Int. J. Nonlinear Anal. Appl. 9 (2018) No. 1, 203-210
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2018.3093

Endpoints of multi-valued cyclic contraction mappings

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(Communicated by M. Eshaghi)

Abstract

Endpoint results are presented for multi–valued cyclic contraction mappings on complete metric spaces (X, d). Our results extend previous results given by Nadler (1969), Daffer–Kaneko (1995), Harandi (2010), Moradi and Kojasteh (2012) and Karapinar (2011).

Keywords: Hausdorff metric; multi–valued mapping; generalized weak contraction; endpoint.

2010 MSC: Primary 26A25; Secondary 39B62.

1. Introduction

Let (X, d) be a metric space and $P(X)$ denotes the class of all subsets of X. Define

$$P_f(X) = \Big\{A \subseteq X : A \neq \emptyset \text{ has property f}\Big\}.$$

Thus $P_{bd}(X)$, $P_{cl}(X)$, $P_{cp}(X)$ and $P_{cl,bd}(X)$ denote the classes of bounded, closed, compact and closed bounded subsets of X, respectively. Also $T : X \longrightarrow P_f(X)$ is called a multi–valued mapping on X. A point x is called a fixed point of T if $x \in Tx$. Denote $Fix(T) = \{x \in X : x \in Tx\}$. An element $x \in X$ is said to be an endpoint of multi–valued mapping T, if $Tx = \{x\}$. The set of all endpoints of T denotes by $End(T)$. Obviously, $End(T) \subseteq Fix(T)$. In recent years many authors studied the existence and uniqueness of endpoints for a multi–valued mappings in metric spaces, see for example [1, 5, 6, 11, 12, 14, 15, 19] and references therein.

A multi–valued mapping $T : X \longrightarrow P_{cl,bd}(X)$ is said to be contraction if there exists $0 \leq \alpha < 1$ such that

$$H(Tx, Ty) \leq \alpha d(x, y) \tag{1.1}$$

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Received: November 2016 *Revised:* May 2017

for all $x, y \in X$, where H denotes the Hausdorff metric on $P_{cl,bd}(X)$ induced by d, that is,

$$H(A, B) := \max\left\{\sup_{x\in B} d(x, A), \sup_{x\in A} d(x, B)\right\} \tag{1.2}$$

for all $A, B \in P_{cl,bd}(X)$.

Rhoades [17, Theorem 2] proved the following fixed point theorem for φ–weak contractive single valued mappings, giving another generalization of the Banach Contraction Principle.

Theorem 1.1. Let (X, d) be a complete metric space, and let $T : X \longrightarrow X$ be a mapping such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.3}$$

for every $x, y \in X$ (i.e. φ–weak contractive), where $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous and nondecreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

In the following theorem, Nadler [16] extended the Banach contraction principle to multivalued mappings.

Theorem 1.2. Let (X, d) be a complete metric space. Suppose that $T : X \longrightarrow P_{cl,bd}(X)$ is a contraction mapping in the sense that for some $0 \leq \alpha < 1$,

$$H(Tx, Ty) \leq \alpha d(x, y) \tag{1.4}$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x \in Tx$.

In 2010 Amini–Harandi [1] proved the following endpoint result for a multi–valued mappings of a complete metric space X into $P_{cl,bd}(X)$.

Theorem 1.3. (Amini–Harandi [1, Theorem 2.1]) Let (X, d) be a complete metric space. Suppose that $T : X \longrightarrow P_{cl,bd}(X)$ is a multi–valued mapping that satisfies

$$H(Tx, Ty) \leq \psi(d(x, y)) \tag{1.5}$$

for each $x, y \in X$, where $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ is upper semicontinuous, $\psi(t) < t$ for all $t > 0$, and satisfies $\liminf_{t\longrightarrow\infty}(t - \psi(t)) > 0$. Then T has a unique endpoint if and only if T has the approximate endpoint property.

In 2012 Moradi and Khojasteh [15] proved the following endpoint result and extended the Amini–Harandi's theorem.

Theorem 1.4. Let (X, d) be a complete metric space and let $T : X \longrightarrow P_{cl,bd}(X)$ be a multi–valued mapping that satisfies

$$H(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.6}$$

for each $x, y \in X$, where $\varphi \in \Psi$(i.e., multi–valued φ–weak contractive). Then T has a unique endpoint if and only if T has the approximate endpoint property. Moreover, $End(T) = Fix(T)$.

In (2010) Păcurar [19] presented the following definitions.

Definition 1.5. Let X be a non–empty set, m a positive integer and $T : X \to X$ an operator. By definition, $X = \cup_{i=1}^{m} X_i$ is a cyclic representation on X with respect to T if

(1) $X_i, i = 1, \ldots, m$ are non–empty sets;

(2) $T(X_1) \subseteq X_2, T(X_2) \subseteq X_3, \ldots, T(X_{m-1}) \subseteq X_m, T(X_m) \subseteq X_1$.

Definition 1.6. Let (X, d) be a metric space, m a positive integer, $A_1, A_2, \ldots, A_m$ closed non–empty subsets of X and $Y = \cup_{i=1}^m A_i$. An operator $T : Y \to Y$ is called a cyclic weak φ–contraction if

(1) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T, and

(2) there exists a continuous, non–decreasing function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$, such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.7}$$

for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$.

In 2011, Karapinar [9] proved the following theorem on the existence of fixed point for cyclic weak φ–contraction mappings.

Theorem 1.7. Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ closed non–empty subsets of X and $Y = \bigcup_{i=1}^m A_i$. Let $T : Y \to Y$ be a cyclic weak ϕ–contractive mapping, where $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\phi(t) > 0$ is a continuous function for $t \in (0, +\infty)$, and $\phi(0) = 0$. Then, T has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.

Recently, Moradi [13] extended the Karapinar's theorem. There are another results on the existence of fixed point for cyclic mappings, see for example [2, 3, 4, 7, 8, 10, 18]. In Section 3 we extended Amini–Harandi, Moradi, Karapinar and Moradi and Khojasteh's results for cyclic mappings.

2. Preliminaries

In this work, (X, d) denotes a complete metric space and H denotes the Hausdorff metric on $P_{cl,bd}(X)$ induced by d. We denote by Φ the class of all mappings $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ such that $\varphi^{-1}(0) = \{0\}$ and $\varphi(t) < t$ for all $t > 0$ and satisfies the following condition:

$$\varphi(t_n) \to 0 \text{ implies } t_n \to 0. \tag{2.1}$$

Definition 2.1. Let X be a non–empty set, m a positive integer and $T : X \to P_{cl,bd}(X)$ a multi–valued operator. By definition, $X = \cup_{i=1}^m X_i$ is a cyclic representation on X with respect to T if

(1) $X_i, i = 1, \ldots, m$ are non–empty sets;

(2) $Tx_1 \subseteq X_2, Tx_2 \subseteq X_3, \ldots, Tx_{m-1} \subseteq X_m, Tx_m \subseteq X_1$ for all $x_1 \in X_1, x_2 \in X_2, \ldots, x_m \in X_m$.

Definition 2.2. Let (X, d) be a metric space, m a positive integer, $X_1, X_2, \ldots, X_m$ closed non–empty subsets of X and $X = \cup_{i=1}^m X_i$. An multi–valued operator $T : X \to P_{cl,bd}(X)$ is called a cyclic φ–contraction if

(1) $\cup_{i=1}^{m} X_i$ is a cyclic representation of X with respect to T, and

(2) there exists $\varphi \in \Phi$ such that

$$H(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{2.2}$$

for any $x \in X_i, y \in X_{i+1}, i = 1, 2, \ldots, m$, where $X_{m+1} = X_1$.

Definition 2.3. Let (X, d) be a metric space, m a positive integer, $X_1, X_2, \ldots, X_m$ closed non–empty subsets of X and $X = \cup_{i=1}^{m} X_i$. Suppose that $X = \cup_{i=1}^{m} X_i$ is a cyclic representation on X with respect to multi–valued operator $T : X \to P_{cl,bd}(X)$. We say that T has the approximate cyclic endpoint property if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \in X_n, n = 1, 2, 3, \ldots$ and

$$\lim_{n\to\infty} H(\{x_n\}, Tx_n) = 0,$$

where $X_{m+1} = X_1, X_{m+2} = X_2, \ldots, X_{2m} = X_m, X_{2m+1} = X_1, X_{2m+2} = X_2, \ldots$.

Definition 2.4. Let (X, d) be a metric space, m a positive integer, $X_1, X_2, \ldots, X_m$ closed non–empty subsets of X and $X = \cup_{i=1}^{m} X_i$ be a cyclic representation on X with respect to multi–valued operator $T : X \to P_{cl,bd}(X)$. We say that the fixed point problem is well–posed for T with respect to H if

(1) $End(T) = \{x\}$,

(2) if $\{x_n\} \subset X$ and $\lim_{n\to\infty} H(\{x_n\}, Tx_n) = 0$, then $\lim_{n\to\infty} x_n = x$, where, $x_i \in X_i$ for all $i \in \mathbb{N}$ and where $X_{m+1} = X_1, X_{m+2} = X_2, \ldots, X_{2m} = X_m, X_{2m+1} = X_1, X_{2m+2} = X_2, \ldots$.

3. The main results

The following theorem is the main theorem of this paper, providing a new type of endpoint theorem for cyclic multi-valued operator. This theorem extends the Amoni–Harandi and Moradi and Khojasteh' theorems.

Theorem 3.1. *Let (X, d) be a metric space, m a positive integer, $X_1, X_2, \ldots, X_m$ closed non–empty subsets of X and $X = \cup_{i=1}^{m} X_i$ be a cyclic representation on X with respect to multi–valued operator $T : X \to P_{cl,bd}(X)$. Let T be a cyclic φ–contraction for some $\varphi \in \Phi$, that is,*

$$H(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{3.1}$$

for all $x \in X_i, y \in X_{i+1}$, where $X_{m+1} = X_1$. Then T has a unique endpoint if and only if T has the approximate cyclic endpoint property. Moreover, $Fix(T) = End(T)$. Also the fixed point problem is well–posed for T.

Proof . At first we consider that $X_{m+1} = X_1$, $X_{m+2} = X_2, \ldots$, $X_{2m} = X_m$, $X_{2m+1} = X_1$, $X_{2m+2} = X_2, \ldots$. It is clear that, if T has an endpoint $x \in X$ then $x \in \cap_{i=1}^{m} X_i$. By define $x_1 = x_2 = x_3 = \cdots = x$ we have $x_1 \in X_1$, $x_2 \in X_2$, $x_3 \in X_3, \ldots$ and $\lim_{n\to\infty} H(\{x_n\}, Tx_n) = 0$. Therefore T has the approximate cyclic endpoint property.

Conversely, suppose that T has the approximate cyclic endpoint property; then there exists a sequence $\{x_n\}$ in X such that for all $n \in \mathbb{N}$; $x_n \in X_n$ and

$$\lim_{n\to\infty} H(\{x_n\}, Tx_n) = 0.$$

For all $n \in \mathbb{N}$ from $x_n \in X_n, x_{n+1} \in X_{n+1}$;

$$\begin{aligned} d(x_{n+1}, x_n) &= H(\{x_{n+1}\}, \{x_n\}) \leq H(\{x_{n+1}\}, Tx_{n+1}) + H(Tx_{n+1}, Tx_n) + H(\{x_n\}, Tx_n) \\ &\leq H(\{x_{n+1}\}, Tx_{n+1}) + d(x_{n+1}, x_n) - \varphi(d(x_{n+1}, x_n)) + H(\{x_n\}, Tx_n). \end{aligned}$$

So

$$\varphi(d(x_{n+1}, x_n)) \leq H(\{x_{n+1}\}, Tx_{n+1}) + H(\{x_n\}, Tx_n).$$

This shows that $\lim_{n\to\infty} \varphi(d(x_{n+1}, x_n)) = 0$ and since $\varphi \in \Phi$ we get

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.2}$$

We claim that $\{x_n\}$ is a Cauchy sequence. Indeed, if it is false, then there exist $a > 0$ and the subsequence $\{n(k)\}_{k=1}^{\infty}$ such that $n(k+1) > n(k)$ is minimal in the sense that $d(x_{n(k+1)}, x_{n(k)}) > a$. Obviously, $n(k) \geq k$ for all $k \in \mathbb{N}$. Using (3.2) there exists $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$, $d(x_{k+1}, x_k) < \frac{a}{3}$. So for all $k \geq N_0$, $n(k+1) - n(k) \geq 2$ and by using the triangle inequality, we obtain

$$\begin{aligned} a < d(x_{n(k+1)}, x_{n(k)}) &\leq d(x_{n(k+1)}, x_{n(k+1)-1}) + d(x_{n(k+1)-1}, x_{n(k)}) \\ &\leq d(x_{n(k+1)}, x_{n(k+1)-1}) + a. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.2), we get

$$\lim_{k\to\infty} d(x_{n(k+1)}, x_{n(k)}) = a. \tag{3.3}$$

For all $k \geq N_0$, there exists $l(k) \in \{0, 1, 2, \ldots, m\}$ such that $n(k+1) - l(k) \equiv n(k) + 1 (mod\, m)$. Now we show that $\lim_{k\to\infty} d(x_{n(k+1)-l(k)}, x_{n(k)}) = a$. For all $k \geq N_0$,

$$\begin{aligned} & d(x_{n(k+1)}, x_{n(k)}) - d(x_{n(k+1)}, x_{n(k+1)-1}) - \cdots - d(x_{n(k+1)-l(k)+1}, x_{n(k+1)-l(k)}) \\ & \leq d(x_{n(k+1)-l(k)}, x_{n(k)}) \\ & \leq d(x_{n(k+1)}, x_{n(k)}) + d(x_{n(k+1)}, x_{n(k+1)-1}) + \cdots + d(x_{n(k+1)-l(k)+1}, x_{n(k+1)-l(k)}). \end{aligned} \tag{3.4}$$

Letting $k \to \infty$ in (3.4) and using (3.2) and (3.3), we get

$$\lim_{k\to\infty} d(x_{n(k+1)-l(k)}, x_{n(k)}) = a. \tag{3.5}$$

Also for all $k \geq N_0$,

$$\begin{aligned} & d(x_{n(k+1)}, x_{n(k+1)-1}) - d(x_{n(k+1)-1}, x_{n(k+1)-2}) - \cdots - d(x_{n(k+1)-l(k)+1}, x_{n(k+1)-l(k)}) \\ & \leq d(x_{n(k+1)}, x_{n(k+1)-l(k)}) \\ & \leq d(x_{n(k+1)}, x_{n(k+1)-1}) + d(x_{n(k+1)-1}, x_{n(k+1)-2}) + \cdots + d(x_{n(k+1)-l(k)+1}, x_{n(k+1)-l(k)}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.2) and $0 \le l(k) \le m$ for all $k \ge N_0$, we conclude that

$$\lim_{k\to\infty} d(x_{n(k+1)}, x_{n(k+1)-l(k)}) = 0. \tag{3.6}$$

Since for all $k \ge N_0$, $n(k+1) - l(k) \equiv n(k) + 1 (mod\, m)$; from (3.1)

$$\begin{aligned} d(x_{n(k+1)}, x_{n(k)}) &\le d(x_{n(k+1)}, x_{n(k+1)-l(k)}) + d(x_{n(k+1)-l(k)}, x_{n(k)}) \\ &= d(x_{n(k+1)}, x_{n(k+1)-l(k)}) + H(\{x_{n(k+1)-l(k)}\}, \{x_{n(k)}\}) \\ &\le d(x_{n(k+1)}, x_{n(k+1)-l(k)}) + H(\{x_{n(k+1)-l(k)}\}, Tx_{n(k+1)-l(k)}) \\ &\quad + H(Tx_{n(k+1)-l(k)}, Tx_{n(k)}) + H(\{x_{n(k)}\}, Tx_{n(k)}) \\ &\le d(x_{n(k+1)}, x_{n(k+1)-l(k)}) + H(\{x_{n(k+1)-l(k)}\}, Tx_{n(k+1)-l(k)}) \\ &\quad + d(x_{n(k+1)-l(k)}, x_{n(k)}) - \varphi(d(x_{n(k+1)-l(k)}, x_{n(k)})) \\ &\quad + H(\{x_{n(k)}\}, Tx_{n(k)}). \end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned} 0 &\le \varphi(d(x_{n(k+1)-l(k)}, x_{n(k)})) \\ &\le d(x_{n(k+1)}, x_{n(k+1)-l(k)}) + H(\{x_{n(k+1)-l(k)}\}, Tx_{n(k+1)-l(k)}) + d(x_{n(k+1)-l(k)}, x_{n(k)}) \\ &\quad + H(\{x_{n(k)}\}, Tx_{n(k)}) - d(x_{n(k+1)}, x_{n(k)}). \end{aligned} \tag{3.8}$$

Letting $k \to \infty$ in (3.8) and using (3.3), (3.5) and (3.6) we get

$$\lim_{k\to\infty} \varphi(d(x_{n(k+1)-l(k)}, x_{n(k)})) = 0. \tag{3.9}$$

Since $\varphi \in \Phi$ then $\lim_{k\to\infty} d(x_{n(k+1)-l(k)}, x_{n(k)}) = 0$ and this is a contradiction. Thus $\{x_n\}$ is Cauchy. Since (X, d) is complete and $\{x_n\}$ is Cauchy, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. From $\lim_{n\to\infty} x_{nm+i} = x$ and $\{x_{nm+i} : n \in \mathbb{N}\} \subseteq X_i$ we conclude that $x \in X_i$ for $i = 1, 2, \ldots, m$. Hence $x \in \cap_{i=1}^m X_i$. Now for all $n \in \mathbb{N}$,

$$\begin{aligned} H(\{x\}, Tx) &\le H(\{x\}, \{x_n\}) + H(\{x_n\}, Tx_n) + H(Tx_n, Tx) \\ &\le d(x, x_n) + H(\{x_n\}, Tx_n) + d(x_n, x) - \varphi(d(x_n, x)). \end{aligned} \tag{3.10}$$

Letting $n \to \infty$ in (3.10) we get $Tx = \{x\}$. Hence T has an endpoint. Uniqueness of endpoint in $\cap_{i=1}^m X_i$ follows from (3.1). Also every endpoint of T belong to $\cap_{i=1}^m X_i$. Therefore $End(T) = \{x\}$. Now suppose $y \in Fix(T)$ is arbitrary. Obviously, $y \in \cap_{i=1}^m X_i$. We need to show that $y = x$. Suppose that $y \ne x$. Then from $y \in Ty$

$$d(x, y) \le H(\{x\}, Ty) = H(Tx, Ty) \le d(x, y) - \varphi(d(x, y)) < d(x, y), \tag{3.11}$$

and this is a contradiction. Therefore $End(T) = Fix(T)$. Also the proof of theorem shows that the fixed point problem is well–posed and this completes the proof. □

Remark 3.2. By taking $X_1 = X_2 = \cdots = X_m = X$ and using Theorem 3.1 we can generalized Moradi and Khojasteh's theorem (Theorem 1.6). Also by define $\varphi(t) = t - \psi(t)$ and using Theorem 3.1 we conclude Amini-Harandi's theorem (Theorem 1.3).

The following corollary is a direct result of Theorem 3.1.

Corollary 3.3. *Let (X,d) be a metric space, m a positive integer, $X_1, X_2, \dots, X_m$ closed non–empty subsets of X and $X = \cup_{i=1}^m X_i$ be a cyclic representation on X with respect to single–valued operator $f : X \to X$. Let f be a cyclic φ–contraction for some $\varphi \in \Phi$, that is,*

$$d(fx, fy) \le d(x,y) - \varphi(d(x,y)) \tag{3.12}$$

for all $x \in X_i, y \in X_{i+1}$, where $X_{m+1} = X_1$. Then f has a unique fixed point if and only if f has the approximate cyclic fixed point property.

Proof . Let $Tx = \{fx\}$ and apply Theorem 3.1. □

The following theorem shows that for single–valued mappings the condition (3.12) is sufficient for f to have the approximate cyclic fixed point property.

Theorem 3.4. *Let (X,d) be a metric space, m a positive integer, $X_1, X_2, \dots, X_m$ closed non–empty subsets of X and $X = \cup_{i=1}^m X_i$ be a cyclic representation on X with respect to single–valued operator $f : X \to X$. Let f be a cyclic φ–contraction for some $\varphi \in \Phi$, that is,*

$$d(fx, fy) \le d(x,y) - \varphi(d(x,y)) \tag{3.13}$$

for all $x \in X_i, y \in X_{i+1}$, where $X_{m+1} = X_1$. Then f has the approximate cyclic fixed point property.

Proof . Let $x_1 \in X, x_2 = fx_1, x_3 = fx_2, \dots$. We may assume that $x_1 \in X_1$. Hence $x_2 \in X_2, x_3 \in X_3, \dots$. For all $n \in \mathbb{N}$, from $x_{n-1} \in X_{n-1}$ and $x_n \in X_n$;

$$d(x_n, fx_n) = d(fx_{n-1}, fx_n) \le d(x_{n-1}, x_n) - \varphi(d(x_{n-1}, x_n)). \tag{3.14}$$

Therefore the sequence $\{d(x_n, x_{n+1})\}$ is monotone non-increasing and bounded below. So, there exists $r > 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. By using (3.14) we conclude that

$$\varphi(d(x_{n-1}, x_n)) \le d(x_{n-1}, x_n) - d(x_n, x_{n+1}), \tag{3.15}$$

and so $\lim_{n\to\infty} \varphi(d(x_{n-1}, x_n)) = 0$. Hence $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$. Thus $\lim_{n\to\infty} d(x_{n-1}, fx_{n-1}) = 0$ and so $\inf_{x\in X} d(x, fx) = 0$. Therefore f has the approximate cyclic fixed point property. □

Remark 3.5. As an application of Corollary 3.3 and Theorem 3.4, we obtain the following fixed point results. This corollary extends Rhoades and Karapinar theorems.

Corollary 3.6. Let (X,d) be a metric space, m a positive integer, $X_1, X_2, \dots, X_m$ closed non–empty subsets of X and $X = \cup_{i=1}^m X_i$ be a cyclic representation on X with respect to single–valued operator $f : X \to X$. Let f be a cyclic φ–contraction for some $\varphi \in \Phi$, that is,

$$d(fx, fy) \le d(x,y) - \varphi(d(x,y)) \tag{3.16}$$

for all $x \in X_i, y \in X_{i+1}$, where $X_{m+1} = X_1$. Then f has a unique fixed point and for every $x_0 \in X$, the sequence $\{f^n(x_0)\}$ converges to this fixed point. Also the fixed point problem for f is well–posed.

The following corollary extends the Nadler's theorem (Theorem 1.2).

Corollary 3.7. *Let (X,d) be a metric space, m a positive integer, $X_1, X_2, \ldots, X_m$ closed non–empty subsets of X, $\cap_{i=1}^m X_i \neq \emptyset$ and $X = \cup_{i=1}^m X_i$ be a cyclic representation on X with respect to multi–valued operator $T : X \to P_{cl,bd}(X)$. Let T be a cyclic contraction mapping, that is,*

$$H(Tx,Ty) \leq \alpha d(x,y) \tag{3.17}$$

for all $x \in X_i, y \in X_{i+1}$ and for some $\alpha \in [0,1)$, where $X_{m+1} = X_1$. Then there exists a point $x \in \cap_{i=1}^m X_i$ such that $x \in X$. Also if T has the approximate cyclic endpoint property, then $Fix(T) = End(T) = \{x\}$. (So the fixed point is unique.)

Proof . Let $Y = \cap_{i=1}^m X_i$. Obviously, $Ty \in P_{cl,bd}(Y)$ for all $y \in Y$. Using Theorem 1.2, there exists $x \in Y$ such that $x \in Tx$. If T has the approximate endpoint property, then T has the unique endpoint and $End(T) = Fix(T) = \{x\}$. □

Acknowledgments

This research was in part supported by a grant from Arak University (no. 93/13596). The author would like to thank this support.

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