



Hilbert–Schmidt Frames: Duality, Weaving and Stability

Mehdi Choubin^{a,*}, Mohammad Bagher Ghaemi^b, Gwang Hui Kim^{c,*}

^aDepartment of Mathematics, Velayat University, Iranshahr, Iran

^bDepartment of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran

^cDepartment of Mathematics, Kangnam University, Yongin, Gyeonggi, 16979, Republic of Korea

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, we give some sufficient conditions under which perturbations preserve Hilbert-Schmidt frames. Also show that the canonical dual of a perturbed Hilbert-Schmidt frame is a perturbation of the canonical dual (alternative dual respectively) of the original Hilbert-Schmidt frame and discuss best approximation in the set of all dual Hilbert-Schmidt frames. Next, we apply the woven principle to Hilbert-Schmidt frames and study the stability of weaving Hilbert-Schmidt frames under perturbations. Finally, we present sufficient conditions under which perturbations preserve weaving Hilbert-Schmidt frames and weaving dual Hilbert-Schmidt frames.

Keywords: Hilbert-Schmidt frames, Weaving frames, Dual frames, Perturbation.

2010 MSC: Primary 46C50; Secondary 42C15.

1. Introduction

The von Neumann-Schatten frames in a separable Banach space was first proposed by Sadeghi and Arefijamaal [29] to deal with all the existing frames as a united object. In fact, von Neumann-Schatten frames is an extension of g -frames [30], bounded quasi-projectors [19], fusion frames [3, 8], pseudo-frames [22], weighted frames [4], oblique frames [10, 18], outer frames [1], p -frames for separable Banach spaces [12] and in the context of numerical analysis the stable space splittings [24, 25]. As an important class of von Neumann-Schatten p -frames, Hilbert-Schmidt frames have interested

*Corresponding author

Email addresses: m.choubin@gmail.com, m.choubin@velayat.ac.ir (Mehdi Choubin), mghaemi@iust.ac.ir (Mohammad Bagher Ghaemi), ghkim@kangnam.ac.kr (Gwang Hui Kim)

some mathematicians due to having the inner product structure [11, 12]. For more information on Hilbert-Schmidt frames, see Refs. [20, 26, 27, 35].

Weaving frames in a separable Hilbert space was first proposed by Bemrose et al. [5]. This frames are an important concept for applications in wireless sensor networks that require distributed processing under different frames, as well as pre-processing of signals using Gabor frames. Casazza and Lynch [6] studied the fundamental properties of weaving frames. Also, Casazza et al. [7] extended the concept of weaving Hilbert space frames to the Banach space setting. They introduced and studied weaving Schauder frames in Banach spaces. Many generalizations of the notion of weaving Hilbert space frames such as Weaving g -frames and fusion frames [14, 15, 21, 33], weaving K -Frames and K - g -Frames [16, 34], continuous weaving frames [31, 32], as well as weaving Gabor frames in $L^2(\mathbb{R})$ [17] were presented by many authors. In this study, we will apply the woven principle to Hilbert-Schmidt frames. We first show that small perturbations of a Hilbert-Schmidt frame give rise to another Hilbert-Schmidt frame. Also, we prove that if we do a sufficiently small perturbation of a Hilbert-Schmidt frame, the canonical dual of the new Hilbert-Schmidt frame is also a small perturbation of the canonical dual of the first one. We then obtain a similar result for the case of alternative dual Hilbert-Schmidt frames. Using this, we present sufficient conditions under which perturbations preserve weaving Hilbert-Schmidt frames and weaving dual Hilbert-Schmidt frames.

The rest of this paper is organized as follows. Section 2 gives an overview of some notions and related results for later use. In Section 3, we give various results about stability under perturbations for Hilbert-Schmidt frames and dual Hilbert-Schmidt frames. Finally, in Section 4, we introduce the notion of weaving Hilbert-Schmidt frames and study the stability of weaving Hilbert-Schmidt frames under perturbations.

2. Background on von Neumann-Schatten and Hilbert-Schmidt frames

In this section, we give some basic notations of von Neumann-Schatten p -Bessel sequences in the sense of Sadeghi and Arefijamaal [29]. Nevertheless, we shall require some facts about the theory of von Neumann-Schatten p -class \mathcal{C}_p . For background on this theory, we use [23, 28] as reference and adopt that book's notation. Moreover, our notation and terminology are standard and, concerning frames in Hilbert and Banach spaces, they are in general those of the book [9].

Let \mathcal{H} be a separable Hilbert space with orthonormal basis $\mathcal{E} = \{e_n\}_{n \in \mathbb{N}}$ and $B(\mathcal{H})$ denotes the C^* -algebra of all bounded linear operators on \mathcal{H} . For a compact operator $\mathcal{A} \in B(\mathcal{H})$, let $s_1(\mathcal{A}) \geq s_2(\mathcal{A}) \geq \dots \geq 0$ denote the singular values of \mathcal{A} , that is, the eigenvalues of the positive operator $|\mathcal{A}| = (\mathcal{A}^* \mathcal{A})^{\frac{1}{2}}$, arranged in a decreasing order and repeated according to multiplicity. For $1 \leq p < \infty$, the von Neumann-Schatten p -class \mathcal{C}_p is defined to be the set of all compact operators \mathcal{A} for which $\sum_{i=1}^{\infty} s_i^p(\mathcal{A}) < \infty$. For $\mathcal{A} \in \mathcal{C}_p$, the von Neumann Schatten p -norm of \mathcal{A} is defined by

$$\|\mathcal{A}\|_{\mathcal{C}_p} = \left(\sum_{i=1}^{\infty} s_i^p(\mathcal{A}) \right)^{\frac{1}{p}} = \left(\mathbf{tr} |\mathcal{A}|^p \right)^{\frac{1}{p}}, \quad (2.1)$$

where \mathbf{tr} is the trace functional which defines as $\mathbf{tr}(\mathcal{A}) = \sum_{n \in \mathbb{N}} \langle \mathcal{A}(e_n), e_n \rangle$. It is convenient to let \mathcal{C}_{∞} denote the class of compact operators, and in this case $\|\mathcal{A}\|_{\mathcal{C}_{\infty}} = s_1(\mathcal{A})$ is the usual operator norm. In what follows, the notations $\|\cdot\|_{\mathcal{C}_p}$ and $\|\cdot\|_{\mathcal{C}_{\infty}}$ denote the norm of the Banach spaces \mathcal{C}_p and \mathcal{C}_{∞} , respectively. The special case \mathcal{C}_2 is called the Hilbert-Schmidt class. Recall from [36, Theorem 1.4.6] that an operator \mathcal{A} is in \mathcal{C}_p if and only if $\mathcal{A}^p \in \mathcal{C}_1$. In particular, $\|\mathcal{A}\|_{\mathcal{C}_p}^p = \|\mathcal{A}^p\|_{\mathcal{C}_1}$. It is proved that \mathcal{C}_p is a two sided $*$ -ideal of $B(\mathcal{H})$, that is, a Banach algebra under the norm (2.1) and the finite rank operators are dense in $(\mathcal{C}_p, \|\cdot\|_{\mathcal{C}_p})$. Moreover, for $\mathcal{A} \in \mathcal{C}_p$, one has $\|\mathcal{A}\|_{\mathcal{C}_p} = \|\mathcal{A}^*\|_{\mathcal{C}_p}$, $\|\mathcal{A}\| \leq \|\mathcal{A}\|_{\mathcal{C}_p}$

and if $\mathcal{B} \in B(\mathcal{H})$, then $\|\mathcal{B}\mathcal{A}\|_{\mathcal{C}_p} \leq \|\mathcal{B}\| \|\mathcal{A}\|_{\mathcal{C}_p}$ and $\|\mathcal{A}\mathcal{B}\|_{\mathcal{C}_p} \leq \|\mathcal{B}\| \|\mathcal{A}\|_{\mathcal{C}_p}$. In particular, $\mathcal{C}_p \subseteq \mathcal{C}_q$ if $1 \leq p \leq q \leq \infty$. We also recall that \mathcal{C}_2 is a Banach space with respect to the norm $\|\cdot\|_{\mathcal{HS}}$. It is shown that the space \mathcal{C}_2 with the inner product $[T, S]_{\text{tr}} := \text{tr}(S^*T)$ is a Hilbert space.

Now for a fixed $1 < p < \infty$, following Conway [13, p. 74], we define the Banach spaces

$$\oplus \mathcal{C}_p = \left\{ \mathcal{A} = \{\mathcal{A}_i\}_{i=1}^\infty : \mathcal{A}_i \in \mathcal{C}_p \quad \forall i \in \mathbb{N} \text{ and } \|\mathcal{A}\|_p := \left(\sum_i \|\mathcal{A}_i\|_{\mathcal{C}_p}^p \right)^{\frac{1}{p}} < \infty \right\}.$$

In particular, $\oplus \mathcal{C}_2$ is a Hilbert space with the inner product

$$\langle \mathcal{A}, \mathcal{A}' \rangle := \sum_{i=1}^\infty [\mathcal{A}_i, \mathcal{A}'_i]_{\text{tr}},$$

and so $\|\mathcal{A}\|_2^2 = \langle \mathcal{A}, \mathcal{A} \rangle$.

If x and y are elements of a Hilbert spaces \mathcal{H} we define the operator $x \otimes y$ on \mathcal{H} by

$$(x \otimes y)(z) = \langle z, y \rangle x.$$

It is obvious that $\|x \otimes y\| = \|x\| \|y\|$ and the rank of $x \otimes y$ is one if x and y are non-zero. If $x, x', y, y' \in \mathcal{H}$ and $u \in B(\mathcal{H})$, then the following equalities are easily verified:

$$\begin{aligned} (x \otimes x')(y \otimes y') &= \langle y, x' \rangle (x \otimes y') \\ (x \otimes y)^* &= y \otimes x \\ u(x \otimes y) &= u(x) \otimes y \\ (x \otimes y)u &= x \otimes u^*(y). \end{aligned}$$

Note that if $x, y \in \mathcal{H}$, then $\|x \otimes y\|_{\mathcal{C}_p} = \|x \otimes y\|_{\mathcal{C}_q} = \|x\| \|y\|$ and $\text{tr}(x \otimes y) = \langle x, y \rangle$ so $x \otimes y$ is in \mathcal{C}_p for all $p \geq 1$. The operator $x \otimes x$ is a rank-one projection if and only if $\langle x, x \rangle = 1$, that is, x is a unit vector. Conversely, every rank-one projection is of the form $x \otimes x$ for some unit vector x . If $\{\eta_i : i \in I\}$ and $\{\zeta_i : i \in I\}$ are orthonormal bases in \mathcal{H} , then $\{\eta_i \otimes \zeta_j : i, j \in I\}$ is an orthonormal basis of \mathcal{C}_2 ; see [28] for more details.

Recall from [29] that a countable family $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^\infty$ of bounded linear operators from \mathcal{X} to $\mathcal{C}_p \subseteq B(\mathcal{H})$ is a von Neumann-Schatten p -frame for the Banach space \mathcal{X} with respect to \mathcal{H} ($1 \leq p < \infty$) if constants $A, B > 0$ exist such that

$$A\|f\|_{\mathcal{X}} \leq \left(\sum_{i \geq 1} \|\mathcal{G}_i(f)\|_{\mathcal{C}_p}^p \right)^{\frac{1}{p}} \leq B\|f\|_{\mathcal{X}} \tag{2.2}$$

for all $f \in \mathcal{X}$. It is called a von Neumann-Schatten p -Bessel sequence with bound B if the second inequality holds. In particular, the authors of [29] showed that the von Neumann-Schatten p -frame condition is satisfied if and only if $\{\mathcal{A}_i\}_{i=1}^\infty \mapsto \sum_{i=1}^\infty \mathcal{A}_i \mathcal{G}_i$ is a well defined mapping from $\oplus \mathcal{C}_q$ onto \mathcal{X}^* , and motivated by this fact, they considered the following operators:

$$T_{\mathcal{G}} : \oplus \mathcal{C}_q \rightarrow \mathcal{X}^*; \quad \{\mathcal{A}_i\}_{i=1}^\infty \mapsto \sum_{i=1}^\infty \mathcal{A}_i \mathcal{G}_i, \tag{2.3}$$

and

$$T_{\mathcal{G}}^* : \mathcal{X} \rightarrow \oplus \mathcal{C}_p; \quad f \mapsto \{\mathcal{G}_i(f)\}_{i=1}^\infty. \tag{2.4}$$

As usual, the operator $T_{\mathcal{G}}$ is called the synthesis operator, and $T_{\mathcal{G}}^*$ is the analysis operator of \mathcal{G} . The reader will remark that if $\mathcal{H} = \mathbb{C}$, then $B(\mathcal{H}) = \mathcal{C}_p = \mathbb{C}$ and thus $\oplus \mathcal{C}_p = \ell^p$ ($1 \leq p \leq \infty$), and thus the above definitions is consistent with the corresponding definitions in the concept of p -frames for separable Banach spaces.

In the case where $p = 2$ the spaces $\mathcal{C}_2 := {}_2(\mathcal{H})$ and $\oplus_2 \mathcal{C}_2$ are Hilbert and motivated by this fact the authors of [2, 29] provided a detailed study of the duals of a von Neumann-Schatten 2-frame, called Hilbert–Schmidt frame for Hilbert space \mathcal{K} with respect to Hilbert space \mathcal{H} .

Definition 2.1. A sequence $\mathcal{G} := \{\mathcal{G}_i\}_{i=1}^{\infty} \subseteq B(\mathcal{K}, \oplus \mathcal{C}_2)$ is said to be a Hilbert-Schmidt frame or simply a HS-frame for \mathcal{K} with respect to \mathcal{H} , whenever there exist two positive numbers $A_{\mathcal{G}}$ and $B_{\mathcal{G}}$ such that

$$A_{\mathcal{G}}\|f\|^2 \leq \sum_{i=1}^{\infty} \|\mathcal{G}_i(f)\|_{\mathcal{C}_2}^2 \leq B_{\mathcal{G}}\|f\|^2 \quad (2.5)$$

for all $f \in \mathcal{K}$. The constants $A_{\mathcal{G}}$ and $B_{\mathcal{G}}$ are called the lower and upper HS-frame bounds of \mathcal{G} and \mathcal{G} is called to be a HS-Bessel sequence for \mathcal{K} with respect to \mathcal{H} , if the right-hand side of (2.5) holds.

Particularly, by using the Hilbert properties of the spaces, they observed that

$$T_{\mathcal{G}}(\{\mathcal{A}_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \mathcal{G}_i^* \mathcal{A}_i \quad \text{and} \quad T_{\mathcal{G}}^*(f) = \{\mathcal{G}_i(f)\}_{i=1}^{\infty},$$

where $f \in \mathcal{K}$ and $\{\mathcal{A}_i\}_{i=1}^{\infty} \in \oplus \mathcal{C}_2$ and the mapping

$$S_{\mathcal{G}} : \mathcal{K} \rightarrow \mathcal{K} \quad , \quad S_{\mathcal{G}}(f) := T_{\mathcal{G}} T_{\mathcal{G}}^*(f) = \sum_{i=1}^{\infty} \mathcal{G}_i^* \mathcal{G}_i(f)$$

is an invertible, self-adjoint, positive and bounded linear operator and

$$A_{\mathcal{G}} Id_{\mathcal{K}} \leq S_{\mathcal{G}} \leq B_{\mathcal{G}} Id_{\mathcal{K}}.$$

From this, they were able to characterize all dual frames of a HS-frame. It is worthwhile to mention that a HS-frame is a more general version of the g -frame, an important generalization of ordinary frames.

3. The Perturbation on the Dual Hilbert-Schmidt frames

In what follows we shall frequently make use of the following notation for a HS-frame \mathcal{F} :

$$\text{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}}) := \left\{ \Phi \in B(\mathcal{K}, \oplus \mathcal{C}_2) : T_{\mathcal{F}} \Phi = 0 \right\},$$

the set of all right annihilators of the operator $T_{\mathcal{F}}$ in $B(\mathcal{K}, \oplus \mathcal{C}_2)$.

Definition 3.1. Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ be a HS-frame for \mathcal{K} with respect to \mathcal{H} . A HS-frame $\{\mathcal{G}_i\}_{i=1}^{\infty}$ is called Hilbert-Schmidt dual frame or simply a HS-dual frame for \mathcal{F} if $f = \sum_{i=1}^{\infty} \mathcal{F}_i^* \mathcal{G}_i(f)$ for all $f \in \mathcal{K}$, i.e. $T_{\mathcal{F}} T_{\mathcal{G}}^* = Id_{\mathcal{K}}$.

By using the properties of the Hilbert-Schmidt frame operator $S_{\mathcal{F}}$, we observe that for all $f \in \mathcal{K}$

$$f = S_{\mathcal{F}}(S_{\mathcal{F}}^{-1}f) = \sum_{i=1}^{\infty} \mathcal{F}_i^* \mathcal{F}_i S_{\mathcal{F}}^{-1}(f) \text{ and } f = S_{\mathcal{F}}^{-1}(S_{\mathcal{F}}f) = \sum_{i=1}^{\infty} S_{\mathcal{F}}^{-1} \mathcal{F}_i^* \mathcal{F}_i(f).$$

The sequence $\tilde{\mathcal{F}} = \{\tilde{\mathcal{F}}_i\}_{i=1}^{\infty} := \{\mathcal{F}_i S_{\mathcal{F}}^{-1}\}_{i=1}^{\infty}$ is a HS-dual frame for \mathcal{K} with respect to \mathcal{H} with the lower and upper HS-bounds $B_{\tilde{\mathcal{F}}}^{-1}$ and $A_{\tilde{\mathcal{F}}}^{-1}$, where $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ are the lower and upper HS-frame bounds of \mathcal{F} (see Ref. [2]). The HS-dual frame $\tilde{\mathcal{F}}$ is called the canonical HS-dual frame of \mathcal{F} . The authors in [2] characterized all HS-duals of \mathcal{F} by using the canonical HS-dual. Indeed, if \mathcal{F} be a HS-frame of \mathcal{H} , then $\mathcal{F}^d = \{\mathcal{F}_i^d\}_{i=1}^{\infty}$ is a HS-dual of \mathcal{F} if and only if

$$\mathcal{F}_i^d = \tilde{\mathcal{F}}_i + \pi_i \Phi = \mathcal{F}_i S_{\mathcal{F}}^{-1} + \pi_i \Phi,$$

where $\Phi \in \text{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}})$ and

$$\pi_i : \oplus_2 \rightarrow_2, \quad \pi_i(\{\mathcal{A}_j\}_j) = \mathcal{A}_i.$$

In what follows, the notation $\tilde{\mathcal{F}}(\Phi)$ denote the HS-dual $\{\tilde{\mathcal{F}}_i + \pi_i \Phi\}_{i=1}^{\infty}$ of \mathcal{F} .

Theorem 3.2. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ and $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ be HS-frames for \mathcal{K} with respect to \mathcal{H} . Then the following statements hold:*

- (a) *Let $\{\nu_{n,m} : n, m \in \mathbb{N}\}$ be an orthonormal basis for \mathcal{C}_2 . Then, \mathcal{G} is a HS-dual frame of \mathcal{F} if and only if the ordinary frame $\{\mathcal{G}_i^*(\nu_{n,m})\}$ be a dual frame of $\{\mathcal{F}_i^*(\nu_{n,m})\}$.*
- (b) *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for \mathcal{H} , then \mathcal{G} is a HS-dual frame of \mathcal{F} if and only if and only $\{\mathcal{G}_i^*(e_n \otimes e_m)\}$ is a dual frame of $\{\mathcal{F}_i^*(e_n \otimes e_m)\}$.*

Proof . Let $\{\mathcal{F}_i\}_{i=1}^{\infty}$ be a HS-frame for \mathcal{K} with respect to \mathcal{H} and $\{\nu_{n,m} : n, m \in \mathbb{N}\}$ be a orthonormal basis of ${}_2$. Define a bounded linear functional on \mathcal{K} as follows

$$f \mapsto [\mathcal{F}_j f, \nu_{n,m}]_{\text{tr}} \quad (f \in \mathcal{K}).$$

By Riesz representation theorem, there exists $f_{j,n,m} \in \mathcal{K}$ such that

$$[\mathcal{F}_j f, \nu_{n,m}]_{\text{tr}} = \langle f, f_{j,n,m} \rangle \quad (f \in \mathcal{K}).$$

Hence

$$\mathcal{F}_j f = \sum_{n,m \in \mathbb{N}} \langle f, f_{j,n,m} \rangle \nu_{n,m} \quad (f \in \mathcal{K}).$$

The sequence $\{f_{j,n,m}\}$ is a Bessel sequence, since for all $f \in \mathcal{K}$

$$\begin{aligned} \sum_{n,m \in \mathbb{N}} |\langle f, f_{j,n,m} \rangle|^2 &= \|\mathcal{F}_j f\|_2^2 \\ &\leq \|\mathcal{F}_j\|^2 \|f\|^2. \end{aligned} \tag{3.1}$$

Now, for any $f \in \mathcal{K}$ and $\mathcal{A} \in {}_2$, we get

$$\begin{aligned} \langle f, \mathcal{F}_j^* \mathcal{A} \rangle &= [\mathcal{F}_j f, \mathcal{A}]_{\text{tr}} = \left[\sum_{n,m \in \mathbb{N}} \langle f, f_{j,n,m} \rangle \nu_{n,m}, \mathcal{A} \right]_{\text{tr}} \\ &= \left\langle f, \sum_{n,m \in \mathbb{N}} [\mathcal{A}, \nu_{n,m}]_{\text{tr}} f_{j,n,m} \right\rangle. \end{aligned}$$

Therefore

$$\mathcal{F}_j^* \mathcal{A} = \sum_{n,m \in \mathbb{N}} [\mathcal{A}, \nu_{n,m}]_{\text{tr}} f_{j,n,m} \quad (\mathcal{A} \in \mathcal{E}_2). \tag{3.2}$$

In particular, $\mathcal{F}_j^*(\nu_{n,m}) = f_{j,n,m}$. Therefore by (3.2), we have

$$\mathcal{F}_j^* \mathcal{A} = \sum_{n,m \in \mathbb{N}} [\mathcal{A}, \nu_{n,m}]_{\text{tr}} \mathcal{F}_j^*(\nu_{n,m}) \quad (\mathcal{A} \in \mathcal{E}_2). \tag{3.3}$$

Similarly, there exists $g_{j,n,m} \in \mathcal{H}$ such that $\mathcal{G}_j^*(\nu_{n,m}) = g_{j,n,m}$. By [2, Theorem 3.3], $F = \{\mathcal{F}_j^*(\nu_{n,m})\}$ and $G = \{\mathcal{G}_j^*(\nu_{n,m})\}$ are frames for Hilbert space \mathcal{K} . Using (3.3), we obtain that

$$\begin{aligned} T_F T_G^*(f) &= \sum_{i=1}^{\infty} \sum_{n,m \in \mathbb{N}} \langle f, \mathcal{G}_i^*(\nu_{n,m}) \rangle \mathcal{F}_i^*(\nu_{n,m}) \\ &= \sum_{i=1}^{\infty} \mathcal{F}_i^* \left(\sum_{n,m \in \mathbb{N}} [\mathcal{G}_i(f), \nu_{n,m}]_{\text{tr}} \nu_{n,m} \right) \\ &= \sum_{i=1}^{\infty} \mathcal{F}_i^* \mathcal{G}_i(f) = T_{\mathcal{F}} T_G^*(f), \end{aligned}$$

which proves part (a) of the theorem. To prove the part (b), it is sufficient to put $\nu_{n,m} := e_n \otimes e_m$ in part (a). \square

Definition 3.3. Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ and $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ be HS-Bessel sequences for \mathcal{K} with respect to \mathcal{H} . For $\mu > 0$, we say that \mathcal{G} is a μ -perturbation of \mathcal{F} if

$$\|T_{\mathcal{F}} - T_{\mathcal{G}}\| \leq \mu.$$

Theorem 3.4. Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ be HS-frame for \mathcal{K} with respect to \mathcal{H} with the lower and upper HS-bounds A, B and let $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ be a μ -perturbation of \mathcal{F} . If $\mu < \sqrt{A}$, then \mathcal{G} is a HS-frame for \mathcal{K} with respect to \mathcal{H} with frame HS-bounds

$$(\sqrt{A} - \mu)^2, \quad (\sqrt{B} + \mu)^2.$$

Proof . Since A and B are the lower and upper HS-frame bounds for \mathcal{F} , so that

$$\sqrt{A} \|f\| \leq \|T_{\mathcal{F}}^*(f)\|_2 \leq \sqrt{B} \|f\| \quad (f \in \mathcal{K}).$$

Therefore, for all $f \in \mathcal{K}$ we have

$$\|T_{\mathcal{G}}^*(f)\|_2 \geq \|T_{\mathcal{F}}^*(f)\|_2 - \|(T_{\mathcal{F}} - T_{\mathcal{G}})^*(f)\|_2 \geq (\sqrt{A} - \mu) \|f\|,$$

and

$$\|T_{\mathcal{G}}^*(f)\|_2 \leq \|(T_{\mathcal{F}} - T_{\mathcal{G}})^*(f)\|_2 + \|T_{\mathcal{F}}^*(f)\|_2 \leq (\mu + \sqrt{B}) \|f\|.$$

Hence, for all $f \in \mathcal{K}$ we obtain

$$(\sqrt{A} - \mu)^2 \|f\|^2 \leq \sum_{i=1}^{\infty} \|\mathcal{G}_i(f)\|_2^2 \leq (\sqrt{B} + \mu)^2 \|f\|^2.$$

\square

Lemma 3.5. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^\infty$ and $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^\infty$ be HS-frame for \mathcal{K} with respect to \mathcal{H} and let $\Phi \in \text{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}})$, $\Psi \in \text{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{G}})$. Then*

$$T_{\tilde{\mathcal{G}}(\Psi)} - T_{\tilde{\mathcal{F}}(\Phi)} = T_{\tilde{\mathcal{F}}(\Phi)} (T_{\mathcal{F}} - T_{\mathcal{G}})^* T_{\tilde{\mathcal{G}}} + \Psi^* - T_{\tilde{\mathcal{F}}(\Phi)} P_{\ker T_{\mathcal{G}}}. \tag{3.4}$$

Proof . At first, note that $\mathcal{R}(T_{\mathcal{G}}^*)$ is closed , since $\|T_{\mathcal{G}}^*(f)\|_2 \geq \sqrt{A}\|f\|$ where A is lower HS-frame bound of \mathcal{G} . Therefore $\mathcal{R}(T_{\mathcal{G}}^*) \oplus \ker T_{\mathcal{G}} = Id_{\oplus_2}$. Thus

$$P_{\mathcal{R}(T_{\mathcal{G}}^*)} \oplus P_{\ker T_{\mathcal{G}}} = Id_{\oplus_2} \quad \text{and} \quad T_{\mathcal{G}}^* T_{\tilde{\mathcal{G}}} = P_{\mathcal{R}(T_{\mathcal{G}}^*)}.$$

Also $T_{\tilde{\mathcal{F}}(\Phi)} T_{\mathcal{F}}^* = Id_{\mathcal{K}}$, Since $\tilde{\mathcal{F}}(\Phi)$ is a HS-dual frame of \mathcal{F} . For an arbitrary $\{\mathcal{A}_i\}_{i=1}^\infty \in \oplus_2$, we have

$$\begin{aligned} T_{\tilde{\mathcal{G}}(\Psi)} (\{\mathcal{A}_i\}_{i=1}^\infty) &= \sum_{i=1}^\infty (\tilde{\mathcal{G}}_i + \pi_i \Psi)^* \mathcal{A}_i \\ &= \sum_{i=1}^\infty \tilde{\mathcal{G}}_i^* \mathcal{A}_i + \sum_{i=1}^\infty \Psi^* \pi_i^* \mathcal{A}_i \\ &= T_{\tilde{\mathcal{G}}} (\{\mathcal{A}_i\}_{i=1}^\infty) + \Psi^* \left(\sum_{i=1}^\infty \pi_i^* \mathcal{A}_i \right) \\ &= T_{\tilde{\mathcal{G}}} (\{\mathcal{A}_i\}_{i=1}^\infty) + \Psi^* (\{\mathcal{A}_i\}_{i=1}^\infty). \end{aligned}$$

So $T_{\tilde{\mathcal{G}}(\Psi)} = T_{\tilde{\mathcal{G}}} + \Psi^*$. Therefor, we obtain

$$\begin{aligned} &T_{\tilde{\mathcal{F}}(\Phi)} (T_{\mathcal{F}}^* - T_{\mathcal{G}}^*) T_{\tilde{\mathcal{G}}} + \Psi^* - T_{\tilde{\mathcal{F}}(\Phi)} P_{\ker T_{\mathcal{G}}} \\ &= T_{\tilde{\mathcal{G}}} - T_{\tilde{\mathcal{F}}(\Phi)} P_{\mathcal{R}(T_{\mathcal{G}}^*)} + \Psi^* - T_{\tilde{\mathcal{F}}(\Phi)} P_{\ker T_{\mathcal{G}}} \\ &= (T_{\tilde{\mathcal{F}}(\Phi)} T_{\mathcal{F}}^*) T_{\tilde{\mathcal{G}}} - T_{\tilde{\mathcal{F}}(\Phi)} (T_{\mathcal{G}}^* T_{\tilde{\mathcal{G}}}) + \Psi^* - T_{\tilde{\mathcal{F}}(\Phi)} P_{\ker T_{\mathcal{G}}} \\ &= (T_{\tilde{\mathcal{G}}} + \Psi^*) - T_{\tilde{\mathcal{F}}(\Phi)} (P_{\mathcal{R}(T_{\mathcal{G}}^*)} + P_{\ker T_{\mathcal{G}}}) \\ &= T_{\tilde{\mathcal{G}}(\Psi)} - T_{\tilde{\mathcal{F}}(\Phi)}, \end{aligned}$$

and the lemma is proven. \square

Theorem 3.6. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^\infty$ be HS-frame for \mathcal{K} with respect to \mathcal{H} with the lower HS-frame bound A and let $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^\infty$ be a μ -perturbation of \mathcal{F} . If $\mu < \sqrt{A}$, then \mathcal{G} is a HS-frame and the canonical dual $\tilde{\mathcal{G}}$ of \mathcal{G} is a λ -perturbation of $\tilde{\mathcal{F}}$, where*

$$\lambda = \frac{1}{\sqrt{A} - \mu}.$$

Proof . By Theorem 3.4, \mathcal{G} is a HS-frame for \mathcal{K} with respect to \mathcal{H} with the lower bound $(\sqrt{A} - \mu)^2$. Putting $\Phi = \Psi := 0$ in Lemma 3.5, we get

$$T_{\tilde{\mathcal{G}}} - T_{\tilde{\mathcal{F}}} = T_{\tilde{\mathcal{F}}} (T_{\mathcal{F}} - T_{\mathcal{G}})^* T_{\tilde{\mathcal{G}}} - T_{\tilde{\mathcal{F}}} P_{\ker T_{\mathcal{G}}},$$

and so

$$\|T_{\tilde{\mathcal{G}}} - T_{\tilde{\mathcal{F}}}\| \leq \|T_{\tilde{\mathcal{F}}}\| \|T_{\mathcal{F}} - T_{\mathcal{G}}\| \|T_{\tilde{\mathcal{G}}}\| + \|T_{\tilde{\mathcal{F}}}\|.$$

Since A^{-1} and $(\sqrt{A} - \mu)^{-2}$ are lower HS-bauds of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$, respectively, therefore

$$\|T_{\tilde{\mathcal{G}}} - T_{\tilde{\mathcal{F}}}\| \leq \frac{1}{\sqrt{A} - \mu},$$

which completes the proof. \square

Theorem 3.7. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^\infty$ be HS-frame for \mathcal{K} with respect to \mathcal{H} with lower HS-frame bound A and let $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^\infty$ be a μ -perturbation of \mathcal{F} . If $\mu < \sqrt{A}$, then \mathcal{G} is a HS-frame and for every $\Phi \in \text{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}})$ the HS-dual $\tilde{\mathcal{G}}(\text{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}(\Phi)}^*)$ of \mathcal{G} is a λ -perturbation of $\tilde{\mathcal{F}}(\Phi)$, where*

$$\lambda = \frac{\mu\sqrt{1+A\|\Phi\|_2^2}}{\sqrt{A}(\sqrt{A}-\mu)}. \tag{3.5}$$

Moreover, for every $\Phi \in \text{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}})$ the HS-dual $\tilde{\mathcal{G}}(\text{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}(\Phi)}^*)$ of \mathcal{G} is a best approximation of $\tilde{\mathcal{F}}(\Phi)$ in the set of all HS-duals of \mathcal{G} .

Proof . By Theorem 3.4, \mathcal{G} is a HS-frame for \mathcal{K} with respect to \mathcal{H} with the lower bound frame $(\sqrt{A}-\mu)^2$ and so $\|T_{\tilde{\mathcal{G}}}\| \leq (\sqrt{A}-\mu)^{-1}$. Since A^{-1} is the lower HS-bound frame of $\tilde{\mathcal{F}}$, we conclude that

$$\begin{aligned} \|T_{\tilde{\mathcal{F}}(\Phi)}^*(f)\|_2^2 &= \sum_{i=1}^\infty \|\tilde{\mathcal{F}}_i + \pi_i \Phi(f)\|_2^2 \\ &= \sum_{i=1}^\infty \langle \tilde{\mathcal{F}}_i + \pi_i \Phi(f), \tilde{\mathcal{F}}_i + \pi_i \Phi(f) \rangle \\ &= \sum_{i=1}^\infty \left(\|\tilde{\mathcal{F}}_i(f)\|_2^2 + \langle \tilde{\mathcal{F}}_i, \pi_i \Phi(f) \rangle + \langle \pi_i \Phi(f), \tilde{\mathcal{F}}_i \rangle + \|\pi_i \Phi(f)\|_2^2 \right) \\ &= \|T_{\tilde{\mathcal{F}}}^*(f)\|_2^2 + \|\Phi(f)\|_2^2 \\ &\leq \frac{1}{A} \|f\|^2 + \|\Phi\|^2 \|f\|^2 \end{aligned} \tag{3.6}$$

for all $f \in \mathcal{K}$ and $\Phi \in \text{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}})$. So that $\|T_{\tilde{\mathcal{F}}(\Phi)}\| \leq \sqrt{A^{-1} + \|\Phi\|_2^2}$. For every $\Phi \in \text{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}})$, we denote $\Psi_0 := \text{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}(\Phi)}^* \in \text{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{G}})$. Putting $\Psi := \Psi_0$ in Lemma 3.5, we get

$$\begin{aligned} T_{\tilde{\mathcal{G}}(\Psi_0)} - T_{\tilde{\mathcal{F}}(\Phi)} &= T_{\tilde{\mathcal{F}}(\Phi)} (T_{\mathcal{F}} - T_{\mathcal{G}})^* T_{\tilde{\mathcal{G}}} + \Psi_0^* - T_{\tilde{\mathcal{F}}(\Phi)} \text{P}_{\ker T_{\mathcal{G}}} \\ &= T_{\tilde{\mathcal{F}}(\Phi)} (T_{\mathcal{F}} - T_{\mathcal{G}})^* T_{\tilde{\mathcal{G}}}, \end{aligned}$$

which implies $\|T_{\tilde{\mathcal{G}}(\Psi_0)} - T_{\tilde{\mathcal{F}}(\Phi)}\| \leq \lambda$, where λ given by (3.5).

Moreover, since $\text{P}_{\mathcal{R}(T_{\mathcal{G}}^*)} \oplus \text{P}_{\ker T_{\mathcal{G}}} = Id_{\oplus_2}$, we observe that

$$=(T_{\tilde{\mathcal{G}}} - T_{\tilde{\mathcal{F}}(\Phi)})\text{P}_{\mathcal{R}(T_{\mathcal{G}}^*)} + (\Psi^* - T_{\tilde{\mathcal{F}}(\Phi)})\text{P}_{\ker T_{\mathcal{G}}}$$

for all $\Psi \in \text{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{G}})$. By setting $\Psi := \Psi_0$ in (??), we have

$$T_{\tilde{\mathcal{G}}(\Psi_0)} - T_{\tilde{\mathcal{F}}(\Phi)} = (T_{\tilde{\mathcal{G}}} - T_{\tilde{\mathcal{F}}(\Phi)})\text{P}_{\mathcal{R}(T_{\mathcal{G}}^*)}.$$

Therefore, for all $\Upsilon \in \oplus_2$ we have

$$\|(T_{\tilde{\mathcal{G}}(\Psi)} - T_{\tilde{\mathcal{F}}(\Phi)})\Upsilon\|^2 \geq \|(T_{\tilde{\mathcal{G}}} - T_{\tilde{\mathcal{F}}(\Phi)})\text{P}_{\mathcal{R}(T_{\mathcal{G}}^*)}(\Upsilon)\|^2 = \|(T_{\tilde{\mathcal{G}}(\Psi_0)} - T_{\tilde{\mathcal{F}}(\Phi)})\Upsilon\|^2,$$

which implies the HS-dual $\tilde{\mathcal{G}}(\text{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}(\Phi)}^*)$ of \mathcal{G} is a best approximation of $\tilde{\mathcal{F}}(\Phi)$ in the set of all HS-duals of \mathcal{G} . \square

Theorem 3.8. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^\infty$ be HS-frame for \mathcal{K} with respect to \mathcal{H} with lower HS-frame bound A and let $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^\infty$ be a μ -perturbation of \mathcal{F} . If $\mu < \frac{\sqrt{A}}{2}$, then \mathcal{G} is a HS-frame and the set of all HS-duals of \mathcal{F} and the set of all HS-duals of \mathcal{G} are isomorphic.*

Proof . We show that for every $\Phi \in \mathbf{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}})$, $\Lambda : \tilde{\mathcal{F}}(\varphi) \mapsto \tilde{\mathcal{G}}(\mathbf{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}(\Phi)}^*)$ is a bijective map from the set of all HS-duals of \mathcal{F} onto the set of all HS-duals of \mathcal{G} . Suppose that $\Lambda(\tilde{\mathcal{F}}(\varphi_1)) = \Lambda(\tilde{\mathcal{F}}(\varphi_2))$ for two $\Phi_1, \Phi_2 \in \mathbf{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}})$. Hence, we have $\tilde{\mathcal{G}}(\mathbf{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}(\Phi_1)}^*) = \tilde{\mathcal{G}}(\mathbf{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}(\Phi_2)}^*)$. Therefore $\mathbf{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}(\Phi_1)}^* = \mathbf{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}(\Phi_2)}^*$, which yields $\mathbf{P}_{\ker T_{\mathcal{G}}} \Phi_1 = \mathbf{P}_{\ker T_{\mathcal{G}}} \Phi_2$. Clearly, $\mathbf{P}_{\ker T_{\mathcal{G}}} \Phi = (\mathbf{P}_{\ker T_{\mathcal{G}}} | \ker T_{\mathcal{F}}) \Phi$ for all $\Phi \in \mathbf{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}})$. It will thus be sufficient to prove that $\mathbf{P}_{\ker T_{\mathcal{G}}} | \ker T_{\mathcal{F}}$ is injective and so that $\Phi_1 = \Phi_2$. By Theorem 3.4, $\|T_{\mathcal{G}}^*(g)\|_2 \geq (\sqrt{A} - \mu)\|g\|$ for all $g \in \mathcal{K}$ which, together with $\mu < \frac{\sqrt{A}}{2}$, yields

$$\begin{aligned} &= \sup \left\{ \inf_{\Omega \in \mathcal{R}(T_{\mathcal{G}}^*)} \|\Omega - \Upsilon\|_2 : \Upsilon \in \mathcal{R}(T_{\mathcal{G}}^*), \|\Upsilon\|_2 = 1 \right\} \\ &= \sup \left\{ \inf_{f \in \mathcal{K}} \|T_{\mathcal{F}}^*(f) - T_{\mathcal{G}}^*(g)\|_2 : g \in \mathcal{K}, \|T_{\mathcal{G}}^*(g)\|_2 = 1 \right\} \\ &\leq \sup \left\{ \|T_{\mathcal{F}}^*(g) - T_{\mathcal{G}}^*(g)\|_2 : g \in \mathcal{K}, \|T_{\mathcal{G}}^*(g)\|_2 = 1 \right\} \\ &\leq \|T_{\mathcal{F}}^* - T_{\mathcal{G}}^*\| \sup \{ \|g\| : g \in \mathcal{K}, \|T_{\mathcal{G}}^*(g)\|_2 = 1 \} \\ &\leq \frac{\mu}{\sqrt{A} - \mu} < 1. \end{aligned}$$

For all $\Upsilon = \{\mathcal{A}_i\}_{i=1}^\infty \in \oplus_2$ with $\|\Upsilon\|_2 = 1$, we can use the Pythagorean relation to show that

$$\|\mathbf{P}_{\ker T_{\mathcal{G}}}(\Upsilon)\|_2^2 = 1 - \|(Id_{\oplus_2} - \mathbf{P}_{\ker T_{\mathcal{G}}})(\Upsilon)\|_2^2.$$

So that by (??) we have

$$\begin{aligned} \inf_{\Upsilon \in \ker T_{\mathcal{F}}, \|\Upsilon\|_2=1} \|\mathbf{P}_{\ker T_{\mathcal{G}}}(\Upsilon)\|_2^2 &= 1 - \sup_{\Upsilon \in \ker T_{\mathcal{F}}, \|\Upsilon\|_2=1} \|(Id_{\oplus_2} - \mathbf{P}_{\ker T_{\mathcal{G}}})(\Upsilon)\|_2^2 \\ &= 1 - \|(Id_{\oplus_2} - \mathbf{P}_{\ker T_{\mathcal{G}}})| \ker T_{\mathcal{F}}\|^2 \\ &= 1 - \|\mathbf{P}_{\mathcal{R}(T_{\mathcal{G}}^*)}| \ker T_{\mathcal{F}}\|^2 \\ &= 1 - \|(\mathbf{P}_{\mathcal{R}(T_{\mathcal{G}}^*)}| \ker T_{\mathcal{F}})^*\|^2 \\ &= 1 - \|\mathbf{P}_{\ker T_{\mathcal{F}}}| \mathcal{R}(T_{\mathcal{G}}^*)\|^2 > 0 \end{aligned}$$

Thus, $\mathbf{P}_{\ker T_{\mathcal{G}}} | \ker T_{\mathcal{F}}$ is bounded below and so is injective.

To prove Λ is surjective, suppose that $\tilde{\mathcal{G}}(\Psi)$ be a HS-duals of \mathcal{G} , where $\Psi \in \mathbf{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{G}})$. We set

$$\Phi := (\mathbf{P}_{\ker T_{\mathcal{G}}} | \ker T_{\mathcal{F}})^{-1}(\Psi - \mathbf{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}}^*).$$

Then, we have

$$\begin{aligned} \mathbf{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}}^*(\Phi) &= \mathbf{P}_{\ker T_{\mathcal{G}}}(T_{\tilde{\mathcal{F}}}^* + \Phi) \\ &= \mathbf{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}}^* + \Psi - \mathbf{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}}^* \\ &= \Psi, \end{aligned}$$

which completes the proof. \square

4. Weaving Hilbert-Schmidt frames and Perturbations

The concept of weaving was recently proposed by Bemrose et al [5] to simulate a question in distributed signal processing. In this section, we first recall the definition of weaving Hilbert space frames ([5]), and apply the woven principle to Hilbert-Schmidt frames. Next, we discuss the erasures and perturbations of weaving for Hilbert-Schmidt frames.

For a given natural number m , let $[m] := \{1, 2, \dots, m\}$.

Definition 4.1. A family of frames $\{\{f_{ij}\}_{i \in \mathbb{N}} : j \in [m]\}$ for a separable Hilbert space \mathcal{H} is said to be woven if there are universal constants A and B so that, for every partition $\{\sigma_j\}_{j \in [m]}$ of \mathbb{N} , the family $\{f_{ij}\}_{i \in \sigma_j, j \in [m]}$ is a frame for \mathcal{H} with lower and upper frame bounds A and B , respectively.

Definition 4.2. A family of HS-frames $\{\{\mathcal{F}_{ij}\}_{i \in \mathbb{N}} : j \in [m]\}$ for a separable Hilbert space \mathcal{K} with respect to \mathcal{H} is said to be HS-woven if there are universal constants A and B so that, for every partition $\{\sigma_j\}_{j \in [m]}$ of \mathbb{N} , the family $\{\mathcal{F}_{ij}\}_{i \in \sigma_j, j \in [m]}$ is a HS-frame for \mathcal{K} with respect to \mathcal{H} with lower and upper HS-frame bounds A and B , respectively, and each $\cup_{j \in [m]} \{\mathcal{F}_{ij}\}_{i \in \sigma_j}$ is called a weaving.

Theorem 4.3. Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^\infty$ and $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^\infty$ be HS-frames for \mathcal{K} with respect to \mathcal{H} and let $\{\nu_{n,m} : n, m \in \mathbb{N}\}$ be an orthonormal basis for \mathcal{C}_2 . Then, \mathcal{F} and \mathcal{G} are HS-woven for \mathcal{K} with respect to \mathcal{H} if and only if for every $\sigma \subset \mathbb{N}$, the family

$$\{\mathcal{F}_i^*(\nu_{n,m})\}_{n,m \in \mathbb{N}, i \in \sigma} \cup \{\mathcal{G}_i^*(\nu_{n,m})\}_{n,m \in \mathbb{N}, i \in \sigma^c}$$

is a frame for \mathcal{K} . In particular, if $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for \mathcal{H} , then \mathcal{F} and \mathcal{G} are HS-woven for \mathcal{K} with respect to \mathcal{H} if and only if for every $\sigma \subset \mathbb{N}$, the family

$$\{\mathcal{F}_i^*(e_n \otimes e_m)\}_{n,m \in \mathbb{N}, i \in \sigma} \cup \{\mathcal{G}_i^*(e_n \otimes e_m)\}_{n,m \in \mathbb{N}, i \in \sigma^c}$$

is a frame for \mathcal{K} .

Proof . Let $\{\mathcal{F}_i\}_{i=1}^\infty$ be a HS-frame for \mathcal{K} with respect to \mathcal{H} and $\{\nu_{n,m} : n, m \in \mathbb{N}\}$ be a orthonormal basis of \mathcal{C}_2 . As in the proof of Theorem 3.2, there exists $f_{j,n,m}, g_{j,n,m} \in \mathcal{H}$ such that $\mathcal{F}_j^*(\nu_{n,m}) = f_{j,n,m}$, $\mathcal{G}_j^*(\nu_{n,m}) = g_{j,n,m}$ and also by (3.1),

$$\|\mathcal{F}_i f\|_2^2 = \sum_{n,m \in \mathbb{N}} |\langle f, f_{i,n,m} \rangle|^2, \quad \|\mathcal{G}_i f\|_2^2 = \sum_{n,m \in \mathbb{N}} |\langle f, g_{i,n,m} \rangle|^2.$$

Now, for every subset $\sigma \subset \mathbb{N}$ and $f \in \mathcal{H}$ we have

$$\begin{aligned} \sum_{i \in \sigma} \|\mathcal{F}_i f\|_2^2 + \sum_{i \in \sigma^c} \|\mathcal{G}_i f\|_2^2 &= \sum_{i \in \sigma} \sum_{n,m \in \mathbb{N}} |\langle f, f_{i,n,m} \rangle|^2 + \sum_{i \in \sigma^c} \sum_{n,m \in \mathbb{N}} |\langle f, g_{i,n,m} \rangle|^2 \\ &= \sum_{i \in \sigma} \sum_{n,m \in \mathbb{N}} |\langle f, \mathcal{F}_i^*(\nu_{n,m}) \rangle|^2 + \sum_{i \in \sigma^c} \sum_{n,m \in \mathbb{N}} |\langle f, \mathcal{G}_i^*(\nu_{n,m}) \rangle|^2. \end{aligned}$$

Therefore, $\{\mathcal{F}_i\}_{i \in \sigma} \cup \{\mathcal{G}_i\}_{i \in \sigma^c}$ is HS-frame for \mathcal{K} with respect to \mathcal{H} if and only if $\{\mathcal{F}_i^*(\nu_{n,m})\}_{n,m \in \mathbb{N}, i \in \sigma} \cup \{\mathcal{G}_i^*(\nu_{n,m})\}_{n,m \in \mathbb{N}, i \in \sigma^c}$ is frames for \mathcal{K} . This completes the proof. \square

As application of Theorem 4.3, we now present the following example which use a finite index set J instead of \mathbb{N} .

Example 4.4. Let \mathcal{H} be a two-dimensional Hilbert space and $\{e_1, e_2\}$ be an orthonormal basis of H . Choose

$$\{f_i\}_{i=1}^3 = \{e_1, e_2, 2e_2\}, \quad \{g_i\}_{i=1}^3 = \{e_1, e_1 + e_2, e_1 - e_2\}.$$

Since both of $\{f_i\}_{i=1}^3$ and $\{g_i\}_{i=1}^3$ span \mathcal{H} , then those are frames. Define

$$\begin{aligned} \mathcal{F}_i : \mathcal{H} &\rightarrow_2 & \mathcal{G}_i : \mathcal{H} &\rightarrow_2 \\ \mathcal{F}_i(f) &:= f \otimes f_i & \mathcal{G}_i(f) &:= f \otimes g_i \end{aligned} \quad \text{and}$$

for all $f \in \mathcal{H}$ and $i \in J := \{1, 2, 3\}$. We first show that for every $\sigma \subseteq J$, $\{\mathcal{F}_i\}_{i \in \sigma} \cup \{\mathcal{G}_i\}_{i \in \sigma^c} \subseteq B(\mathcal{H},_2)$ are HS-frames, i.e. $\{\mathcal{F}_i\}_{i=1}^3$ and $\{\mathcal{G}_i\}_{i=1}^3$ are HS-woven.

Note that $\{\mathcal{F}_i\}_{i=1}^3$ and $\{\mathcal{G}_i\}_{i=1}^3$ are HS-frames, because for all $f \in \mathcal{H}$

$$\begin{aligned} \left(\sum_{i=1}^3 \|f_i\|^2\right) \|f\|^2 &\geq \sum_i \|\mathcal{F}_i f\|_2^2 = \sum_{i=1}^3 \sum_{n,m=1}^2 |[\mathcal{F}_i(f), e_n \otimes e_m]_{\text{tr}}|^2 \\ &= \sum_{i=1}^3 \sum_{n,m=1}^2 |\text{tr}((e_m \otimes e_n)(f \otimes f_i))|^2 \\ &= \sum_{i=1}^3 \sum_{n,m=1}^2 \left| \sum_{k=1}^2 \langle \langle f, e_n \rangle (e_m \otimes f_i)(e_k), e_k \rangle \right|^2 \\ &= \sum_{i=1}^3 \sum_{n,m=1}^2 \left| \sum_{k=1}^2 \langle f, e_n \rangle \langle e_k, f_i \rangle \langle e_m, e_k \rangle \right|^2 \\ &= \sum_{i=1}^3 \sum_{n,m=1}^2 |\langle f, e_n \rangle \langle e_m, f_i \rangle|^2 \\ &\geq \sum_{i=1}^3 \sum_{n=1}^2 |\langle f, e_n \rangle \langle e_n, f_i \rangle|^2 \\ &= \sum_{i=1}^3 |\langle f, f_i \rangle|^2 \end{aligned}$$

and similarly $\sum_{i=1}^3 |\langle f, g_i \rangle|^2 \leq \sum_{i=1}^3 \|\mathcal{G}_i f\|_2^2 \leq (\sum_{i=1}^3 \|g_i\|^2) \|f\|^2$.

Let \mathcal{A} be an Hilbert-Schmidt operator. We compute

$$\begin{aligned} [\mathcal{F}_i(f), \mathcal{A}]_{\text{tr}} &= \text{tr}(\mathcal{A}^* f \otimes f_i) \\ &= \sum_{k=1}^2 \langle (\mathcal{A}^* f \otimes f_i)(e_k), e_k \rangle \\ &= \sum_{k=1}^2 \langle e_k, f_i \rangle \langle \mathcal{A}^* f, e_k \rangle \\ &= \langle f, \mathcal{A} \left(\sum_{k=1}^2 \langle f_i, e_k \rangle e_k \right) \rangle \\ &= \langle f, \mathcal{A}(f_i) \rangle. \end{aligned}$$

Thus $\mathcal{F}_i^*(\mathcal{A}) = \mathcal{A}(f_i)$. Similarly $\mathcal{G}_i^*(\mathcal{A}) = \mathcal{A}(g_i)$. So that

$$\{\mathcal{F}_i^*(e_n \otimes e_m) : n, m = 1, 2\}_{i=1}^3 = \left\{ \underbrace{e_1, 0, e_2, 0}_{i=1}; \underbrace{0, e_1, 0, e_2}_{i=2}; \underbrace{0, 2e_1, 0, 2e_2}_{i=3} \right\}$$

and

$$\{\mathcal{G}_i^*(e_n \otimes e_m) : n, m = 1, 2\}_{i=1}^3 = \left\{ \underbrace{e_1, 0, e_2, 0}_{i=1}; \underbrace{e_1, e_1, e_2, e_2}_{i=2}; \underbrace{e_1, -e_1, e_2, -e_2}_{i=3} \right\}.$$

Now, since for any $\sigma \subset J$, $\{\mathcal{F}_i^*(e_n \otimes e_m) : n, m = 1, 2\}_{i \in \sigma} \cup \{\mathcal{G}_i^*(e_n \otimes e_m) : n, m = 1, 2\}_{i \in \sigma^c}$ is a frame for \mathcal{H} , hence by Theorem 4.3, $\{\mathcal{F}_i\}_{i=1}^3$ and $\{\mathcal{G}_i\}_{i=1}^3$ are HS-woven.

Two ordinary frames $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$ for a Hilbert space \mathcal{H} are weakly woven if for every $\sigma \subset \mathbb{N}$, the family $\{f_i\}_{i \in \sigma} \cup \{g_i\}_{i \in \sigma^c}$ is a frame for \mathcal{H} . In [5], Bemrose et al. proved that woven frames are equivalent to weakly woven frames.

Definition 4.5. A family of HS-frames $\{\{\mathcal{G}_{ni}\}_{n \in \mathbb{N}} : i \in [m]\}$ for a separable Hilbert space \mathcal{K} with respect to \mathcal{H} is said to be weakly HS-woven if for every partition $\{\sigma_i\}_{i \in [m]}$ of \mathbb{N} , the family $\{\mathcal{G}_{ni}\}_{n \in \sigma_i, i \in [m]}$ is a HS-frame for \mathcal{K} with respect to \mathcal{H} .

Using a technique given in Theorem 4.5. of [5], we have the following result for woven and weakly woven HS-frames.

Theorem 4.6. Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^\infty$ and $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^\infty$ be HS-frames for \mathcal{K} with respect to \mathcal{H} . Then, the following are equivalent:

- (a) \mathcal{F} and \mathcal{G} are HS-woven frames.
- (b) \mathcal{F} and \mathcal{G} are weakly HS-woven frames.

Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^\infty$ be HS-frames for \mathcal{K} with respect to \mathcal{H} . For all $\sigma \subseteq \mathbb{N}$, let $\oplus_{i \in \sigma} \mathcal{C}_2$ denote the Hilbert space

$$\left\{ \mathcal{A} = \{\mathcal{A}_i\}_{i \in \sigma} : \mathcal{A}_i \in \mathcal{C}_2 \quad \forall i \in \sigma \text{ and } \|\mathcal{A}\|_2 := \left(\sum_{i \in \sigma} \|\mathcal{A}_i\|_{\mathcal{C}_2}^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

We define $\pi_\sigma : \oplus_{i \in \mathbb{N}} \mathcal{C}_2 \rightarrow \oplus_{i \in \sigma} \mathcal{C}_2$ by $\pi_\sigma(\{\mathcal{A}_i\}_{i \in \mathbb{N}}) := \{\mathcal{A}_i\}_{i \in \sigma}$ and

$$T_{\mathcal{F}, \sigma}(\{\mathcal{A}_i\}_{i \in \mathbb{N}}) := T_{\mathcal{F}} \pi_\sigma(\{\mathcal{A}_i\}_{i \in \mathbb{N}}) = \sum_{i \in \sigma} \mathcal{F}_i^* \mathcal{A}_i, \quad (\{\mathcal{A}_i\}_{i \in \mathbb{N}} \in \oplus_{i \in \mathbb{N}} \mathcal{C}_2).$$

Theorem 4.7. Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^\infty$ be a HS-frame for \mathcal{K} with respect to \mathcal{H} with lower and upper HS-bounds $A_{\mathcal{F}}, B_{\mathcal{F}}$, respectively, and let $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^\infty$ be a HS-Bessel sequences for \mathcal{K} with respect to \mathcal{H} with upper HS-bound $B_{\mathcal{G}}$. If \mathcal{G} be a μ -perturbation of \mathcal{F} and $\mu < \frac{A_{\mathcal{F}}}{\sqrt{B_{\mathcal{F}}} + \sqrt{B_{\mathcal{G}}}}$, then \mathcal{G} is a HS-frame for \mathcal{K} with respect to \mathcal{H} and \mathcal{F} and \mathcal{G} are HS-woven.

Proof . The fact that \mathcal{G} is a HS-frame for \mathcal{K} with respect to \mathcal{H} follows directly from Theorem 3.4, since

$$\mu < \frac{A_{\mathcal{F}}}{\sqrt{B_{\mathcal{F}}} + \sqrt{B_{\mathcal{G}}}} \leq \frac{A_{\mathcal{F}}}{\sqrt{B_{\mathcal{F}}}} \leq \sqrt{A_{\mathcal{F}}}.$$

Let σ be an arbitrary subset of \mathbb{N} . For every $f \in \mathcal{K}$, we have

$$\sum_{i \in \sigma^c} \|\mathcal{F}_i(f)\|_2^2 + \sum_{i \in \sigma} \|\mathcal{G}_i(f)\|_2^2 \leq B_{\mathcal{F}}\|f\|^2 + B_{\mathcal{G}}\|f\|^2 \leq (B_{\mathcal{F}} + B_{\mathcal{G}})\|f\|^2.$$

Therefore, the upper HS-frame bound of $\{\mathcal{G}_i\}_{i \in \sigma} \cup \{\mathcal{F}_i\}_{i \in \sigma^c}$ is at most $B_{\mathcal{F}} + B_{\mathcal{G}}$. Since $\|\pi_{\sigma}\| \leq 1$, observe that

$$\|T_{\mathcal{F},\sigma}^*\| = \|T_{\mathcal{F},\sigma}\| = \|T_{\mathcal{F}}\pi_{\sigma}\| \leq \|T_{\mathcal{F}}\| \leq \sqrt{B_{\mathcal{F}}}$$

and similarly $\|T_{\mathcal{G},\sigma}\| \leq \sqrt{B_{\mathcal{F}}}$, $\|T_{\mathcal{G},\sigma}^* - T_{\mathcal{F},\sigma}^*\| = \|T_{\mathcal{G},\sigma} - T_{\mathcal{F},\sigma}\| \leq \|T_{\mathcal{G}} - T_{\mathcal{F}}\| \leq \mu$. Therefore, for every $f \in \mathcal{K}$ we have

$$\begin{aligned} \left\| \sum_{i \in \sigma} \mathcal{G}_i^* \mathcal{G}_i(f) - \sum_{i \in \sigma} \mathcal{F}_i^* \mathcal{F}_i(f) \right\| &= \|T_{\mathcal{G},\sigma} T_{\mathcal{G},\sigma}^*(f) - T_{\mathcal{F},\sigma} T_{\mathcal{F},\sigma}^*(f)\| \\ &\leq \|(T_{\mathcal{G},\sigma} T_{\mathcal{G},\sigma}^* - T_{\mathcal{G},\sigma} T_{\mathcal{F},\sigma}^*)(f)\| + \|(T_{\mathcal{G},\sigma} T_{\mathcal{F},\sigma}^* - T_{\mathcal{F},\sigma} T_{\mathcal{F},\sigma}^*)(f)\| \\ &\leq (\|T_{\mathcal{G},\sigma}\| \|T_{\mathcal{G},\sigma}^* - T_{\mathcal{F},\sigma}^*\| + \|T_{\mathcal{G},\sigma} - T_{\mathcal{F},\sigma}\| \|T_{\mathcal{F},\sigma}^*\|) \|f\| \\ &\leq \mu(\sqrt{B_{\mathcal{F}}} + \sqrt{B_{\mathcal{G}}}) \|f\| \end{aligned}$$

which implies

$$\begin{aligned} \left\| \sum_{i \in \sigma} \mathcal{G}_i^* \mathcal{G}_i(f) + \sum_{i \in \sigma^c} \mathcal{F}_i^* \mathcal{F}_i(f) \right\| &= \left\| \sum_{i=1}^{\infty} \mathcal{F}_i^* \mathcal{F}_i(f) + \left(\sum_{i \in \sigma} \mathcal{G}_i^* \mathcal{G}_i(f) - \sum_{i \in \sigma} \mathcal{F}_i^* \mathcal{F}_i(f) \right) \right\| \\ &\geq \left\| \sum_{i=1}^{\infty} \mathcal{F}_i^* \mathcal{F}_i(f) \right\| - \left\| \sum_{i \in \sigma} \mathcal{G}_i^* \mathcal{G}_i(f) - \sum_{i \in \sigma} \mathcal{F}_i^* \mathcal{F}_i(f) \right\| \\ &\geq \|S_{\mathcal{F}}(f)\| - \|T_{\mathcal{G},\sigma} T_{\mathcal{G},\sigma}^*(f) - T_{\mathcal{F},\sigma} T_{\mathcal{F},\sigma}^*(f)\| \\ &\geq \left(A_{\mathcal{F}} - \mu(\sqrt{B_{\mathcal{F}}} + \sqrt{B_{\mathcal{G}}}) \right) \|f\|. \end{aligned}$$

Thus $A_{\mathcal{F}} - \mu(\sqrt{B_{\mathcal{F}}} + \sqrt{B_{\mathcal{G}}})$ is the lower HS-frame bound of $\{\mathcal{G}_i\}_{i \in \sigma} \cup \{\mathcal{F}_i\}_{i \in \sigma^c}$, which proves the theorem. \square

Corollary 4.8. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ be a HS-frame for \mathcal{K} with respect to \mathcal{H} with HS-bounds $A_{\mathcal{F}}, B_{\mathcal{F}}$ and let $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ be a μ -perturbation of \mathcal{F} . If $\mu < \frac{A_{\mathcal{F}}}{2\sqrt{B_{\mathcal{F}}} + \mu}$, then \mathcal{G} is a HS-frame for \mathcal{K} with respect to \mathcal{H} and \mathcal{F} and \mathcal{G} are HS-woven. Moreover, if also*

$$\mu < \frac{\sqrt{A_{\mathcal{F}}}(\sqrt{A_{\mathcal{F}}} - \mu)^2}{2B_{\mathcal{F}}(2 + A_{\mathcal{F}}\|\Phi\|^2)},$$

for some operator $\Phi \in \mathbf{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}})$, then the HS-dual $\tilde{\mathcal{F}}(\Phi)$ of \mathcal{F} and HS-dual $\tilde{\mathcal{G}}(\mathbf{P}_{\ker T_{\mathcal{G}}} T_{\tilde{\mathcal{F}}(\Phi)}^*)$ of \mathcal{G} are HS-woven.

Proof . The fact that \mathcal{G} is a HS-frame for \mathcal{K} with respect to \mathcal{H} follows directly from Theorem 3.4, since

$$\mu < \frac{A_{\mathcal{F}}}{2\sqrt{B_{\mathcal{F}}} + \mu} \leq \frac{A_{\mathcal{F}}}{2\sqrt{B_{\mathcal{F}}}} \leq \frac{\sqrt{A_{\mathcal{F}}}}{2}.$$

This theorem also yields that \mathcal{G} has the lower and upper HS-frame bounds $A_{\mathcal{G}} := (\sqrt{A} - \mu)^2$ and $B_{\mathcal{G}} := (\sqrt{B} + \mu)^2$, respectively. By (3.6), $\tilde{\mathcal{F}}(\Phi)$ has the upper HS-frame bound $A_{\tilde{\mathcal{F}}(\Phi)} := A^{-1} + \|\Phi\|^2$. Also $B_{\tilde{\mathcal{F}}(\Phi)} := B^{-1}$ is the lower HS-bound frame of $\tilde{\mathcal{F}}(\Phi)$, since

$$\|T_{\tilde{\mathcal{F}}(\Phi)}^*(f)\|_2^2 = \|T_{\mathcal{F}}^*(f)\|_2^2 + \|\Phi(f)\|_2^2 \geq \frac{1}{B}\|f\|^2 + \|\Phi(f)\|_2^2 \geq \frac{1}{B}\|f\|^2.$$

Similarly, we conclude

$$\begin{aligned} \left\| T_{\tilde{\mathcal{G}}(\Psi)}^*(f) \right\|_2^2 &= \|T_{\tilde{\mathcal{G}}}(f)\|_2^2 + \|\mathbb{P}_{\ker T_{\tilde{\mathcal{F}}(\Phi)}} T_{\tilde{\mathcal{F}}(\Phi)}^*(f)\|_2^2 \\ &\leq \frac{1}{(\sqrt{A} - \mu)^2} \|f\|^2 + \|T_{\tilde{\mathcal{F}}(\Phi)}^*(f)\|_2^2 \\ &\leq \left(\frac{1}{(\sqrt{A} - \mu)^2} + \frac{1}{A} + \|\Phi\|^2 \right) \|f\|^2 \end{aligned}$$

where, $\Psi := \mathbb{P}_{\ker T_{\tilde{\mathcal{F}}(\Phi)}} T_{\tilde{\mathcal{F}}(\Phi)}^*$. Thus $B_{\tilde{\mathcal{G}}(\Psi)} := (\sqrt{A} - \mu)^{-2} + A^{-1} + \|\Phi\|^2$ is the upper HS-frame bound of $\tilde{\mathcal{G}}(\Psi)$. Theorem 3.7 yields that $\tilde{\mathcal{G}}(\Psi)$ is a λ -perturbation of $\tilde{\mathcal{F}}(\Phi)$, where λ given by (3.5). Since $\mu < \frac{A_{\mathcal{F}}}{\sqrt{B_{\mathcal{F}}} + \sqrt{B_{\mathcal{G}}}}$ and

$$\begin{aligned} \frac{A_{\tilde{\mathcal{F}}(\Phi)}}{\sqrt{B_{\tilde{\mathcal{F}}(\Phi)}} + \sqrt{B_{\tilde{\mathcal{G}}(\Psi)}}} &\geq \frac{1}{2B_{\mathcal{F}} \sqrt{(\sqrt{A_{\mathcal{F}}} - \mu)^{-2} + A_{\mathcal{F}}^{-1} + \|\Phi\|^2}} \\ &\geq \frac{\sqrt{A_{\mathcal{F}}} - \mu}{2B_{\mathcal{F}} \sqrt{2 + (\sqrt{A_{\mathcal{F}}} - \mu)^2 \|\Phi\|^2}} \\ &\geq \frac{\sqrt{A_{\mathcal{F}}} - \mu}{2B_{\mathcal{F}} \sqrt{2 + A_{\mathcal{F}} \|\Phi\|^2}} \\ &> \frac{\mu \sqrt{2 + A_{\mathcal{F}} \|\Phi\|^2}}{\sqrt{A_{\mathcal{F}}} (\sqrt{A_{\mathcal{F}}} - \mu)} \\ &> \lambda, \end{aligned}$$

then by Theorem 4.7 we have the result. \square

Corollary 4.9. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^\infty$ be HS-frame for \mathcal{K} with respect to \mathcal{H} with HS-bounds $A_{\mathcal{F}}, B_{\mathcal{F}}$ and let $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^\infty$ be a μ -perturbation of \mathcal{F} . If $\mu < \frac{A_{\mathcal{F}}}{\sqrt{A_{\mathcal{F}}} + 2\sqrt{B_{\mathcal{F}}}}$, then \mathcal{G} is a HS-frame for \mathcal{K} with respect to \mathcal{H} and \mathcal{F} and \mathcal{G} are HS-woven. Moreover, if also*

$$\mu < \frac{A_{\mathcal{F}} \sqrt{A_{\mathcal{F}}}}{8B_{\mathcal{F}} (2 + A_{\mathcal{F}} \|\Phi\|^2)},$$

for some operator $\Phi \in \mathbf{rann}_{B(\mathcal{K}, \oplus_2)}(T_{\mathcal{F}})$, then the HS-dual $\tilde{\mathcal{F}}(\Phi)$ of \mathcal{F} and HS-dual $\tilde{\mathcal{G}}(\mathbb{P}_{\ker T_{\tilde{\mathcal{F}}(\Phi)}} T_{\tilde{\mathcal{F}}(\Phi)}^*)$ of \mathcal{G} are HS-woven.

References

[1] A. Aldroubi, C. Cabrelli and U. Molter, *Wavelets on irregular grids with arbitrary dilation matrices and frame atomics for $L^2(\mathbb{R}^d)$* , Appl. Comput. Harmon. Anal. 17 (2004), 119–140.
 [2] A. Arefijamaal and G. Sadeghi, *von Neumann–Schatten Dual Frames and their Perturbations*, Results in Mathematics (2015), 1–11.
 [3] M.S. Asgari and A. Khosravi, *Frames and bases of subspaces in Hilbert spaces*, J. Math. Anal. Appl. 308 (2005), 541–553.
 [4] P. Balazs, J.P. Antoine and A. Grybos, *Weighted and controlled frames: mutual relationship and first numerical properties*, Int. J. Wavelets Multiresolut. Inf. Process. 8 (2010), 109–132.

- [5] . Bemrose, P.G. Casazza, K. Gröchenig, M.C. Lammers and R.G. Lynch, *Weaving frames*, Appl. Comput. Harmon. Anal. Oper. Matrices 10 (2016), 1093–1116.
- [6] P.G. Casazza and R.G. Lynch, *Weaving properties of Hilbert space frames*, In Proceedings of the SampTA, (2015), 110–114
- [7] P.G. Casazza, D. Freeman and R.G. Lynch, *Weaving Schauder frames*, J. Approx. Theory 211 (2016), 42–60.
- [8] P. Casazza and G. Kutyniok, *Frames of subspaces*, in: *Wavelets, Frames and Operator Theory*, in: *Contemp. Math.*, Amer. Math. Soc. 345 (2004), 87–113.
- [9] O. Christensen, *Introduction to frames and Riesz bases*, Birkhäuser, 2003.
- [10] O. Christensen and Y.C. Eldar, *Oblique dual frames and shift-invariant spaces*, Appl. Comput. Harmon. Anal. 17 (2004), 48–68.
- [11] O. Christensen and R. S. Laugesen, *Approximately dual frame pairs in Hilbert spaces and applications to Gabor frames*, Sampl. Theor. Signal Image Process. 9 (2010), 77–89.
- [12] O. Christensen and D. Stoeva, *p -frames in separable Banach spaces*, Adv. Comput. Math. 18 (2003), 117–126.
- [13] J. B. Conway, *A Course in Functional Analysis*, Springer–Verlag, 1985.
- [14] Deepshikha, L.K. Vashisht and G. Verma, *Generalized weaving frames for operators in Hilbert spaces*, Results Math. 72(3) (2017), 1369–1391.
- [15] Deepshikha, S. Garg, L.K. Vashisht and G. Verma, *On weaving fusion frames for Hilbert spaces*, In Proceedings of the SampTA, (2017), 381–385.
- [16] Deepshikha and L. K. Vashisht, *Weaving K -frames in Hilbert spaces*, Results Math. 73(2) (2018), Art. 81, 20 pp.
- [17] M. Dörfler and M. Faulhuber, *Multi-window weaving frames*, In: Proceedings of the SampTA (2017), 373–376.
- [18] Y. Eldar, *Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors*, J. Fourier Anal. Appl. 9 (2003), 77–96.
- [19] M. Fornasier, *Quasi-orthogonal decompositions of structured frames*, J. Math. Anal. Appl. 289 (2004), 180–199.
- [20] H. Javanshiri and M. Choubin, *Multipliers for von Neumann-Schatten Bessel sequences in separable Banach spaces*, Linear Algebra Appl. 545 (2018), 108–138.
- [21] A. Khosravi and J. Sohrabi Banyarani, *Weaving g -frames and weaving fusion frames*, Bull. Malays. Math. Sci. Soc. 42(6) (2019), 3111–3129.
- [22] S. Li and H. Ogawa, *Pseudo-frames for subspaces with applications*, J. Fourier Anal. Appl. 10 (2004), 409–431.
- [23] R. Meise and D. Vogt, *Introduction to functional analysis*, Oxford Graduate Texts in Mathematics, The Clarendon Press, Oxford University Press, New York, 1997.
- [24] P. Oswald, *Multilevel Finite Element Approximation: Theory and Application*, Teubner Skr. Nu-merik, Teubner, Stuttgart, 1994.
- [25] P. Oswald, *Frames and space splittings in Hilbert spaces*, Lecture Notes Part. 1997,1:1–32.
- [26] A. Poria, *Some identities and inequalities for Hilbert-Schmidt frames*, Mediterr. J. Math. 14(2) (2017), Art. 59, 14 pp.
- [27] A. Poria, *Approximation of the inverse frame operator and stability of Hilbert-Schmidt frames*, Mediterr. J. Math. 14(4) (2017), Art. 153, 22 pp.
- [28] J. R. Ringrose, *Compact Non-Self-Adjoint Operators*, Van Nostrand Reinhold Company, 1971.
- [29] G. Sadeghi and A. Arefijamaal, *von Neumann-Schatten frames in separable Banach spaces*, Mediterr. J. Math. 9 (2012) 525–535.
- [30] W. C. Sun, *G -frames and G -Riesz bases*, J. Math. Anal. Appl. 322 (2006) 437–452.
- [31] L. K. Vashisht and Deepshikha, *On continuous weaving frames*, Adv. Pure Appl. Math. 8(1) (2017), 15–31.
- [32] L. K. Vashisht and Deepshikha, *Weaving properties of generalized continuous frames generated by an iterated function system*, J. Geom. Phys. 110 (2016), 282–295.
- [33] L. K. Vashisht, S. Garg, Deepshikha and P. K. Das, *On generalized weaving frames in Hilbert spaces*, Rocky Mountain J. Math. 48(2) (2018), 661–685.
- [34] X.C. Xiao, G.R. Zhoua and Y.C. Zhub, *Weaving of K - g -frames in Hilbert spaces*, ScienceAsia 45 (2019), 285–291.
- [35] W. Zhang, *Dual and approximately dual Hilbert-Schmidt frames in Hilbert spaces*, Results Math. 73(1) (2018), Art. 4, 20 pp.
- [36] K. Zhuo, *Operator Theory in Function Spaces*, Marcel Dekker, Inc, 1990.