



Extension of Darbo fixed-point theorem to illustrate existence of the solutions of some nonlinear functional stochastic integral equations

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Abstract

In this paper, Darbo fixed-point theorem is employed as a mathematical tool to examine existence of the solution of some nonlinear functional stochastic integral equations which arise in many physical, chemical and biological problems. Throughout this paper, we consider $(C[0, 1], \|\cdot\|)$ as Banach space equipped with uniform norm.

Keywords: Nonlinear functional integral equations; Stochastic integral equations; Fixed-point theorem; Banach space; Measure of non-compactness.

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1. Introduction

Many problems in different branches of science can be modeled by using nonlinear functional integral equations [1-4]. Solutions of some functional-integral equations in Banach algebra have been discussed in [5]. Maleknejad et al. have studied existence of solutions for some nonlinear integral equations and nonlinear functional- integral equations in [6] and [7], respectively. The existence and uniqueness for Volterra-Fredholm type integral equations have been investigated by using the coupled fixed point theorems in the framework of Banach space $C([a, b], \mathbb{R})$ in [8]. Petryshyn's fixed point theorem has been used to establish existence of solutions for some nonlinear Volterra integral equations in [9]. A

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fixed point theorem for an appropriate operator on the cartesian product of the given spaces endowed with directed graphs has been applied in [10]. The concept of contraction via the measure of noncompactness on the Banach space have been employed to investigate the existence of the solution of fractional integral equations in the paper [11]. Furthermore, some extensions of Darbo fixed point theorem has been given using the technique of measures of noncompactness. Then it used to prove an existence result for a quadratic integral equation of Hammerstein type on an unbounded interval in two variables [12].

In recent years, it has become obvious that different problems classically are modeled by deterministic integral equations, can be more satisfactorily modeled by using various stochastic integral equations such as stochastic integral equations [13] or stochastic integro-differential equations [14]. Recently, development, analysis and implementation of stable methods for providing the numerical solution of various kinds of stochastic integral equations have attracted the attention of many researchers. An efficient numerical technique has been applied to provide the approximate solution of nonlinear stochastic Itô-Volterra integral equations driven by fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ in the paper [15]. Maleknejad et al. have used block pulse functions and their stochastic operational matrix of integration to solve stochastic Volterra integral equations, numerically [16]. Meshless methods based on radial basis functions have been employed to solve two dimensional linear stochastic integral equations and fractional stochastic integro-differential equations in the papers [17] and [18], respectively. Saffarzadeh et al. have proposed a numerical iterative approach for obtaining approximate solutions of nonlinear stochastic Itô-Volterra integral equations [19]. Their proposed method is based on a combination of the successive approximations method, the linear spline interpolation and Itô approximation. The shifted Legendre spectral collocation method, which is based on P panels M -point Newton-Cotes rules with M fixed for estimating Itô integrals, has been utilized to solve stochastic fractional integro-differential equations [20]. A new numerical method based on triangular functions for solving nonlinear stochastic differential equations has been presented by Asgari et al in [21].

Before starting to solve every equations, we should first be sure that these equations have exact solutions and then start to solve them. But, in none of the published paper on numerical solution of stochastic integral equations, existence and uniqueness of solution to these equations have not been investigated.

In this paper, we investigate existence of the solution of nonlinear functional stochastic integral equations which have the following form

$$x(t) = f\left(t, \int_0^t u(t, s, x(s))dB(s), x(\alpha(t))\right) \cdot g\left(t, \int_0^1 v(t, s, x(s))dB(s), x(\beta(t))\right), \quad (1.1)$$

where $t \in [0, 1]$, $f(t, y, x)$, $g(t, y, x)$, $u(t, s, x)$ and $v(t, s, x)$ are known stochastic process defined on probability space $(\Lambda, \mathcal{F}, \mathbf{P})$, $x(t)$ is unknown function which is called solution of integral equations, and $B(t)$ is Brownian motion process defined on the same probability space $(\Lambda, \mathcal{F}, \mathbf{P})$.

Eq. (1.1) is extension of some particular stochastic integral equations. For example

- (i) if $g(t, y, x) \equiv 1$ and $f(t, y, x) = a(t) + y$, then Eq. (1.1) converts to the Itô-Volterra integral equations of the second kind as follows

$$x(t) = a(t) + \int_0^t u(t, s, x(s))dB(s),$$

- (ii) if $f(t, y, x) \equiv 1$ and $g(t, y, x) = b(t) + y$, then Eq. (1.1) converts to the Itô-Fredholm integral

equations of the second kind as follows

$$x(t) = b(t) + \int_0^1 v(t, s, x(s)) dB(s).$$

In this paper, we prove an existence theorem of solutions to some nonlinear functional stochastic integral equations which are extension of some particular stochastic integral equations such as nonlinear Itô-Volterra integral equations [22] or stochastic quadratic integral equations. The main tool applied in this work is Darbo fixed-point theorem for the product of two operators.

2. Darbo fixed-point theorem

The first measure of non-compactness, the function α , was introduced by C. Kuratowski [23] in 1930. Later in 1955, G. Darbo [24] was the first person who applied the function α to examine operators whose properties can be described as being intermediate between those of contraction and compact mappings. He proved that if S be a continuous operator mapping nonempty, bounded, closed and convex subset C of a Banach space E to itself such that

$$\alpha(SX) \leq h\alpha(X), \quad \text{for all } X \subset C, \quad (2.1)$$

where $0 \leq h < 1$ is a constant number, then the operator S has at least one fixed point in the set C . If Eq. (2.1) be satisfied, we say the operator S satisfies Darbo fixed-point theorem with respect to measure α and constant h . Darbo fixed-point theorem is a generalization of well-known Schauder's fixed-point theorem and it involves the existence part of Banach's fixed-point theorem.

Theorem 2.1. [25] Suppose that Ω be a nonempty, bounded, convex and closed subset of $C[0, 1]$ and continuous operators F and G transform the set Ω into $C[0, 1]$ in such a way that $F(\Omega)$ and $G(\Omega)$ are bounded. Furthermore, suppose that the operator $T = F.G$ transform Ω into itself. If the operators F and G satisfies the Darbo's condition with constant h_1 and h_2 , respectively, then the operator T satisfies the Darbo's condition on Ω with the constant $\|F(\Omega)\|h_2 + \|G(\Omega)\|h_1$.

In the following, we consider the Banach space $C[0, 1]$ which is equipped with the uniform norm defined as

$$\|f\| = \sup\{|f(t)|; t \in [0, 1]\}.$$

J. Banas and K. Goebel introduced and studied a special measure of non-compactness in space $C[0, 1]$ in [26]. Let $\mathbf{m}_{C[0,1]}$ denotes the family of all non-empty and bounded subset of $C[0, 1]$. The modulus of continuity of $x \in X (X \in \mathbf{m}_{C[0,1]})$ is denoted by $\omega(x, \epsilon)$ and is defined as follows

$$\omega(x, \epsilon) = \sup\{|x(t_1) - x(t_2)|; t_1, t_2 \in [0, 1], |t_1 - t_2| \leq \epsilon\}.$$

Furthermore, we let

$$\begin{aligned} \omega(X, \epsilon) &= \sup\{\omega(x, \epsilon); x \in X\}, \\ \omega_0(X) &= \lim_{\epsilon \rightarrow 0} \omega(X, \epsilon). \end{aligned}$$

The function $\omega_0(X)$ is a regular measure of non-compactness in the Banach space $C[0, 1]$.

3. Brownian motion process

In this section, we present basic definitions and properties of a important stochastic process, Brownian motion process, which are needed in the next Section.

The motion of pollen particle suspended in fluid was studied by botanist R. Brown in 1828. He observed that particle has an irregular and random movement. He failed to model this phenomena and released it as an open problem. A. Einstein ,in 1905, argued that this random motion is due to bombardment of the particle by the molecules of the fluid and provided the equations for describing this movement. He named this process as Brownian motion due to R. Brown's studies. Brownian motion is used as a mathematical model for movement of stock prices by L. Bachelier in 1900. The mathematical base for Brownian motion as a stochastic process was done by N. Wiener in 1931. For this reason, Brownian motion process is also named the Wiener process.

The Brownian motion process $B(t)$ serves as a basic model for the cumulative effect of pure noise. $B(t)$ and the displacement $B(t) - B(0)$ denote the position of a particle at time t and the effect of the purely random bombardment by the molecules of the fluid or the effect of noise over time t , respectively.

Definition 3.1. Brownian motion $\{B(t)\}$ is a stochastic process with the following properties [27]

1. (Independence of increments) $B(t) - B(s)$, for $t > s$, is independent of the past, that is, of B_u , $0 \leq u \leq s$, or of \mathcal{F}_s , the σ -field generated by $B(u)$, $u \leq s$.
2. (Normal increments) $B(t) - B(s)$ has Normal distribution with mean 0 and variance $t - s$. This implies (taking $s = 0$) that $B(t) - B(0)$ has $N(0, t)$ distribution.
3. (Continuity of paths) $B(t)$, $t \geq 0$ are continuous functions of t .

The initial position of Brownian motion is not specified in the above definition. When $B(0) = x$, then the Brownian motion process is started at x . If $x = 0$, i.e. $B(0) = 0$ then Brownian motion process is called standard Brownian motion.

The next theorem obtain some conditions to exist the Itô integrals.

Definition 3.2. [27] A process Y is called adapted to the filtration $\mathcal{F} = (\mathcal{F}_t)$, if for all t , $Y(t)$ is \mathcal{F}_t -measurable.

Theorem 3.3. [27] If Y is a continuous adapted process then the Itô integral $\int_0^T Y(t)dB(t)$ exists.

4. Existence of the solution

In this section, we illustrate the nonlinear stochastic integral equation (1.1) has at least one solution in $C[0, 1]$ under the following mild hypotheses

(H1) $f, g : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions and there exists a nonnegative constant l such that for every $t \in [0, 1]$, we have

$$\begin{aligned} |f(t, 0, 0)| &\leq l, \\ |g(t, 0, 0)| &\leq l. \end{aligned}$$

(H2) $u, v : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions with the continuous first derivative.

(H3) There exist four continuous functions $a_1, a_2, b_1, b_2 : [0, 1] \longrightarrow [0, 1]$, such that

$$\begin{aligned} |f(t, y_1, x_1) - f(t, y_2, x_2)| &\leq a_1(t)|y_1 - y_2| + a_2(t)|x_1 - x_2|, \\ |g(t, y_1, x_1) - g(t, y_2, x_2)| &\leq b_1(t)|y_1 - y_2| + b_2(t)|x_1 - x_2|. \end{aligned}$$

(H4) There is a nonnegative constant number k such that for every $t \in [0, 1]$, we have

$$a_1(t), a_2(t), b_1(t), b_2(t) \leq k.$$

(H5) There exist positive constant numbers α and β such that for every $t, s \in [0, 1]$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} |u(t, s, x)| &\leq \alpha + \beta|x|, \\ |v(t, s, x)| &\leq \alpha + \beta|x|. \end{aligned}$$

(H6) $4\alpha'\beta' < 1$, where $\alpha' = k\eta\alpha + l$ and $\beta' = k(\eta\beta + 1)$ and $\eta = \sup\{B(t); t \in [0, 1]\}$.

Theorem 4.1. Under the assumptions (H1)-(H6), the nonlinear stochastic integral equation (1.1) has at least one solution in Banach space $C[0, 1]$.

Proof . We define operators F and G as follows

$$\begin{aligned} (Fx)(t) &= f\left(t, \int_0^t u(t, s, x(s))dB(s), x(\alpha(t))\right), \\ (Gx)(t) &= g\left(t, \int_0^1 v(t, s, x(s))dB(s), x(\beta(t))\right). \end{aligned}$$

Therefore, the needed operator is obtained as follows

$$Tx = (Fx)(Gx).$$

We consider $B_r \subset C[0, 1]$ which is defined as follows

$$B_r = \{x : x \in C[0, 1], \|x\| \leq r\}.$$

By using hypothesis (H1)-(H5), the following estimate can be obtained for fix $x \in C[0, 1]$ and $t \in [0, 1]$:

$$\begin{aligned} |(Fx)(t)| &= \left| f\left(t, \int_0^t u(t, s, x(s))dB(s), x(\alpha(t))\right) \right| \\ &\leq \left| f\left(t, \int_0^t u(t, s, x(s))dB(s), x(\alpha(t))\right) - f(t, 0, 0) \right| + |f(t, 0, 0)| \\ &\leq a_1(t) \left| \int_0^t u(t, s, x(s))dB(s) \right| + a_2(t)|x(\alpha(t))| + |f(t, 0, 0)| \\ &\leq a_1(t) \int_0^t |u(t, s, x(s))|dB(s) + a_2(t)|x(\alpha(t))| + |f(t, 0, 0)| \\ &\leq k\eta(\alpha + \beta|x(t)|) + k|x(\alpha(t))| + l \\ &\leq k\eta(\alpha + \beta\|x\|) + k\|x\| + l = \beta'\|x\| + \alpha', \end{aligned} \tag{4.1}$$

where $\alpha' = k\eta\alpha + l$ and $\beta' = k(\eta\beta + 1)$ and $\eta = \sup\{B(t); t \in [0, 1]\}$. From Eq. (4.1), we get

$$\|Fx\| \leq \beta'\|x\| + \alpha'. \tag{4.2}$$

Similarly, we can show

$$\|Gx\| \leq \beta'\|x\| + \alpha'. \tag{4.3}$$

By using definition of operator T and Eqs. (4.2) and (4.3), for every $x \in C[0, 1]$, we get

$$\|Tx\| \leq (\beta'\|x\| + \alpha')^2. \tag{4.4}$$

From Eq. (4.4), we deduce that the operator T transforms the ball B_r to itself, $T : B_r \rightarrow B_r$, for $r_1 \leq r \leq r_2$, where

$$r_1 = \frac{1 - 2\alpha'\beta' - \sqrt{1 - 4\alpha'\beta'}}{2\beta'^2},$$

$$r_2 = \frac{1 - 2\alpha'\beta' + \sqrt{1 - 4\alpha'\beta'}}{2\beta'^2}.$$

In the following, we consider $r = r_1$.

Now, we should prove that T is a continuous operator on B_r . For this aim, first we establish that F and G are continuous on B_r . Consider fix $\epsilon > 0$ and $x, y \in B_r$ such that $\|x - y\| \leq \epsilon$. Then for every $t \in [0, 1]$, we have

$$\begin{aligned} |(Fx)(t) - (Fy)(t)| &= \left| f\left(t, \int_0^t u(t, s, x(s))dB(s), x(\alpha(t))\right) - f\left(t, \int_0^t u(t, s, y(s))dB(s), y(\alpha(t))\right) \right| \\ &\leq a_1(t) \left| \int_0^t u(t, s, x(s))dB(s) - \int_0^t u(t, s, y(s))dB(s) \right| + a_2(t) |x(\alpha(t)) - y(\alpha(t))| \\ &\leq a_1(t) \int_0^t |u(t, s, x(s)) - u(t, s, y(s))| dB(s) + a_2(t) |x(\alpha(t)) - y(\alpha(t))| \\ &\leq k\eta\omega(u, \epsilon) + k\epsilon, \end{aligned} \tag{4.5}$$

where

$$\omega(u, \epsilon) = \sup\left\{ |u(t, s, x) - u(t, s, y)|; t, s \in [0, 1], x, y \in [-r, r], |x - y| \leq \epsilon \right\}.$$

Since every continuous function defined on the closed interval is a uniformly continuous function, so the function $u(t, s, x)$ is uniformly continuous on the bounded subset $[0, 1] \times [0, 1] \times [-r, r]$. It follows that $\omega(u, \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. From estimate (4.5), we deduce that the operator F is continuous on B_r . Similarly, we can show that the operator G is continuous and therefore T is a continuous operator on B_r .

In the following, we show operators F and G satisfy the Darbo fixed-point theorem with respect to the measure ω_0 on B_r . Consider the non-empty subset X of B_r and $x \in X$, then for a fixed $\epsilon > 0$

and $t_1, t_2 \in [0, 1]$ where $t_1 \leq t_2$ and $t_2 - t_1 \leq \epsilon$, we have

$$\begin{aligned}
 |(Fx)(t_2) - (Fx)(t_1)| &= \left| f\left(t_2, \int_0^{t_2} u(t_2, s, x(s))dB(s), x(\alpha(t_2))\right) - f\left(t_1, \int_0^{t_1} u(t_1, s, x(s))dB(s), x(\alpha(t_1))\right) \right| \\
 &\leq \left| f\left(t_2, \int_0^{t_2} u(t_2, s, x(s))dB(s), x(\alpha(t_2))\right) - f\left(t_2, \int_0^{t_1} u(t_1, s, x(s))dB(s), x(\alpha(t_2))\right) \right| \\
 &\quad + \left| f\left(t_2, \int_0^{t_1} u(t_1, s, x(s))dB(s), x(\alpha(t_2))\right) - f\left(t_1, \int_0^{t_1} u(t_1, s, x(s))dB(s), x(\alpha(t_1))\right) \right| \\
 &\leq a_1(t) \left| \int_0^{t_2} u(t_2, s, x(s))dB(s) - \int_0^{t_1} u(t_1, s, x(s))dB(s) \right| \\
 &\quad + \left| f\left(t_2, \int_0^{t_1} u(t_1, s, x(s))dB(s), x(\alpha(t_2))\right) - f\left(t_1, \int_0^{t_1} u(t_1, s, x(s))dB(s), x(\alpha(t_2))\right) \right| \\
 &\quad + \left| f\left(t_1, \int_0^{t_1} u(t_1, s, x(s))dB(s), x(\alpha(t_2))\right) - f\left(t_1, \int_0^{t_1} u(t_1, s, x(s))dB(s), x(\alpha(t_1))\right) \right| \\
 &\leq a_1(t) \left[\int_0^{t_1} |u(t_2, s, x(s)) - u(t_1, s, x(s))|dB(s) + \int_{t_1}^{t_2} |u(t_2, s, x(s))|dB(s) \right] \\
 &\quad + \left| f\left(t_2, \int_0^{t_1} u(t_1, s, x(s))dB(s), x(\alpha(t_2))\right) - f\left(t_1, \int_0^{t_1} u(t_1, s, x(s))dB(s), x(\alpha(t_2))\right) \right| \\
 &\quad + a_2(t) |x(\alpha(t_2)) - x(\alpha(t_1))|. \tag{4.6}
 \end{aligned}$$

Consider the following notations

$$\begin{aligned}
 \omega_u(\epsilon, \dots) &= \sup \left\{ |u(t_1, s, x) - u(t_2, s, x)|; t_1, t_2, s \in [0, 1], |t_1 - t_2| \leq \epsilon, x \in [-r, r] \right\}, \\
 k' &= \sup \left\{ |u(t, s, x)|; t, s \in [0, 1], x \in [-r, r] \right\}, \\
 \omega_f(\epsilon, \dots) &= \sup \left\{ |f(t_1, y, x) - f(t_2, y, x)|; t_1, t_2 \in [0, 1], |t_1 - t_2| \leq \epsilon, x \in [-r, r], y \in [-k'\eta, k'\eta] \right\}, \\
 \omega(B, \epsilon) &= \sup \left\{ |B(t_2) - B(t_1)|; t_1, t_2 \in [0, 1], |t_1 - t_2| \leq \epsilon \right\}.
 \end{aligned}$$

From Eq. (4.6) and above notations, we yield

$$|(Fx)(t_2) - (Fx)(t_1)| \leq k[\eta\omega_u(\epsilon, \dots) + k'\omega(B, \epsilon)] + \omega_f(\epsilon, \dots) + k|x(\alpha(t_2)) - x(\alpha(t_1))|.$$

Thus,

$$\omega(Fx, \epsilon) \leq k[\eta\omega_u(\epsilon, \dots) + k'\omega(B, \epsilon)] + \omega_f(\epsilon, \dots) + k\omega(x, \omega(\alpha, \epsilon)). \tag{4.7}$$

Since $f(t, y, x), u(t, s, x)$ and $B(t)$ are uniformly continuous, so $\omega_u(\epsilon, \dots) \rightarrow 0, \omega_f(\epsilon, \dots) \rightarrow 0$ and $\omega(B, \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Hence, from Eq. (4.7), we have

$$\omega_0(FX) \leq k\omega_0(X). \tag{4.8}$$

By using a similar way, we can show

$$\omega_0(GX) \leq k\omega_0(X). \tag{4.9}$$

Finally, by using Eqs. (4.2), (4.3), (4.8) and (4.9) and Theorem 2.1, we deduce that the operator T satisfies the Darbo condition with respect to measure ω_0 and constant $2k(\beta'r + \alpha')$ on the ball B_r . Also, we have

$$\begin{aligned}
 2k(\beta'r + \alpha') &= 2k(\beta'r_1 + \alpha') = 2k\left(\beta' \frac{1 - 2\alpha'\beta' - \sqrt{1 - 4\alpha'\beta'}}{2\beta'^2} + \alpha'\right) \\
 &= \frac{k}{\beta'}(1 - \sqrt{1 - 4\alpha'\beta'}).
 \end{aligned}$$

Under assumption (H6), we have $1 - \sqrt{1 - 4\alpha'\beta'} < 1$. By using assumption (H5), $\beta > 0$ and definition $\eta > 0$, so $\frac{k}{\beta'} = \frac{k}{k(\eta^{\beta+1})} = \frac{1}{\eta^{\beta+1}} < 1$. So, the operator T is a contraction on B_r with respect to measure ω_0 and has at least one fixed point on ball B_r . Consequently, the nonlinear functional stochastic integral equation (1.1) has at least one solution on B_r and this completes the proof. \square

5. Application

In this section, we present an example of stochastic integral equation which satisfies hypothesis (H1)-(H6).

Example 5.1. Let us take

$$f, g : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, u : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R},$$

defined as

$$f(t, y, x) = \frac{1}{8} \sin\left(\frac{1}{1+t}\right) + y, g(t, y, x) = \frac{1}{8}, u(t, s, x) = \frac{1}{9}ts \sin(x).$$

So, Eq. (1.1) can be written as

$$x(t) = \frac{1}{64} \sin\left(\frac{1}{1+t}\right) + \frac{t}{72} \int_0^t s \sin(x(s)) dB(s). \quad (5.1)$$

It is obvious that the functions f, g are continuous and u is continuous function with the continuous first derivative and satisfy hypothesis (H1)-(H2) with $l = \frac{1}{8}$. Also, $a_1(t) = 1, a_2(t) = 0, b_1(t) = 0, b_2(t) = 0$ and therefore $k = \max\{a_1(t), a_2(t), b_1(t), b_2(t)\} = 1$. Moreover, it is clear that u satisfies the (H5) with $\alpha = 0$ and $\beta = \frac{1}{9}$. With these constants, we obtain that the inequality (H6) is correct. So, the Itô-Volterra integral equation (5.1) has at least one solution in Banach space $C[0, 1]$.

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References

- [1] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
- [2] F. Mirzaee and E. Hadadiyan, *Application of modified hat functions for solving nonlinear quadratic integral equations*, Iran. J. Numer. Anal. Opt. 6(2) (2016) 65–84.
- [3] L.N. Mishra and M. Sen, *On the concept of existence and local attractivity of solutions for some quadratic Volterra integral equation of fractional order*, Appl. Math. Comput. 285 (2016) 174–183.
- [4] E. Gokmen, G. Yuksel and M. Sezer, *A numerical approach for solving Volterra type functional integral equations with variable bounds and mixed delays*, J. Comput. Appl. Math. 311 (2017) 354–363.
- [5] J. Banas and K. Sadarangani, *Solutions of some functional-integral equations in Banach algebra*, Math. Comput. Model. 38 (2003) 245–250.
- [6] K. Maleknejad, K. Nouri and R. Mollapourasl, *Existence of solutions for some nonlinear integral equations*, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 2559–2564.
- [7] K. Maleknejad, R. Mollapourasl and K. Nouri, *Study on existence of solutions for some nonlinear functional-integral equations*, Nonlinear Anal. 69 (2008) 2582–2588.
- [8] S. Moradi, M. Mohammadi Anjedani and E. Analoei, *On existence and uniqueness of solutions of a nonlinear Volterra-Fredholm integral equation*, Int. J. Nonlinear Anal. Appl. 6(1) (2015) 62–68.

- [9] M. Kazemi and R. Ezzati, *Existence of solutions for some nonlinear Volterra integral equations via petryshyn's fixed point theorem*, Int. J. Nonlinear Anal. Appl. 9(1) (20018) 1–12.
- [10] C. Chifu and G. Petrusel, *Existence and uniqueness of the solution for a general system of operator equations in b - metric spaces endowed with a graph*, Int. J. Nonlinear Anal. Appl. 8(2) (2017) 263–276.
- [11] M. Rabbani and R. Arab, *Existence of some theorems to find solution of nonlinear integral equation and homotopy perturbation method to solve it*, Math. Sci. 11 (2017) 87–94.
- [12] R. Allahyari and A. Aghajani, *Some extensions of Darbos theorem and solutions of integral equations of Hammerstein type*, Int. J. Nonlinear Anal. Appl. 8(1) (2017) 337–351.
- [13] F. Mirzaee, N. Samadyar and S.F. Hosseini, *A new scheme for solving nonlinear Stratonovich Volterra integral equations via Bernoulli's approximation*, Appl. Anal. 96(13) (2017) 2163–2179.
- [14] F. Mirzaee and N. Samadyar, *Application of orthonormal Bernstein polynomials to construct a efficient scheme for solving fractional stochastic integro-differential equation*, Optik Int. J. Light Electron Opt. 132 (2017) 262–273.
- [15] F. Mirzaee and N. Samadyar, *Numerical solution of nonlinear stochastic Itô-Volterra integral equations driven by fractional Brownian motion*, Math. Meth. Appl. Sci. 41 (2018) 1410–1423.
- [16] K. Maleknejad, M. Khodabin and M. Rostami, *Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block pulse functions*, Math. Comput. Model. 55(3-4) (2012) 791–800.
- [17] F. Mirzaee and N. Samadyar, *Using radial basis functions to solve two dimensional linear stochastic integral equations on non-rectangular domains*, Eng. Anal. Bound. Elem. 92 (2018) 180–195.
- [18] F. Mirzaee and N. Samadyar, *On the numerical solution of fractional stochastic integro-differential equations via meshless discrete collocation method based on radial basis functions*, Eng. Anal. Bound. Elem. (2018) In Press.
- [19] M. Saffarzadeh, G.B. Loghmani and M. Heydari, *An iterative technique for the numerical solution of nonlinear stochastic Itô-Volterra integral equations*, J. Comput. Appl. Math. 333 (2018) 74–86.
- [20] Z. Taheri, S. Javadi and E. Babolian, *Numerical solution of stochastic fractional integro-differential equation by the spectral collocation method*, J. Comput. Appl. Math. 321 (2017) 336–347.
- [21] M. Asgari and M. Khodabin, *Computational method based on triangular operational matrices for solving nonlinear stochastic differential equations*, Int. J. Nonlinear Anal. Appl. 8(2) (2017) 169–179.
- [22] X. Zhang, *Euler schemes and large deviations for stochastic Volterra equations with singular kernels*, J. Differ. Equ. 244(9) (2008) 2226–2250.
- [23] C. Kuratowski, *Sur les espaces complets*, Fundam. Math. 15 (1930) 301–309.
- [24] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, Rend. Sem. Mat. Univ. Padova. 24 (1955) 84–92.
- [25] D. Guo, V. Lakshmikantham and X. Liu, *Nonlinear integral equations in abstract space*, Springer Science & Business Media, 2013.
- [26] N.G. Lloyd, *Measures of noncompactness in Banach space*, Bulletin of the London Mathematical Society, 13(6) (1981) 583–584.
- [27] F.C. Klebaner, *Introduction to stochastic calculus with applications*, World Scientific Publishing Company, 2012.