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On the metric triangle inequality

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Abstract

A non-contradictible axiomatic theory is constructed under the local reversibility of the metric triangle inequality. The obtained notion includes the metric spaces as particular cases and the generated metric topology is T_1-separated and generally, non-Hausdorff.

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1. Introduction

As it is well-known, a metric space is a pair (X,d) with the metric $d : X \times X \to \mathbb{R}_+$ having the following properties:
(M1) $d(x,x) = 0,$ for all $x \in X$ (reflexivity)
(M2) $d(x,y) = 0 \implies x = y$ (separation, identity of indiscernibles)
(M3) $d(x,y) = d(y,x),$ for all $x, y \in X$ (symmetry)
(M4) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$ (triangle inequality).

Several extensions of this notion were proposed by replacing the set $\mathbb{R}_+$ of metric values with other structures (such as $\mathbb{R}_+ \cup \{\infty\}$, a positive cone in a Banach space, an ordered semigroup, a complete lattice, a lattice ordered group, or the set of fuzzy numbers with positive support, as can be viewed in [4]), or by considering various variants of the axioms (M1)-(M4) and obtaining more modifications of a metric: semi-metric, quasi-metric, quasi-pseudometric, relaxed metric, relaxed pseudo-metric, weak quasi-metric, and so on (see [3] and [4]). Regarding the generalizations of the triangle inequality we can mention:
(i) the tetrahedral inequality: $d(x,y) \le d(x,z) + d(z,w) + d(w,y)$, (see [1]);
(ii) the well-known ultrametric inequality: $d(x,y) \le \max\{d(x,z), d(y,z)\}$;

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(iii) the near-metric inequality: $d(x,y) \leq \rho \cdot (d(x,z) + d(y,z))$, with given $\rho \geq 1$, (see [8]);
(iv) the inframetric inequality: $d(x,y) \leq \rho \cdot \max\{d(x,z), d(y,z)\}$, (see [4]);
(v) the sharp triangle inequality (arising at partial metric spaces [2]): $d(x,z) \leq d(x,y) + d(y,z) - d(y,y)$, $\forall x,y,z \in X$.

Concerning the reversibility of the triangle inequality, from

$$d(x,z) \geq d(x,y) + d(y,z), \quad \forall x,y,z \in X,$$

under the conditions (M1)-(M3), we obtain only the trivial zero-distance $d(x,y) = 0$ for all $x,y \in X$. Although, there are known at least two nontrivial examples for which the triangle inequality is violated: the l_p-metric on $\mathbb{R}^n$ for $0 < p < 1$:

$$d_p(x,y) = \|x-y\|_p = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}},$$

and the time-like metric arising in Cosmology.

In the Relativity Theory, it appears the Einstein time inequality:

$$\|x+y\| \geq \|x\| + \|y\|$$

where the "distance" is the length of the time-like line joining two events in the frame of the Minkowski space-time. The set X of space-time consists of events $x = (x_0, x_1)$ where, $x_0 \in \mathbb{R}$ is the time and $x_1 \in \mathbb{R}^3$ is the spatial location of the event x. A kinematic metric (or *time-like metric*, [4]) is a function $\tau : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ such that, for all $x,y,z \in X$:

$$\tau(x,x) = 0 \tag{1.1}$$

$$\tau(x,y) > 0 \text{ implies } \tau(y,x) = 0 \quad \text{(anti-symmetry)} \tag{1.2}$$

and

$$\begin{gathered}\tau(x,y) > 0,\ \tau(y,z) > 0 \text{ implies } \tau(x,z) > \tau(x,y) + \tau(y,z) \\ \text{(inverse triangle inequality).}\end{gathered} \tag{1.3}$$

Here, the inequality $\tau(x,y) > 0$ means causality (x can influence y) and the value $\tau(x,y)$ is understood as the largest (speed dependent) proper time of moving from x to y.

In the case of a kinematic metric the triangle inequality was reversed replacing the symmetry axiom by (1.2), while for the l_p-metric are preserved all the other axioms (M1)-(M3). The l_p-metrics are applied in Psychology for the problem of geometric representation of stimuli in a perceptual space (see [5] and [7]).

Related to the triangle inequality for l_p-metrics in the case $0 < p < 1$, we can see that for each pair $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ there exist $z, w \in \mathbb{R}^n$ such that $d_p(x,y) \leq d_p(x,w) + d_p(w,y)$ and $d_p(x,y) > d_p(x,z) + d_p(z,y)$. For instance, taking $n = 2$, $x = (0,1)$, $y = (1,0)$, $z = (0,0)$, $w = (2,1)$, and $p = \frac{1}{2}$, we have $4 = d_p(x,y) > d_p(x,z) + d_p(z,y) = 2$ and $4 = d_p(x,y) \leq d_p(x,z) + d_p(z,y) = 6$.

The purpose of this short paper is to provide a noncontradictible axiomatic theory which is verified in spaces like $(\mathbb{R}^n, d_p)$, with $0 < p < 1$, considering an appropriate triangle inequality variant.

2. Extrametric spaces

As we have observed before in the case $(\mathbb{R}^n, d_p)$ with $0 < p < 1$, we can replace the triangle inequality by the following variant: for each pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ there exist two subsets $H(x, y), N(x, y) \subset \mathbb{R}^n$ such that $\mathbb{R}^n = N(x, y) \cup H(x, y)$,

$$d_p(x, y) \leq d_p(x, z) + d_p(z, y), \quad \forall z \in N(x, y)$$

and

$$d_p(x, y) > d_p(x, z) + d_p(z, y), \quad \forall z \in H(x, y).$$

In this way, we arrive to the following definition where the triangle inequality is locally reversed preserving the symmetry axiom. For the obtained notion we have chosen the name *extrametric*.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is an extrametric if the following properties hold:
(E1) $d(x, y) \geq 0$, for all $x, y \in X$;
(E2) $d(x, y) = 0$ iff $x = y$;
(E3) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(E4) for each pair $(x, y) \in X \times X$, there exist two subsets $H(x, y), N(x, y) \subset X$ such that $X = N(x, y) \cup H(x, y)$,

$$d(x, y) \leq d(x, z) + d(z, y), \quad \forall z \in N(x, y)$$

and

$$d(x, y) > d(x, z) + d(z, y), \quad \forall z \in H(x, y).$$

The pair (X, d) is called extrametric space.

Remark 2.2. (i) In the case $x = y$ the triangle inequality is obviously valid having $H(x, x) = \varnothing$ and $N(x, x) = X$ for all $x \in X$. Therefore, the axiom (E4) becomes interesting for the case $x \neq y$. Any metric space is an extrametric space with $H(x, y) = \varnothing$ for all pairs $(x, y) \in X \times X$.
(ii) The set $N(x, y)$ can be divided in two parts $E(x, y)$ and $C(x, y)$ such that

$$z \in E(x, y) \Longleftrightarrow d(x, z) < d(x, y) + d(y, z)$$

and

$$z \in C(x, y) \Longleftrightarrow d(x, z) = d(x, y) + d(y, z).$$

The notion of extrametric space is noncontradictible iff the generated axiomatic theory is consistent, and this fact can be proved by constructing a model that satisfies the axioms (E1)-(E4) of Definition 2.1.

Theorem 2.3. *(i) The topology of an extrametric space is T_1-separated.*
(ii) There exists at least one nontrivial model of extrametric space with $X = \mathbb{R}^n$, where for any pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $x \neq y$, the geometry of the region $H(x, y)$ is non-Euclidean.

Proof . (i) The topology of an extrametric space (X, d) can be generated by open balls. The open ball centered in $u_0 \in X$ and with radius $r > 0$ is the set

$$B(u_0, r) = \{u \in X : d(u_0, u) < r\}.$$

It is obvious that for $r_1 < r_2$ we have $B(u_0, r_1) \subset B(u_0, r_2)$. A set $A \subset X$ is called open iff for any $x \in A$ there exists $r > 0$ such that $B(x, r) \subset A$.
Denoting $\Im = \{A \subset X : A \text{ is open set}\} \cup \{X, \varnothing\}$, we can prove in the classical way that $\Im$ is a topology on X, called the natural topology generated by the extrametric. For instance, let $\{A_i : i = \overline{1,n}\} \subset \Im$ and $x \in \bigcap_{i-1}^{n} A_i$. Since $A_i \in \Im$, $\forall i = \overline{1,n}$, it follows that there exist $r_i > 0$, $i = \overline{1,n}$, such that $B(x, r_i) \subset A_i$, $\forall i = \overline{1,n}$. For $r = \min\{r_i : i = \overline{1,n}\}$ we have $B(x, r) \subset B(x, r_i) \subset A_i$, $\forall i = \overline{1,n}$. Then $B(x, r) \subset \bigcap_{i-1}^{n} A_i$ and $\bigcap_{i-1}^{n} A_i \in \Im$. It is easy to prove that $\bigcup_{i \in I} A_i \in \Im$ for arbitrary family of open sets $\{A_i\}_{i \in I} \subset \Im$. So, $\Im$ is a topology on X. Now, let us consider arbitrary $x, y \in X$, $x \neq y$. Then $r = d(x, y) > 0$ and we have $x \notin B\left(y, \frac{r}{2}\right)$, $y \notin B\left(x, \frac{r}{2}\right)$. We conclude that $(X, \Im)$ is a T_1-separated topological space.
(ii) Consider the Euclidean plane $X = \mathbb{R}^2$ and $D : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined as follows: for arbitrary $x, y \in \mathbb{R}^2$ we construct the segment $[x, y]$ joining the points x and y, and using the Euclidean norm $\|\cdot\|_e$ let

$$D(x, y) = \frac{\pi}{4} \|x - y\|_e^2 . \tag{2.1}$$

$D(x, y)$ is the area of the disk having as diameter the segment $[x, y]$.
Of course, $D(x, y) \geq 0$, for all $x, y \in \mathbb{R}^2$. If $x = y$, then the disk collapses into a point and $D(x, y) = 0$. If $D(x, y) = 0$ then the length of the diameter $[x, y]$ is zero, and thus $x = y$. The equality $D(x, y) = D(y, x)$, for all $x, y \in \mathbb{R}^2$ is obvious.
Now, let arbitrary $x, y \in \mathbb{R}^2$, with $x \neq y$, and $A(x)$, $B(y)$ be the corresponding points in the Euclidean plane. Since these points are distinct we can construct the disk having as diameter the segment $[x, y] = AB$. Denoting the boundary of this disk by $C(x, y)$ and its interior by $H(x, y)$, we can consider the set $E(x, y) = \mathbb{R}^2 \setminus \{C(x, y) \cup H(x, y)\}$ which contains all the points situated outside the disk. Thus $C(x, y) \cup H(x, y)$ represents the corresponding closed disk. According to the Pythagora's generalized identity, for any $z \in C(x, y)$ we have $\|x - y\|_e^2 = \|x - z\|_e^2 + \|z - y\|_e^2$ and consequently, $D(x, y) = D(x, z) + D(z, y)$. Let $z \in H(x, y)$ and $P(z)$ be the corresponding point in the Euclidean plane. We have

$$\|x - y\|_e^2 = \|x - z\|_e^2 + \|z - y\|_e^2 - 2\|x - z\|_e \cdot \|z - y\|_e \cos\left(\widehat{AP, PB}\right) \tag{2.2}$$

and since $\cos\left(\widehat{AP, PB}\right) < 0$, it follows that $\|x - y\|_e^2 > \|x - z\|_e^2 + \|z - y\|_e^2$. So, $D(x, y) > D(x, z) + D(z, y)$ for any $z \in H(x, y)$. If $z \in E(x, y)$ and $M(z)$ is the corresponding point in the Euclidean plane, then $\cos\left(\widehat{AM, MB}\right) > 0$ and from (2.2) we infer that $\|x - y\|_e^2 < \|x - z\|_e^2 + \|z - y\|_e^2$, that is $D(x, y) < D(x, z) + D(z, y)$. Thus, $D(x, y) \leq D(x, z) + D(z, y)$, $\forall z \in N(x, y) = C(x, y) \cup E(x, y)$, and we conclude that $(\mathbb{R}^2, D)$ is a nontrivial extrametric space. Therefore, we can assert that the axiomatic theory of extrametric spaces in noncontradictible.
Now, we see that the region $H(x, y)$ which reverses the triangle inequality is the interior of the circle having the diameter $[x, y]$ represented by the segment AB. In the set $H(x, y)$ we can consider the Cayley's model of non-Euclidean (hyperbolic) geometry. If $z \in H(x, y)$ is represented by the point $P(z)$ and $P \notin (AB)$, then PA and PB are the two non-Euclidean parallel to the non-Euclidean "straight-line" AB. There are infinitely many straight-lines through P intersecting AB in $H(x, y)$ and infinitely many straight-lines through P which not intersect AB inside $H(x, y)$. So, in $H(x, y)$ the geometry is non-Euclidean hyperbolic for each pair $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ with $x \neq y$. $\square$

Remark 2.4. In order to give a geometric model of an extrametric, our intuition must to leave the image of one-dimensional length and to fancy a two-dimensional area that will correspond here to the "distance".
For $u_0 = (x_0, y_0) \in \mathbb{R}^2$, we can observe a connection between the open ball $B(u_0, r)$ in the extrametric space $(\mathbb{R}^2, D)$ and the corresponding open ball in the Euclidean space $(\mathbb{R}^2, \|\cdot\|_e)$. More exactly, for $u = (x, y) \in \mathbb{R}^2$ we have

$$u \in B(u_0, r) \Longleftrightarrow D(u, u_0) < r \Longleftrightarrow \pi \left(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{2} \right)^2 < r$$

$$\Longleftrightarrow (x - x_0)^2 + (y - y_0)^2 < \frac{4r}{\pi} \Longleftrightarrow \|u - u_0\|_e < 2\sqrt{\frac{r}{\pi}}.$$

According to Theorem 2.3, the topology induced by an extrametric is T_1-separated. This topology is not necessarily T_2-separated (Hausdorff), as can be observed in the following examples. In [6] it is shown that in the case of a quasi-metric space, the corresponding metric topology is T_1-separated but not generally Hausdorff separated, and this fact holds in the absence of the symmetry axiom. Here, in the case of an extrametric space, the missing of the Hausdorff separated property will be proved on an example (see Example 2.6) in the presence of the symmetry axiom.

Remark 2.5. The topology of the extrametric space $(\mathbb{R}^2, D)$ presented above is Hausdorff separated. Indeed, in $(\mathbb{R}^2, D)$, for arbitrary $x, y \in \mathbb{R}^2$, $x \neq y$, we have $D(x, y) = r > 0$. Consider the open balls $B\left(x, \frac{r}{4}\right)$ and $B\left(y, \frac{r}{4}\right)$, and by reduction to a contradiction we suppose that $B\left(x, \frac{r}{4}\right) \cap B\left(y, \frac{r}{4}\right) \neq \varnothing$. Then, there exists $z \in B\left(x, \frac{r}{4}\right) \cap B\left(y, \frac{r}{4}\right)$ and consequently, $D(x, z) < \frac{r}{4}$ and $D(y, z) < \frac{r}{4}$. So, $\frac{\pi}{4}\|x - z\|_e^2 < \frac{r}{4}$ and $\frac{\pi}{4}\|z - y\|_e^2 < \frac{r}{4}$, obtaining $\|x - z\|_e + \|z - y\|_e < 2\sqrt{\frac{r}{\pi}}$ (according to Remark 2.4). Since $D(x, y) = r$, we infer that $\|x - y\|_e = 2\sqrt{\frac{r}{\pi}}$ and

$$2\sqrt{\frac{r}{\pi}} = \|x - y\|_e \leq \|x - z\|_e + \|z - y\|_e < 2\sqrt{\frac{r}{\pi}} \Longrightarrow 1 < 1.$$

With this contradiction we get $B\left(x, \frac{r}{4}\right) \cap B\left(y, \frac{r}{4}\right) = \varnothing$ and $(\mathbb{R}^2, D)$ is Hausdorff separated.

Example 2.6. Denoting the set of all nonzero natural numbers by $\mathbb{N}^*$, let $d : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{R}$ be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ \frac{1}{|x-y|}, & \text{if } x \neq y. \end{cases}$$

The axioms (E1)-(E3) are obviously satisfied. For $x, y \in \mathbb{N}^*$, $x \neq y$, and for arbitrary $z \in \mathbb{N}^*$ we distinguish the cases: (i) $z = x$ or $z = y$; (ii) $z \neq x$ and $z \neq y$.
(i) When $z = x$ or $z = y$, it follows that $d(x, y) = d(x, z) + d(z, y)$.
(ii) If $z \neq x$ and $z \neq y$, then denoting $N(x, y) = \{z \in \mathbb{N}^* \mid \frac{1}{|x-y|} \leq \frac{1}{|x-z|} + \frac{1}{|z-y|}\}$ and $H(x, y) = \{z \in \mathbb{N}^* \mid \frac{1}{|x-y|} > \frac{1}{|x-z|} + \frac{1}{|z-y|}\}$, we see that $\mathbb{N}^* = N(x, y) \cup H(x, y)$. There exists at least one pair (x, y), $x \neq y$ such that $H(x, y) \neq \varnothing$. Indeed, taking $x = 1$, $y = 3$ we have $N(x, y) = \{1, 2, 3, 4, 5, 6\}$ and $H(x, y) = [7, \infty) \cap \mathbb{N}^*$. For instance, if $x = 1$, $y = 3$, $z = 7$ we have

$$d(x, y) = \frac{1}{|x - y|} = \frac{1}{2} > \frac{1}{6} + \frac{1}{4} = \frac{1}{|x - z|} + \frac{1}{|z - y|} = d(x, z) + d(y, z).$$

So, $(\mathbb{N}^*, d)$ is an extrametric space. In order to investigate the structure of the open balls $B(x, r)$ in $(\mathbb{N}^*, d)$ we can observe that in the case $r > 1$ it obtains $B(x, r) = \mathbb{N}^*$, for any $x \in \mathbb{N}^*$. We see that $y \in B(x, 1) \setminus \{x\} \Longleftrightarrow |x - y| > 1$, and if $0 < r < 1$ then there exists a natural number $k \in \mathbb{N}^*$ such that $\frac{1}{k+1} < r \leq \frac{1}{k}$. For arbitrary $x \in \mathbb{N}^*$ and for any $y \neq x$ it follows that $y \in B(x, r) \Longleftrightarrow |x - y| > k$, and therefore $B(x, r) = \{x\} \cup \{y \in \mathbb{N}^* : |x - y| > k\}$.
The space $(\mathbb{N}^*, d)$ is obviously T_1-separated as is stated in Theorem 2.3. In this context, for any $x, y \in \mathbb{N}^*$, $x \neq y$, we have $y \notin B\left(x, \frac{1}{k}\right)$ and $x \notin B\left(y, \frac{1}{k}\right)$, where $|x - y| = k \geq 1$. Now, in order to prove that $(\mathbb{N}^*, d)$ is not T_2-separated, we take a pair $(x, y) \in \mathbb{N}^* \times \mathbb{N}^*$, $x \neq y$, and show that $B(x, r) \cap B(y, r) \neq \varnothing$ for any $r > 0$. In this purpose, consider $x, y \in \mathbb{N}^*$, be consecutive natural numbers, that is $|x - y| = 1$. The case $r > 1$ is obvious since $B(x, r) = B(y, r) = \mathbb{N}^*$. In the case $0 < r \leq 1$, having $k \in \mathbb{N}^*$ with $\frac{1}{k+1} < r \leq \frac{1}{k}$, we observe that $\{[x+k+3, \infty) \cap \mathbb{N}^*\} \subset B(x, r) \cap B(y, r)$. So, $(\mathbb{N}^*, d)$ is not T_2-separated.

3. Conclusions

We have locally reversed the triangle inequality, obtaining a metric type T_1-separated topology (without Hausdorff separation property, in general) and a noncontradictible axiomatic theory which contains the metric spaces as particular cases. For the obtained notion we have proposed the name of *extrametric space*, providing an interesting geometric model that illustrates the existence of the hyperbolic non-Euclidean geometry on the place where the triangle inequality is reversed.

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