New Hermite-Hadamard type inequalities on fractal set

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Abstract

In this study, we present the new Hermite-Hadamard type inequality for functions which are $h$-convex on fractal set $\mathbb{R}^\alpha$ ($0 < \alpha \leq 1$) of real line numbers. Then we provide the special cases of the result using different type of convex mappings.

Keywords: Hermite-Hadamard inequality; fractal set; $h$-convex function.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. If $f$ is a convex function then the following double inequality holds [3]:

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \quad (1.1)$$

The above inequality [1.1] which is well known in the literature as the Hermite–Hadamard inequality, is the most fundamental and interesting inequality for classical convex functions. This inequality provides a lower and an upper estimations for the integral average of any convex functions defined on a compact interval. For numerous interesting results which generalize, improve and extend the classical Hermite-Hadamard inequality see for instance [3], [10] and references therein.

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2. The preliminaries

The concepts of fractional calculus [6] and local fractional calculus (also called fractal calculus) (see, for details, [18], [19] and [20]) are becoming increasingly useful in a wide variety of problems in mathematical, physical and engineering sciences (see, for example, [21] to [24]). We need the following notations and preliminaries to define the local fractional derivative and the local fractional integral.

Recall the set $\mathbb{R}^\alpha$ of real line numbers and use the Gao-Yang-Kang’s idea to describe the definition of the local fractional derivative and local fractional integral, (see [18], [19], [20]) and so on. Recently, the theory of Yang’s fractional sets [19] was introduced as follows:

For $0 < \alpha \leq 1$, we have the following $\alpha$-type set of element sets:

- $\mathbb{Z}^\alpha$: The $\alpha$-type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, \ldots, \pm n^\alpha, \ldots\}$.
- $\mathbb{Q}^\alpha$: The $\alpha$-type set of the rational numbers is defined as the set $\{m^\alpha = (\frac{p}{q})^\alpha : p, q \in \mathbb{Z}, q \neq 0\}$.
- $\mathbb{J}^\alpha$: The $\alpha$-type set of the irrational numbers is defined as the set $\{m^\alpha \neq (\frac{p}{q})^\alpha : p, q \in \mathbb{Z}, q \neq 0\}$.
- $\mathbb{R}^\alpha$: The $\alpha$-type set of the real line numbers is defined as the set $\mathbb{R}^\alpha = \mathbb{Q}^\alpha \cup \mathbb{J}^\alpha$.

If $a^\alpha$, $b^\alpha$ and $c^\alpha$ belongs the set $\mathbb{R}^\alpha$ of real line numbers, then

1. $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belongs the set $\mathbb{R}^\alpha$;
2. $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a + b)^\alpha = (b + a)^\alpha$;
3. $a^\alpha + (b^\alpha + c^\alpha) = (a + b)^\alpha + c^\alpha$;
4. $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
5. $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;
6. $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
7. $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows:

**Definition 2.1.** (Yang [19]) A non-differentiable function $f : \mathbb{R} \to \mathbb{R}^\alpha$, $x \to f(x)$ is called to be local fractional continuous at $x_0$, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in \mathbb{R}$. If $f(x)$ is local continuous on the interval $(a, b)$, we denote $f(x) \in C_{\alpha}(a, b)$.

**Definition 2.2.** (Yang [19]) The local fractional derivative of $f(x)$ of order $\alpha$ at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left. \frac{d^n f(x)}{dx^n} \right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha (f(x) - f(x_0)) = \Gamma(1 + \alpha) (f(x) - f(x_0))$. If there exists $f^{(k+1)\alpha}(x) = D^\alpha_x \cdots D^\alpha_x f(x)$ for any $x \in I \subseteq \mathbb{R}$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \ldots$
Definition 2.3. (Yang [19]) Let \( f(x) \in C_{\alpha [a,b]} \). Then the local fractional integral is defined by,

\[
a I_{b}^{\alpha} f(x) = \frac{1}{\Gamma(1 + \alpha)} \int_{a}^{b} f(t)(dt)^{\alpha} = \frac{1}{\Gamma(1 + \alpha)} \lim_{N \to 0} \sum_{j=0}^{N-1} f(t_{j})(\Delta t_{j})^{\alpha},
\]

with \( \Delta t_{j} = t_{j+1} - t_{j} \) and \( \Delta t = \max \{ \Delta t_{1}, \Delta t_{2}, ..., \Delta t_{N-1} \} \), where \([t_{j}, t_{j+1}]\), \( j = 0, ..., N - 1 \) and \( a = t_{0} < t_{1} < ... < t_{N-1} < t_{N} = b \) is partition of interval \([a,b] \).

Here, it follows that \( a I_{b}^{\alpha} f(x) = 0 \) if \( a = b \) and \( a I_{b}^{\alpha} f(x) = -b I_{a}^{\alpha} f(x) \) if \( a < b \). If for any \( x \in [a, b] \), there exists \( a I_{b}^{\alpha} f(x) \), then we denoted by \( f(x) \in I_{x}^{\alpha [a,b]} \).

Lemma 2.4. (Yang [19])

(i) (Local fractional integration is anti-differentiation) Suppose that \( f(x) = g^{(\alpha)}(x) \in C_{\alpha [a,b]} \), then we have

\[
a I_{b}^{\alpha} f(x) = g(b) - g(a).
\]

(ii) (Local fractional integration by parts) Suppose that \( f(x), g(x) \in D_{\alpha [a,b]} \) and \( f^{(\alpha)}(x), g^{(\alpha)}(x) \in C_{\alpha [a,b]} \), then we have

\[
a I_{b}^{\alpha} f(x)g^{(\alpha)}(x) = f(x)g(x)|_{a}^{b} - a I_{b}^{\alpha} f^{(\alpha)}(x)g(x).
\]

Lemma 2.5. (Yang [19])

(i) \( \frac{d^{\alpha} x^{k\alpha}}{dx^{\alpha}} = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k-1)\alpha)} x^{(k-1)\alpha} \),

(ii) \( \frac{1}{\Gamma(1 + \alpha)} \int_{a}^{b} x^{k\alpha} (dx)^{\alpha} = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k+1)\alpha)} (b^{(k+1)\alpha} - a^{(k+1)\alpha}) \), \( k \in \mathbb{R} \).

Now, we give some definitions which are used in our results:

Definition 2.6. (Mo, Sui, Yu [7]) Let \( f : I \subseteq \mathbb{R} \to \mathbb{R}^{\alpha} \). For any \( x_{1}, x_{2} \in I \) and \( \lambda \in [0, 1] \), if the following inequality

\[
f(\lambda x_{1} + (1-\lambda)x_{2}) \leq \lambda^{\alpha} f(x_{1}) + (1-\lambda)^{\alpha} f(x_{2})
\]

holds, then \( f \) is called a generalized convex function on \( I \). If this inequality reversed, then \( f \) is called a generalized concave function.

Here are two basic examples of generalized convex functions:

(i) \( f(x) = x^{\alpha p} \), \( x \geq 0 \), \( p > 1 \);

(ii) \( f(x) = E_{\alpha}(x^{\alpha}) \), \( x \in \mathbb{R} \) where \( E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{x^{ak}}{\Gamma(1+k\alpha)} \) is the Mittag-Leffler function.

In [7], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 2.7. Let \( f(x) \in I_{x}^{\alpha [a,b]} \) be generalized convex function on \([a, b] \) with \( a < b \). Then

\[
f \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(1 + \alpha)}{(b-a)^{\alpha}} a I_{b}^{\alpha} f(x) \leq \frac{f(a) + f(b)}{2^{\alpha}}.
\]
In [17], the definition of \( h \)-convex functions on fractal sets was established by Vivas et al., as follows:

**Definition 2.8.** Let \( h : J \to \mathbb{R}^\alpha \) be a non-negative function and \( h \neq 0 \), defined over an interval \( J \subset \mathbb{R} \) and such that \( (0, 1) \subset J \). We say that \( f : I \to \mathbb{R}^\alpha \) defined over an interval \( I \subset \mathbb{R} \), is \( h \)-convex if \( f \) is non-negative and we have

\[
f(tx_1 + (1 - t)x_2) \leq h(t)f(x_1) + h(1 - t)f(x_2)
\]

for all \( t \in (0, 1) \) and \( x_1, x_2 \in I \).  

**Example 2.9.** Let \( 0 < s < 1 \), \( h : (0, 1) \to \mathbb{R}^\alpha \) defined as 
\[
h(t) = t^{s\alpha}
\]
and \( a^\alpha, b^\alpha, c^\alpha \in \mathbb{R}^\alpha \). For \( x \in \mathbb{R}_+ = [0, \infty) \), define 
\[
f(x) = \begin{cases} 
  a^\alpha, & x = 0 \\
  b^\alpha x^{s\alpha} + c^\alpha, & x > 0 
\end{cases}
\]

In [8], Mo and Sui introduced the definitions of two kinds of generalized \( s \)-convex functions on fractal sets such as follows:

**Definition 2.10.** (i) Let \( \mathbb{R}_+ = [0, \infty) \). A function \( f : \mathbb{R}_+ \to \mathbb{R}^\alpha \) is said to be generalized \( s \)-convex \( (0 < s < 1) \) in the first sense, if

\[
f(\lambda_1 u + \lambda_2 v) \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v),
\]

for all \( u, v \in \mathbb{R}_+ \) and all \( \lambda_1, \lambda_2 \geq 0 \) with \( \lambda_1^s + \lambda_2^s = 1 \). One denotes by \( f \in GK^1_s \).

(ii) A function \( f : \mathbb{R}_+ \to \mathbb{R}^\alpha \) is said to be generalized \( s \)-convex \( (0 < s < 1) \) in the second sense, if

\[
f(\lambda_1 u + \lambda_2 v) \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v),
\]

for all \( u, v \in \mathbb{R}_+ \) and all \( \lambda_1, \lambda_2 \geq 0 \) with \( \lambda_1 + \lambda_2 = 1 \). One denotes by \( f \in GK^2_s \).

Note that, if \( s = 1 \) in Definition 2.10 and we have the generalized convex function.

For more information and recent developments on local fractional theory, please refer to [1], [2], [4]-[9], [11]-[20], [22], [23].

The main goal of this article is to establish new Hermite-Hadamard type inequalities for \( h \)-convex.

### 3. The main results

We start with the following important theorem for our work.

**Theorem 3.1.** Let \( h : [0, 1] \to \mathbb{R}^\alpha \) be a non-negative function and \( f : I \to \mathbb{R}^\alpha \) be a \( h \)-convex function such that \( h \left( \frac{1}{2} \right) \neq 0^\alpha \) and \( _{0}I_{t}^{\alpha}h(t) \geq \left( \frac{1}{2} \right)^{\alpha} \), then

\[
\frac{1}{2^{2\alpha} \left[ h \left( \frac{1}{2} \right) \right]^2} f \left( \frac{a + b}{2} \right) \leq \Delta_1 \leq \frac{\Gamma(1 + \alpha)}{(b - a)^\alpha} \ _{0}I_{t}^{\alpha} f(x)
\]

\[
\leq \Delta_2 \leq \Gamma(1 + \alpha) \left[ f(a) + f(b) \right] \left\{ h \left( \frac{1}{2} \right) + \left( \frac{1}{2} \right)^{\alpha} \right\} \ _{0}I_{t}^{\alpha}h(t),
\]

where \( \alpha, \beta, \gamma \) are positive numbers and \( \alpha, \beta, \gamma \) are the fractional orders.
Similarly, for 

\[ \Delta_1 = \frac{1}{2^{2\alpha} h(\frac{1}{2})} \left[ f \left( \frac{a + 3b}{4} \right) + f \left( \frac{3a + b}{4} \right) \right] \]

and

\[ \Delta_2 = \Gamma(1 + \alpha) \left[ f(a) + f \left( \frac{b + a}{2} \right) \right] \int_0^h \right. \]

Proof. Firstly, we divide interval \([a, b]\) into \([a, \frac{a+b}{2}]\) and \([\frac{a+b}{2}, b]\). Since \(f\) function is \(h\)-convex function, for \(\left[ a, \frac{a+b}{2} \right] \) we have

\[ f \left( \frac{a + \frac{a+b}{2}}{2} \right) = f \left( ta + (1-t) \frac{a+b}{2} + (1-t) a + t \frac{a+b}{2} \right) \]

\[ \leq h \left( \frac{1}{2} \right) \left[ f \left( ta + (1-t) \frac{a+b}{2} \right) + f \left( (1-t) a + t \frac{a+b}{2} \right) \right] . \]

Integrating both sides of above inequality with respect to \(t\) on \([0, 1]\), we obtain

\[ \frac{1}{2^{2\alpha} h(\frac{1}{2})} f \left( \frac{3a + b}{4} \right) \leq \frac{\Gamma(1 + \alpha)}{(b-a)^{\alpha}} aI_b^a f(x). \] (3.1)

Similarly, for \(\left[ \frac{a+b}{2}, b \right] \) we have

\[ f \left( \frac{\frac{a+b}{2} + b}{2} \right) = f \left( t\frac{a+b}{2} + (1-t)b + (1-t) \frac{a+b}{2} + tb \right) \]

\[ \leq h \left( \frac{1}{2} \right) \left[ f \left( \frac{a+b}{2} + (1-t)b \right) + f \left( (1-t) \frac{a+b}{2} + tb \right) \right] . \]

Integrating both sides of above inequality with respect to \(t\) on \([0, 1]\), we obtain

\[ \frac{1}{2^{2\alpha} h(\frac{1}{2})} f \left( \frac{a + 3b}{4} \right) \leq \frac{\Gamma(1 + \alpha)}{(b-a)^{\alpha}} \frac{a+b}{2} b I^a_b f(x). \] (3.2)

By adding inequalities (3.1) and (3.2), it yields

\[ \Delta_1 = \frac{1}{2^{2\alpha} h(\frac{1}{2})} \left[ f \left( \frac{a + 3b}{4} \right) + f \left( \frac{3a + b}{4} \right) \right] \]

\[ \leq \frac{\Gamma(1 + \alpha)}{(b-a)^{\alpha}} aI_b^a f(x) \]

\[ = \frac{\Gamma(1 + \alpha)}{2^{\alpha}} \left[ \frac{2^{\alpha}}{(b-a)^{\alpha}} aI_{\frac{a+b}{2}}^a f(x) + \frac{2^{\alpha}}{(b-a)^{\alpha}} \frac{a+b}{2} b I^a_b f(x) \right] \]

\[ \leq \frac{\Gamma(1 + \alpha)}{2^{\alpha}} \left[ \left\{ f(a) + f \left( \frac{a+b}{2} \right) \right\} aI_h^a h(t) \right] + \frac{\Gamma(1 + \alpha)}{2^{\alpha}} \left[ \left\{ f \left( \frac{a+b}{2} \right) + f(b) \right\} aI_h^a h(t) \right] \]

\[ = \frac{\Gamma(1 + \alpha)}{2^{\alpha}} \left[ f(a) + f(b) + 2^{\alpha} f \left( \frac{a+b}{2} \right) \right] aI_h^a h(t) = \Delta_2. \]
On the other hand, since \( f \) is \( h \)-convex function and \( \Gamma(1 + \alpha)_{0} I_{1}^{\alpha} h(t) \geq (\frac{1}{2})^{\alpha} \), we deduce that

\[
\frac{1}{2^{2\alpha} [h(\frac{1}{2})]^{2}} f\left(\frac{a + b}{2}\right) = \frac{1}{2^{2\alpha} [h(\frac{1}{2})]^{2}} f\left(\frac{13a + b}{4} + \frac{1a + 3b}{4}\right)
\leq \frac{1}{2^{2\alpha} [h(\frac{1}{2})]^{2}} \left[ h\left(\frac{1}{2}\right) \left\{ f\left(\frac{3a + b}{4}\right) + f\left(\frac{a + 3b}{4}\right) \right\} \right]
= \frac{1}{2^{2\alpha} [h(\frac{1}{2})]^{2}} \left[ f\left(\frac{3a + b}{4}\right) + f\left(\frac{a + 3b}{4}\right) \right] = \Delta_{1}
\leq \frac{1}{2^{2\alpha} [h(\frac{1}{2})]^{2}} \left[ h\left(\frac{1}{2}\right) \left\{ f(a) + f(b) + 2^{\alpha} f\left(\frac{a + b}{2}\right) \right\} \right]
= \left(\frac{1}{2}\right)^{\alpha} \left[ f(a) + f(b) \right] + f\left(\frac{a + b}{2}\right) \]
\leq \Gamma(1 + \alpha) \left[ \frac{f(a) + f(b)}{2^{\alpha}} + f\left(\frac{a + b}{2}\right) \right]_{0} I_{1}^{\alpha} h(t) = \Delta_{2}
\leq \Gamma(1 + \alpha) \left[ \frac{f(a) + f(b)}{2^{\alpha}} + h\left(\frac{1}{2}\right) \left\{ f(a) + f(b) \right\} \right]_{0} I_{1}^{\alpha} h(t)
= \Gamma(1 + \alpha) \left[ f(a) + f(b) \right] \left\{ h\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{\alpha} \right\} \right]_{0} I_{1}^{\alpha} h(t).

This completes the proof. □

**Corollary 3.2.** If we choose \( h(t) = t^{\alpha} \) in Theorem 3.1, we obtain

\[
f\left(\frac{a + b}{2}\right) \leq \Delta_{1} \leq \frac{\Gamma(1 + \alpha)}{(b - a)^{\alpha}} \circ I_{0}^{\alpha} f(x)
\leq \Delta_{2} \leq [f(a) + f(b)] \frac{[\Gamma(1 + \alpha)]^{2}}{\Gamma(1 + 2\alpha)},
\]

where

\[
\Delta_{1} = \frac{1}{2^{\alpha}} \left[ f\left(\frac{a + 3b}{4}\right) + f\left(\frac{3a + b}{4}\right) \right]
\]

and

\[
\Delta_{2} = \left[ \frac{f(a) + f(b)}{2^{\alpha}} + f\left(\frac{a + b}{2}\right) \right] \frac{[\Gamma(1 + \alpha)]^{2}}{\Gamma(1 + 2\alpha)}.
\]

**Corollary 3.3.** Let \( f : I \rightarrow \mathbb{R}^{\alpha} \) be a generalized \( s \)-convex function in the second sense where \( s \in (0, 1] \) such that \( \Gamma(1 + \alpha)_{0} I_{1}^{\alpha} t^{\alpha} \geq (\frac{1}{2})^{\alpha} \), then

\[
2^{(2s - 2)\alpha} f\left(\frac{a + b}{2}\right) \leq \Delta_{1} \leq \frac{\Gamma(1 + \alpha)}{(b - a)^{\alpha}} \circ I_{0}^{\alpha} f(x)
\leq \Delta_{2} \leq [f(a) + f(b)] \left\{ \left(\frac{1}{2^{s}}\right)^{\alpha} + \left(\frac{1}{2}\right)^{\alpha} \right\} \frac{\Gamma(1 + s\alpha)\Gamma(1 + \alpha)}{\Gamma(1 + (s + 1)\alpha)},
\]
where
\[ \Delta_1 = 2^{(s-2)\alpha} \left[ f \left( \frac{a + 3b}{4} \right) + f \left( \frac{3a + b}{4} \right) \right] \]
and
\[ \Delta_2 = \left[ \frac{f(a) + f(b)}{2^\alpha} + f \left( \frac{a + b}{2} \right) \right] \frac{\Gamma(1 + s\alpha) \Gamma(1 + \alpha)}{\Gamma(1 + (s + 1)\alpha)}. \]

Definition 3.4. A function \( f : I \to \mathbb{R}^\alpha \) is said to be generalized \( P \)-convex function, if \( f \) is non-negative and for all \( x, y \in I \) and \( t \in [0, 1] \), we have
\[ f(tx + (1-t)y) \leq f(x) + f(y). \] (3.3)

Corollary 3.5. Let \( f : I \to \mathbb{R}^\alpha \) be a generalized \( P \)-convex function, then
\[ \frac{1}{2^{2\alpha}} f \left( \frac{a + b}{2} \right) \leq \Delta_1 \leq \frac{\Gamma(1 + \alpha)}{(b-a)^\alpha} \frac{a}{b} f(x) \]
\[ \leq \Delta_2 \leq \frac{3}{2} \alpha \left[ f(a) + f(b) \right], \]
where
\[ \Delta_1 = 2^{(s-2)\alpha} \left[ f \left( \frac{a + 3b}{4} \right) + f \left( \frac{3a + b}{4} \right) \right] \]
and
\[ \Delta_2 = \left[ \frac{f(a) + f(b)}{2^\alpha} + f \left( \frac{a + b}{2} \right) \right] \frac{\Gamma(1 + s\alpha) \Gamma(1 + \alpha)}{\Gamma(1 + (s + 1)\alpha)}. \]

Remark 3.6. If we choose \( \alpha = 1 \) in the above results, then we obtain the inequalities given by Noor et al. in [9].

References


