

$$t=\min\{m:\ r_\varepsilon\leq m,\ m\in H(K)\ \}.$$

We note that such t exists as $|H(K)|=\infty$.

Now for any $n\in K\cap A_{r_\varepsilon}$ the relation $s_t^*\in\{\{\mathbf{s}^n\}\}$ holds; that is, $s_{j_n}^n=s_t^*$ for some index j_n. Denote

$$\alpha=\inf_{n\in K\cap A_{r_\varepsilon}}\{\sum_{r=1}^{j_n-1}C(s_r^n,s_{r+1}^n)-\sum_{j=1}^{t-1}C(s_j^*,s_{j+1}^*)\}. \tag{4.6}$$

We note that there are only a finite number of possible different combinations $(s_1^n,\cdots,s_{j_n}^n)$ with the same fixed initial point s_1^* and the end point s_t^*. Then, condition A implies $\alpha>0$.

Denote

$$a^n = \sum_{r=1}^{j_n-1} C(s_r^n, s_{r+1}^n), \;\; a = \sum_{j=1}^{t-1} C(s_j^*, s_{j+1}^*);$$

$$b^n = \sum_{r=j_n}^{\infty} C(s_r^n, s_{r+1}^n), \;\; b = \sum_{j=t}^{\infty} C(s_j^*, s_{j+1}^*).$$

Clearly, $f(\mathbf{s}^n) = a^n + b^n$ and $f(\mathbf{s}^*) = a + b$. From (4.6) we have $a^n \geq a + \alpha$. Since $\mathbf{s}^*$ is optimal and $s_{j_n}^n = s_t^*$ we have $b^n \geq b$. Thus

$$f(\mathbf{s}^n) = a^n + b^n \geq a^n + b \geq a + b + \alpha = f(\mathbf{s}^*) + \alpha, \quad \forall n \in K \cap A_{r_\varepsilon}.$$

This means that $f(\mathbf{s}^n)$ does not converge to $f(\mathbf{s}^*)$; that is, $\mathbf{s}^n$ is not a minimizing sequence. This is a contradiction. Lemma proved.□

In this lemma, condition $|K \cap A| = \infty$ for all $A \notin I$ means that the set K should be quite "large". We describe it in Corollary 4.5 in terms of the density of K.

In the next lemma, we investigate the classical convergence of the sequence $\{\mathbf{s}^n\}_{n\in\mathbf{N}}$ to the optimal trajectory $\mathbf{s}^*$. It is shown that stronger condition is required in comparison to Theorem 4.2.

Corollary 4.3. *Assume that optimal trajectory $\mathbf{s}^*$ satisfies Condition A, $\mathbf{s}^n$ is a minimizing sequence and $|H| = \infty$. Then $\mathbf{s}^n \to \mathbf{s}^*$ as $n \to \infty$.*

Here H is defined in (4.2) which corresponds to $\mathbf{K} = \mathbf{N}$ in terms of Theorem 4.2.
Proof: We apply Theorem 4.2 assuming that $\mathbf{K} = \mathbf{N}$ and the ideal I is the set of finite subsets of $\mathbf{N}$; that is, $I = I_{fin}$. Firstly, we have $|H(\mathbf{K})| = |H| = \infty$. On the other hand, for any $A \notin I_{fin}$ the relation $|A| = \infty$ holds and therefore

$$|\mathbf{K} \cap A| = |\mathbf{N} \cap A| = |A| = \infty.$$

Thus, all the assumptions of Theorem 4.2 hold. The ideal convergence in this case is the classical convergence $\mathbf{s}^n \to \mathbf{s}^*$ as $n \to \infty$.

The corollary is proved. □

The condition $|\mathbf{H}| = \infty$ means the number of nodes in $\mathbf{s}^*$ that is "common" in all trajectories $\mathbf{s}^n$ is infinite; in other words, all trajectories $\mathbf{s}^n$ pass through an infinite number of nodes in $\mathbf{s}^*$.

In the following example, we investigate the necessity of condition A in this corollary.

Example 4.4. Let $\mathbf{s}^* = (s_1, s_3, s_4, \cdots)$, $\mathbf{s}^n = (s_1, s_2, s_3, s_4, \cdots)$ and

$$C(s_1, s_2) + C(s_2, s_3) = C(s_1, s_3).$$

Then $f(\mathbf{s^n}) = f(\mathbf{s}^*)$ for all n; however $\mathbf{s^n}$ does not converge to $\mathbf{s}^*$ as $\rho(\mathbf{s^n}, s^*) \geq 0.5$ for all n. We also mention that in this example, $H = \mathbf{N}$. □

To consider the necessity of condition $|H| = \infty$ in Corollary 4.3, we refer to the decision network in Example 4.1. In this example, given any $k \geq 2$, the relation $s_{2k-1} \notin \{\{\mathbf{s}^n\}\}$ holds for all $n \geq k$. This means that the set H contains just one element; that is, the condition $|H| \neq \infty$ does not hold.

Clearly, $f(\mathbf{s^n}) \to f(\mathbf{s}^*)$ however $\mathbf{s}^n$ does not converge to $\mathbf{s}^*$.

Now we consider a special case of Theorem 4.2 when the ideal convergence is defined by the statistical convergence. We have

Corollary 4.5. *Assume that optimal trajectory* $\mathbf{s}^*$ *satisfies Condition A,* $\mathbf{s}^n$ *is a minimizing sequence, there exists* $K \subset \mathbf{N}$ *such that* $|H(K)| = \infty$ *and* $\delta(K) = 1$. *Then* $\mathbf{s}^n$ *statistically converges to* $\mathbf{s}^*$ *as* $n \to \infty$.

Proof: We apply Theorem 4.2 assuming that the ideal I is the set of subsets of $\mathbf{N}$ having density 0; that is, $I = \{A \subset \mathbf{N} : \delta(A) = 0\}$. The relation $A \notin I$ in this case means A has a nonzero density. Then for any set $\mathbf{K} \subset \mathbf{N}$ with $\delta(\mathbf{K}) = 1$, $\mathbf{K} \cap A$ also has nonzero density. This means that $|\mathbf{K} \cap A| = \infty$ and all the assumptions of Theorem 4.2 hold. The corollary is proved. $\square$

Clearly, Corollary 4.5 is a special case of Corollary 4.3. Next we provide an example where $\mathbf{s}^n$ statistically converges to $\mathbf{s}^*$ however the classical convergence $\mathbf{s}^n \to \mathbf{s}^*$ is not true.

Example 4.6. Assume that the set of states (nodes) in a decision network consists of nodes $\{s_k\}_{k\in\mathbf{N}}$, $\{\xi_k\}_{k\in\mathbf{N}}$ and the cost function is given by

$$C(s_1, s_2) = \frac{1}{2},\ C(s_{2k-1}, s_{2k+1}) = \frac{1}{2^k},\ C(s_{2k}, s_{2k+1}) = C(s_{2k}, s_{2k+2}) = \frac{1}{2^{k+1}},\quad \forall k \geq 1. \tag{4.7}$$

$$C(s_{3^{\kappa(n)-1}}, \xi_n) = C(\xi_n, s_{3^{\kappa(n)-1}+2}) = \frac{1}{2}\left[C(s_{3^{\kappa(n)-1}}, s_{3^{\kappa(n)-1}+2}) + \frac{1}{n}\right],\quad \forall n \in \mathbf{N} \setminus \{3^k\}_{k\in\mathbf{N}}. \tag{4.8}$$

In this example $\mathbf{s}^* = (s_1, s_3, s_5, s_7, \cdots)$ is an optimal trajectory. Consider the function of indices $\kappa : \mathbf{N} \to \mathbf{N}$ defined by

$$\kappa(n) = i,\ \forall n \in \{3^{i-1}, 3^{i-1}+1, \cdots, 3^i - 1\},\ i = 1, 2, \cdots .$$

We construct a sequence of trajectories s^n in the following form:

$$\mathbf{s}^n = (s_1, s_2, s_{2+2}, \cdots, s_{2n}, s_{2n+1}, s_{2n+3}, \cdots),\quad n \in \{3^k\}_{k\in\mathbf{N}};$$

$$\mathbf{s}^n = (s_1, s_3, \cdots, s_{3^{\kappa(n)-1}}, \xi_n, s_{3^{\kappa(n)-1}+2}, s_{3^{\kappa(n)-1}+4}, \cdots),\quad n \in \mathbf{N} \setminus \{3^k\}_{k\in\mathbf{N}}.$$

Now we show that $f(\mathbf{s^n}) \to f(\mathbf{s}^*)$. For any $n \in \{3^k\}_{k\in\mathbf{N}}$, from (4.7) we have

$$f(\mathbf{s^n}) - f(\mathbf{s}^*) = \frac{1}{2^{n+1}}.$$

Similarly, for any $n \in \mathbf{N} \setminus \{3^k\}_{k\in\mathbf{N}}$, it follows from (4.8) that

$$f(\mathbf{s}^n) - f(\mathbf{s}^*) = \frac{1}{n}.$$

Therefore $f(\mathbf{s^n}) \to f(\mathbf{s}^*)$ as $n \to \infty$; that is, s^n is a minimizing sequence.

Now consider the set $K = N \setminus \{3^k\}_{k\in\mathbf{N}}$. Clearly, $H(K) = \{2n-1\}_{n\in\mathbf{N}}$ and $\delta(K) = 1$; that is the conditions of Corollary 4.5 are satisfied. It is not difficult to verify that $\mathbf{s}^n$ statistically converges to $\mathbf{s}^*$. However, it does not converge to $\mathbf{s}^*$ in the sense of classical convergence as $\rho(\mathbf{s^n}, s^*) \geq 0.5$ for all $n \in \{3^k\}_{k\in\mathbf{N}}$. $\square$

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