



# A new proof of Singer-Wermer theorem with some results on $\{g, h\}$ -derivations

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## Abstract

Singer and Wermer proved that if  $\mathcal{A}$  is a commutative Banach algebra and  $d : \mathcal{A} \rightarrow \mathcal{A}$  is a continuous derivation, then  $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$ , where  $\text{rad}(\mathcal{A})$  denotes the Jacobson radical of  $\mathcal{A}$ . In this article, we will establish a new proof of that. Moreover, we prove that every continuous Jordan derivation on a finite dimensional Banach algebra is identically zero under certain conditions. As another objective of this article, we study  $\{g, h\}$ -derivations on algebras. In this regard, we prove that if  $f$  is a  $\{g, h\}$ -derivation on a unital algebra, then  $f$ ,  $g$  and  $h$  are generalized derivations. In addition, we achieve some results concerning the automatic continuity of  $\{g, h\}$ -derivations on Banach algebras. In the last section of this article, we introduce the concept of a  $\{g, h\}$ -homomorphism and then we characterize it under certain conditions. Indeed, we prove that if  $\mathcal{A}$  is an algebra with the identity element  $e$  and  $f : \mathcal{A} \rightarrow \mathcal{A}$  is a  $\{g, h\}$ -homomorphism such that  $g(e)$  and  $h(e)$  are invertible elements of  $\mathcal{A}$ , then there exists a homomorphism  $\theta$  of  $\mathcal{A}$  such that  $f(ab) = f(a)\theta(b) = \theta(a)f(b)$ ,  $g(ab) = g(a)\theta(b) = \theta(a)g(b)$  and  $h(ab) = h(a)\theta(b) = \theta(a)h(b)$  for all  $a, b \in \mathcal{A}$ .

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## 1. Introduction and preliminaries

Throughout the paper,  $\mathcal{A}$  denotes an associative complex algebra. If an algebra is unital, then  $e$  stands for the identity element. We first introduce the basic notions and set the notations that play a fundamental role in what follows. An algebra  $\mathcal{A}$  is said to be a domain if  $\mathcal{A} \neq \{0\}$ , and

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$a = 0$  or  $b = 0$ , whenever  $ab = 0$  in  $\mathcal{A}$ . A commutative algebra which is also a domain is called an integral domain. As usual, the set of all primitive ideals of  $\mathcal{A}$  is denoted by  $\Pi(\mathcal{A})$ . The Jacobson radical of an algebra  $\mathcal{A}$  is the intersection of all primitive ideals of  $\mathcal{A}$  which is denoted by  $rad(\mathcal{A})$ . Indeed,  $rad(\mathcal{A}) = \bigcap_{\mathcal{P} \in \Pi(\mathcal{A})} \mathcal{P}$ . The algebra  $\mathcal{A}$  is called semisimple if  $rad(\mathcal{A}) = \{0\}$ . A nonzero linear functional  $\varphi$  on an algebra  $\mathcal{A}$  is called a *character* if  $\varphi(ab) = \varphi(a)\varphi(b)$  for every  $a, b \in \mathcal{A}$ . By  $\Phi_{\mathcal{A}}$  we denote the set of all characters on  $\mathcal{A}$ . According to [4, Proposition 1.3.37], we know that  $\ker \varphi$  is a maximal ideal of  $\mathcal{A}$  for any  $\varphi \in \Phi_{\mathcal{A}}$ . Recall that a linear mapping  $d : \mathcal{A} \rightarrow \mathcal{A}$  is called a derivation (resp. Jordan derivation) if  $d(ab) = d(a)b + ad(b)$  (resp.  $d(a^2) = d(a)a + ad(a)$ ) for all  $a, b \in \mathcal{A}$ .

This article contains four sections. The main results will be presented in Sections 2, 3 and 4. In the second section, we establish a new proof of Singer-Wermer Theorem. As a pioneering work, Singer and Wermer [17] achieved a fundamental result which started investigation into the range of derivations on Banach algebras. The so-called Singer-Wermer theorem states that any continuous derivation on a commutative Banach algebra maps the algebra into its Jacobson radical. They proved their theorem by using some complex analysis techniques. During the last years, there has been ongoing interest concerning the image of derivations and Jordan derivations. So far, many authors have studied the image of derivations, for instance, see [1, 6, 7, 9, 10, 11, 12, 13, 14, 19, 20]. Hosseini [7] proved that every rank one cubic derivation on a unital integral domain is identically zero. A mapping  $d : \mathcal{A} \rightarrow \mathcal{A}$  is called a cubic derivation if  $d(ab) = d(a)b^3 + a^3d(b)$  for all  $a, b \in \mathcal{A}$ . In this study, we obtain the same result for derivations on any integral domain, with or without identity element. Using this theorem, we establish a new proof of Singer-Wermer Theorem. As another objective, we present some results on the image of Jordan derivations. For instance, we prove the following result:

Let  $\mathcal{A}$  be an algebra and let  $\mathcal{I}$  be a semiprime ideal of  $\mathcal{A}$ . Suppose that  $d : \mathcal{A} \rightarrow \mathcal{A}$  is a Jordan derivation such that  $d(\mathcal{I}) \subseteq \mathcal{I}$ . If  $\dim\{d(a) + \mathcal{I} \mid a \in \mathcal{A}\} \leq 1$ , then  $d(\mathcal{A}) \subseteq \mathcal{I}$ .

We also present some consequences of this result. According to [4, Proposition 1.4.36], if  $\mathcal{A}$  is a commutative algebra, then every primitive ideal of  $\mathcal{A}$  has codimension 1. In this paper, we prove the converse of this result for semisimple Banach algebras. Indeed, we show that if  $\mathcal{A}$  is a semisimple Banach algebra, then  $\mathcal{A}$  is commutative if and only if each primitive ideal of  $\mathcal{A}$  is of codimension 1. Furthermore, under certain conditions, we make clear the status of continuous Jordan derivations on unital finite dimensional Banach algebras as follows. Let  $n$  be a positive integer and let  $\mathcal{A}$  be an  $n$ -dimensional unital Banach algebra with the basis  $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ . Suppose that for every integer  $k$ ,  $1 \leq k \leq n$ , an ideal  $\mathcal{I}_k$  generated by  $\mathcal{B} - \{b_k\}$  is a proper subset of  $\mathcal{A}$ . Then every continuous Jordan derivation on  $\mathcal{A}$  is identically zero.

In the third section of the current study, we present some results on the automatic continuity of  $\{g, h\}$ -derivations. In 2016, the concept of a  $\{g, h\}$ -derivation was introduced by Brešar [1]. Let  $\mathcal{A}$  be an algebra over a field  $\mathbb{F}$  with  $char(\mathbb{F}) \neq 2$ , and let  $f, g, h : \mathcal{A} \rightarrow \mathcal{A}$  be linear maps. We say that  $f$  is a  $\{g, h\}$ -derivation if  $f(ab) = g(a)b + ah(b) = h(a)b + ag(b)$  for all  $a, b \in \mathcal{A}$ , and we say that  $f$  is a Jordan  $\{g, h\}$ -derivation if  $f(a \circ b) = g(a) \circ b + a \circ h(b)$  for all  $a, b \in \mathcal{A}$ , where  $a \circ b = ab + ba$ . We call  $a \circ b$  the Jordan product of  $a$  and  $b$ . Obviously, every  $\{g, h\}$ -derivation is a Jordan  $\{g, h\}$ -derivation, but the converse is in general not true (see [1, Example 2.1]). As an interesting result, Brešar [1, Theorem 4.3] proved that every Jordan  $\{g, h\}$ -derivation of a semiprime algebra  $\mathcal{A}$  is a  $\{g, h\}$ -derivation. He also showed that every Jordan  $\{g, h\}$ -derivation of the tensor product of a semiprime and a commutative algebra is a  $\{g, h\}$ -derivation. In this section, we prove that if  $f$  is a  $\{g, h\}$ -derivation on a unital algebra  $\mathcal{A}$ , then  $f, g, h$  are generalized derivations of  $\mathcal{A}$ . Another objective of this section is concerning the automatic continuity of  $\{g, h\}$ -derivations on Banach algebras. It is noteworthy that the theory of automatic continuity of derivations has a fairly long history. Results on automatic continuity of linear operators defined on Banach algebras

comprise a fruitful area of research intensively developed during the last sixty years. The references [4, 5] review most of the main achievements obtained during the last sixty years. In this article, we achieve the following result on the automatic continuity of  $\{g, h\}$ -derivations:

Let  $\mathcal{A}$  be a Banach algebra and let  $f : \mathcal{A} \rightarrow \mathcal{A}$  be a  $\{g, h\}$ -derivation. Then we have the following statements:

- i) Let  $\text{ann}(\mathcal{A}) = \{0\}$ . If  $f$  is continuous, then both  $g$  and  $h$  are continuous.
- ii) If  $\mathcal{A}$  has the Cohen factorization property and at least one of  $g$  or  $h$  is continuous, then  $f$  is continuous.
- iii) If  $\mathcal{A}$  is unital, then the continuity of one of these mappings (, i.e.  $f$  or  $g$  or  $h$ ) forces the continuity of the other two.

Now, we express the fourth section of this article. In this section, we introduce the concept of a  $\{g, h\}$ -homomorphism. Let  $f, g, h : \mathcal{A} \rightarrow \mathcal{A}$  be linear maps. We say that  $f$  is a  $\{g, h\}$ -homomorphism if  $f(ab) = g(a)h(b) = h(a)g(b)$  for all  $a, b \in \mathcal{A}$ . If a  $\{g, h\}$ -homomorphism is onto and one-to-one, then it is called a  $\{g, h\}$ -automorphism. We prove that if  $f$  is a  $\{g, h\}$ -homomorphism on a unital algebra  $\mathcal{A}$  such that  $g(\mathbf{e})$  and  $h(\mathbf{e})$  are invertible elements of  $\mathcal{A}$ , then there exists a homomorphism  $\theta$  on  $\mathcal{A}$  such that

- $f(ab) = f(a)\theta(b) = \theta(a)f(b)$ ;
- $g(ab) = g(a)\theta(b) = \theta(a)g(b)$ ;
- $h(ab) = h(a)\theta(b) = \theta(a)h(b)$ .

for all  $a, b \in \mathcal{A}$ .

## 2. A new proof of Singer-Wermer Theorem

Before expressing the first theorem of this section, we present the following brief note. An algebra  $\mathcal{A}$  can always be embedded into an algebra with identity as follows. Let  $\mathfrak{A}$  denote the set of all pairs  $(\lambda, a)$ ,  $a \in \mathcal{A}$ ,  $\lambda \in \mathbb{C}$ , that is,  $\mathfrak{A} = \mathbb{C} \oplus \mathcal{A}$ . Then  $\mathfrak{A}$  becomes an algebra if the linear space operations and multiplication are defined by  $(\lambda, a) + (\mu, b) = (\lambda + \mu, a + b)$ ,  $\mu(\lambda, a) = (\mu\lambda, \mu a)$  and  $(\lambda, a)(\mu, b) = (\lambda\mu, ab + \lambda b + \mu a)$  for all  $a, b \in \mathcal{A}$  and  $\lambda, \mu \in \mathbb{C}$ . A simple calculation shows that the element  $\mathbf{e} = (1, 0) \in \mathfrak{A}$  is the identity element of  $\mathfrak{A}$ . Moreover, the mapping  $a \mapsto (0, a)$  is an algebra isomorphism of  $\mathcal{A}$  onto an ideal of codimension one in  $\mathfrak{A}$ . Obviously,  $\mathfrak{A}$  is commutative if and only if  $\mathcal{A}$  is commutative. Now, suppose that  $\mathcal{A}$  is a normed algebra. We introduce a norm on  $\mathfrak{A}$  by  $\|(\lambda, a)\| = |\lambda| + \|a\|$  for all  $a \in \mathcal{A}$  and  $\lambda \in \mathbb{C}$ . It is clear that this norm turns  $\mathfrak{A}$  into a normed algebra and that  $\mathfrak{A}$  is a Banach algebra if and only if so is  $\mathcal{A}$ . Some authors call  $\mathfrak{A}$  the unitization of  $\mathcal{A}$ . Now, let us start with the following theorem which has been motivated by [7].

**Theorem 2.1.** *Let  $\mathcal{A}$  be an integral domain and let  $d : \mathcal{A} \rightarrow \mathcal{A}$  be a derivation. If the rank of  $d$  is at most one, i.e  $\dim(d(\mathcal{A})) \leq 1$ , then  $d$  is identically zero.*

**Proof .** If  $\dim(d(\mathcal{A})) = 0$ , then there is nothing to be proved. Suppose that  $\dim(d(\mathcal{A})) = 1$ . We are going to show that  $d$  is identically zero. First, we introduce a mapping  $D : \mathfrak{A} \rightarrow \mathfrak{A}$  by  $D(\alpha, a) = (0, d(a))$  for all  $(\alpha, a) \in \mathfrak{A}$ , where  $\mathfrak{A}$  is the unitization of  $\mathcal{A}$ . Clearly,  $D$  is a derivation. Since  $\dim(d(\mathcal{A})) = 1$ , there exist a nonzero element  $x$  of  $\mathcal{A}$  and a functional  $\Psi : \mathcal{A} \rightarrow \mathbb{C}$  such that  $d(a) = \Psi(a)x$  for all  $a \in \mathcal{A}$ . So, we see that  $D(\alpha, a) = (0, d(a)) = (0, \Psi(a)x) = \Psi(a)(0, x)$ , which means that  $D$  is also a rank one derivation. According to [4, Corollary 1.3.55],  $\mathfrak{A}$  is an integral domain. So,  $D$  is a rank one derivation on the integral domain  $\mathfrak{A}$ . Similar to the reasoning done in

[7], we prove that  $D$  is identically zero. We have to find a contradiction. To obtain a contradiction, assume that there exists a nonzero element  $(\alpha_0, a_0) \in \mathfrak{A}$  such that  $D(\alpha_0, a_0) \neq (0, 0)$ . Evidently,  $\Psi(a_0) \neq 0$ . Assume that  $D(0, x) = (0, 0)$ . So, we have  $\Psi(x)(0, x) = 0$  which implies that  $\Psi(x) = 0$ . We have the following expressions:

$$\begin{aligned} \Psi(a_0^2)x &= d(a_0^2) \\ &= d(a_0)a_0 + a_0d(a_0) \\ &= 2a_0d(a_0) \\ &= 2\Psi(a_0)a_0x, \end{aligned}$$

which means that  $\Psi(a_0^2)x = 2\Psi(a_0)a_0x$ . Since  $\Psi(x) = 0$ , we see that  $d(x) = 0$ . Therefore, we have

$$\begin{aligned} 0 &= \Psi(a_0^2)d(x) = d(\Psi(a_0^2)x) \\ &= d(2\Psi(a_0)a_0x) \\ &= 2\Psi(a_0)d(a_0)x, \end{aligned}$$

which means that  $2\Psi(a_0)d(a_0)x = 0$ . This equation along with the assumption that  $\mathcal{A}$  is a domain imply that  $x = 0$  or  $d(a_0) = 0$ , a contradiction. Now, suppose that  $D(0, x) \neq (0, 0)$ . Clearly,  $\Psi(x) \neq 0$ , too. Note that

$$\Psi(x^2)x = d(x^2) = 2xd(x) = 2\Psi(x)x^2.$$

Hence,

$$\begin{aligned} (0, 0) &= (0, d(\Psi(x^2)x - 2\Psi(x)x^2)) \\ &= (0, \Psi(x^2)d(x) - 4\Psi(x)xd(x)) \\ &= (0, \Psi(x^2)d(x)) + (0, -4\Psi(x)xd(x)) \\ &= \Psi(x^2)(0, d(x)) - 4\Psi(x)(0, x)(0, d(x)) \\ &= (\Psi(x^2)(1, 0) - 4\Psi(x)(0, x))(0, d(x)) \\ &= (\Psi(x^2), -4\Psi(x)x)(0, d(x)) \end{aligned}$$

Since  $\mathfrak{A}$  is a domain,  $(0, d(x)) = (0, 0)$ , a contradiction or  $(\Psi(x^2), -4\Psi(x)x) = (0, 0)$ . This means that both  $\Psi(x^2)$  and  $4\Psi(x)x$  are zero. But the equation  $4\Psi(x)x = 0$  is not valid, since both  $x$  and  $\Psi(x)$  are nonzero. As observed above, both cases  $D(0, x) = (0, 0)$  and  $D(0, x) \neq (0, 0)$  lead to a contradiction. This contradiction shows that there is no element  $(\alpha_0, a_0)$  of  $\mathfrak{A}$  such that  $D(\alpha_0, a_0) \neq (0, 0)$ . Thereby,  $D$  is identically zero and then so is  $d$ .  $\square$

**Remark 2.2.** Let  $\mathcal{A}$  be an algebra and  $a, b \in \mathcal{A}$ . The quasi-product of  $a$  and  $b$  is denoted by  $a \diamond b$  and is  $a \diamond b = a + b - ab$ . An element  $a$  of  $\mathcal{A}$  is left (resp. right) quasi-invertible if there exists  $b \in \mathcal{A}$  such that  $b \diamond a = 0$  (resp.  $a \diamond b = 0$ ). The element  $a$  is quasi-invertible if it is both left and right quasi-invertible, and a subset of  $\mathcal{A}$  is left quasi-invertible (resp. right quasi-invertible) if each of its elements is left quasi-invertible (resp. right quasi-invertible). The quasi-product is an associative operation with identity  $0$ . Suppose that  $a \diamond b = c \diamond a = 0$ . Then clearly  $b = c$ , and so, if  $a$  is quasi-invertible, there is a unique element  $b \in \mathcal{A}$  such that  $a \diamond b = b \diamond a = 0$ . In this case,  $b$  is called the quasi-inverse of  $a$ . We write  $q - Inv \mathcal{A}$  for the set of quasi-invertible elements of  $\mathcal{A}$ . A topological algebra  $\mathcal{A}$  is called a  $Q - algebra$  if  $q - Inv \mathcal{A}$  is an open subset of  $\mathcal{A}$ . Banach algebras are the main

examples of  $Q$  – algebras. Some incomplete normed algebras are  $Q$  – algebras. For example, the algebra  $(C_{00}(X), |x|)$  is a  $Q$  – algebra for each completely regular space  $X$ , but it is only complete for special spaces  $X$ . Indeed, it is clear that a normed algebra  $(\mathcal{A}, \|\cdot\|)$  is a  $Q$  – algebra if and only if  $\sum_{n=1}^{\infty} a^n$  converges in  $\mathcal{A}$  for each  $a \in \mathcal{A}$  with  $\|a\| < 1$ . It follows from Theorem 2.2.28 (iii) of [4] that every primitive ideal of a  $Q$  – algebra  $\mathcal{A}$  is closed. Therefore, Proposition 2.7.22 of [4] and Lemma 3.2 of [16] are also true in the case that  $\mathcal{A}$  is just a normed  $Q$  – algebra. For more material in this regard, see pages 29, 177 and 178 of [4].

In the following theorem, we do not need both of these assumptions that  $d$  is a continuous derivation and  $\mathcal{A}$  is a Banach space.

**Theorem 2.3.** *Let  $\mathcal{A}$  be a normed  $Q$ -algebra and let  $d : \mathcal{A} \rightarrow \mathcal{A}$  be a derivation. Suppose that there exists a positive real constant  $C$  such that  $\|Q_{ker\varphi}d^n\| \leq C^n$  for all  $n \in \mathbb{N}$ , where  $\varphi$  is an arbitrary character on  $\mathcal{A}$  and  $Q_{ker\varphi} : \mathcal{A} \rightarrow \frac{\mathcal{A}}{ker\varphi}$  is the quotient map. Then  $d(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker\varphi$ . In particular, if  $\mathcal{A}$  is commutative, then  $d(\mathcal{A}) \subseteq rad(\mathcal{A})$ .*

**Proof .** Let  $\varphi$  be an arbitrary character on  $\mathcal{A}$ . It follows from [4, Proposition 1.3.37 (i)] that  $ker\varphi$  is a maximal modular ideal of codimension one in  $\mathcal{A}$ . Moreover, according to [4, Proposition 1.4.34 (iv)]  $ker\varphi$  is a primitive ideal of  $\mathcal{A}$ . Since  $\mathcal{A}$  is a normed  $Q$ -algebra, it follows from [4, Theorem 2.2.28 (i)] that  $ker\varphi$  is a closed ideal in  $\mathcal{A}$ . Since all the assumptions of [4, Proposition 2.7.22] are fulfilled, one can conclude that  $d(ker\varphi) \subseteq ker\varphi$ . So,  $D : \frac{\mathcal{A}}{ker\varphi} \rightarrow \frac{\mathcal{A}}{ker\varphi}$  introduced by  $D(a + ker\varphi) = d(a) + ker\varphi$  is a derivation. It is evident that the quotient algebra  $\frac{\mathcal{A}}{ker\varphi}$  is an integral domain. Using Theorem 2.1, we deduce that the derivation  $D$  is identically zero and consequently,  $d(\mathcal{A}) \subseteq ker(\varphi)$ . Since we are assuming that  $\varphi$  is an arbitrary character on  $\mathcal{A}$ ,  $d(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker\varphi$ . We know that if  $\mathcal{A}$  is a commutative algebra, then  $rad(\mathcal{A}) = \bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker\varphi$  (see Proposition 1.3.37(i) and Proposition 1.4.36 of [4]). This proves the theorem.  $\square$

In the following, we present a new proof of Singer- Wermer theorem.

**Theorem 2.4.** *(Singer-Wermer theorem) Let  $\mathcal{A}$  be a commutative Banach algebra and let  $d : \mathcal{A} \rightarrow \mathcal{A}$  be a continuous derivation. Then  $d(\mathcal{A}) \subseteq rad(\mathcal{A})$ .*

**Proof . The first proof:** Let  $\mathcal{P}$  be an arbitrary primitive ideal of  $\mathcal{A}$ . According to [4, Proposition 1.4.36 (c)],  $\frac{\mathcal{A}}{\mathcal{P}}$  is a field and it is a well-known fact in the abstract algebra that  $\dim(\frac{\mathcal{A}}{\mathcal{P}}) = 1$ . Since  $d$  is a continuous derivation, it follows from [16, Lemma 3.2] that  $d(\mathcal{P}) \subseteq \mathcal{P}$  for any primitive ideal  $\mathcal{P}$  in  $\mathcal{A}$ . Hence,  $D : \frac{\mathcal{A}}{\mathcal{P}} \rightarrow \frac{\mathcal{A}}{\mathcal{P}}$  defined by  $D(a + \mathcal{P}) = d(a) + \mathcal{P}$  is a derivation. Using Theorem 2.1, we conclude that  $D$  is identically zero and consequently,  $d(\mathcal{A}) \subseteq \mathcal{P}$ . Since  $\mathcal{P}$  is an arbitrary primitive ideal of  $\mathcal{A}$ ,  $d(\mathcal{A}) \subseteq rad(\mathcal{A})$ . Thereby, our proof is complete.

**The second proof:** Let  $\varphi$  be an arbitrary character on  $\mathcal{A}$ . It follows from [4, Proposition 1.3.37 (i)] that  $ker\varphi$  is a maximal modular ideal of codimension one in  $\mathcal{A}$ . Clearly, the quotient algebra  $\frac{\mathcal{A}}{ker\varphi}$  is a nonzero, finite-dimensional complex algebra which is also a domain. According to [4, Proposition 1.3.56], the quotient algebra  $\frac{\mathcal{A}}{ker\varphi}$  has an identity  $\mathbf{e}$  and  $\frac{\mathcal{A}}{ker\varphi} = \mathbb{C}\mathbf{e}$ . Moreover, according to [4, Proposition 1.4.34 (iv)],  $ker\varphi$  is a primitive ideal of  $\mathcal{A}$  and using [16, Lemma 3.2], we obtain that  $d(ker\varphi) \subseteq ker\varphi$ . Hence, the mapping  $D : \frac{\mathcal{A}}{ker\varphi} \rightarrow \frac{\mathcal{A}}{ker\varphi}$  defined by  $D(a + ker\varphi) = d(a) + ker\varphi$  is a derivation. From the fact that  $\frac{\mathcal{A}}{ker\varphi} = \mathbb{C}\mathbf{e}$ , we infer that the derivation  $D$  is identically zero, and consequently,  $d(\mathcal{A}) \subseteq ker\varphi$ . Since we are assuming that  $\varphi$  is an arbitrary character on  $\mathcal{A}$ ,  $d(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker\varphi$ . We know that if  $\mathcal{A}$  is a commutative algebra, then  $\bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker\varphi = rad(\mathcal{A})$  and

consequently  $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$ .  $\square$

We denote the set of all invertible elements of a unital algebra  $\mathcal{A}$  by  $\text{Inv}(\mathcal{A})$ . Let  $\mathcal{A}$  be a unital algebra and let  $a \in \mathcal{A}$ . The spectrum of  $a$  is  $\mathfrak{S}(a) = \{\lambda \in \mathbb{C} : \lambda e - a \notin \text{Inv}(\mathcal{A})\}$ , where  $e$  is the identity element of  $\mathcal{A}$ , and the spectral radius of  $a$  is  $\nu(a) = \sup\{|\lambda| : \lambda \in \mathfrak{S}(a)\}$ . An element  $a \in \mathcal{A}$  is called quasi-nilpotent if  $\nu(a) = 0$ , i.e.  $\mathfrak{S}(a) = \{0\}$  or  $\mathfrak{S}(a) = \emptyset$ ; the set of quasi-nilpotents of  $\mathcal{A}$  is denoted by  $Q(\mathcal{A})$ . In the case that  $\mathcal{A}$  is a Banach algebra, then  $a \in Q(\mathcal{A})$  if and only if  $\lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = 0$ . For more material about  $Q(\mathcal{A})$ , see, e.g. [4, 5].

**Remark 2.5.** Let  $\mathcal{P}$  be an arbitrary primitive ideal of  $\mathcal{A}$ . We may think that  $d(\mathcal{P}) \subseteq \mathcal{P}$  is true whenever  $d$  is a continuous (Jordan) derivation (e.g. see [16, Lemma 3.2]). In the following, we give a discontinuous (equivalently, unbounded) derivation  $D$  on a Banach algebra  $\mathfrak{B}$  leaving every primitive ideal of  $\mathfrak{B}$  invariant, i.e.  $D(\mathcal{P}) \subseteq \mathcal{P}$  for every primitive ideal  $\mathcal{P}$  of  $\mathfrak{B}$ . Let  $\mathcal{A}$  be a Banach algebra. Consider  $\mathfrak{B} = \mathbb{C} \oplus \mathcal{A}$  as an algebra with pointwise addition, scalar multiplication and the following product:

$$(\alpha, a) \cdot (\beta, b) = (\alpha\beta, \alpha b + \beta a),$$

for all  $a, b \in \mathcal{A}$  and  $\alpha, \beta \in \mathbb{C}$ . The algebra  $\mathfrak{B}$  with the norm  $\|(\alpha, a)\| = |\alpha| + \|a\|$  is a Banach algebra. Clearly,  $\mathfrak{B}$  is unital and commutative. Using [4, Proposition 1.4.36] and [5, Propositions 2.2.3, 3.2.1], we have  $\bigcap_{\varphi \in \Phi_{\mathfrak{B}}} \ker \varphi = \text{rad}(\mathfrak{B}) = Q(\mathfrak{B})$ . We show that  $\Phi_{\mathfrak{B}}$ , the set of all characters on  $\mathfrak{B}$ , contains only one element. For an arbitrary element  $(\alpha, a) \in Q(\mathfrak{B})$ , we have  $0 = \lim_{n \rightarrow \infty} \|(\alpha, a)^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \sqrt[n]{|\alpha|^n + n|\alpha|^{n-1}\|a\|}$ . This equation shows that  $\alpha = 0$ . Thus, every element of  $Q(\mathfrak{B})$  has the form  $(0, a)$  for all  $a \in \mathcal{A}$ , i.e.  $Q(\mathfrak{B}) = \{0\} \oplus \mathcal{A}$ . We introduce  $\varphi_1 : \mathfrak{B} \rightarrow \mathbb{C}$  by  $\varphi_1(\alpha, a) = \alpha$ . It is observed that  $\ker \varphi_1 = \{0\} \oplus \mathcal{A} = Q(\mathfrak{B}) = \text{rad}(\mathfrak{B}) = \bigcap_{\varphi \in \Phi_{\mathfrak{B}}} \ker \varphi$ . Hence,  $\ker \varphi_1 \subseteq \ker \varphi$  for all  $\varphi \in \Phi_{\mathfrak{B}}$ . This conclusion along with the fact that  $\ker \varphi$  is a maximal ideal for each  $\varphi \in \Phi_{\mathfrak{B}}$  imply that  $\varphi_1 = \varphi$  for all  $\varphi \in \Phi_{\mathfrak{B}}$ . It means that  $\Phi_{\mathfrak{B}}$  contains only the character  $\varphi_1$ . According to [4, Proposition 1.4.36] and [5, Proposition 3.2.1], the unital, commutative Banach algebra  $\mathfrak{B}$  has the unique primitive ideal  $\mathcal{P} = \ker \varphi_1 = \{0\} \oplus \mathcal{A}$ . Suppose that  $T : \mathcal{A} \rightarrow \mathcal{A}$  is an unbounded linear map. We define  $D : \mathfrak{B} \rightarrow \mathfrak{B}$  by  $D(\alpha, a) = (0, T(a))$ . We have

$$\begin{aligned} D((\alpha, a)(\beta, b)) &= D(\alpha\beta, \alpha b + \beta a) \\ &= (0, \alpha T(b) + \beta T(a)) \\ &= (\alpha, a)(0, T(b)) + (\beta, b)(0, T(a)) \\ &= (\alpha, a)D(\beta, b) + D(\alpha, a)(\beta, b), \end{aligned}$$

which means that  $D$  is a discontinuous derivation on  $\mathfrak{B}$ . Note that  $D(\mathcal{P}) = \{D(0, a) \mid a \in \mathcal{A}\} = \{(0, T(a)) \mid a \in \mathcal{A}\} \subseteq \{0\} \oplus \mathcal{A} = \mathcal{P}$ . We see that  $D(\mathcal{P}) \subseteq \mathcal{P}$ , while  $D$  is a discontinuous derivation on  $\mathfrak{B}$ .

Below, we establish a theorem on the image of Jordan derivations which has been motivated by [8, 15].

**Theorem 2.6.** Let  $\mathcal{A}$  be an algebra and  $\mathcal{I}$  be a semiprime ideal of  $\mathcal{A}$ . Suppose that  $d : \mathcal{A} \rightarrow \mathcal{A}$  is a Jordan derivation such that  $d(\mathcal{I}) \subseteq \mathcal{I}$ . If  $\dim\{d(a) + \mathcal{I} \mid a \in \mathcal{A}\} \leq 1$ , then  $d(\mathcal{A}) \subseteq \mathcal{I}$ .

**Proof.** If  $\dim\{d(a) + \mathcal{I} \mid a \in \mathcal{A}\} = 0$ , then obviously  $d(\mathcal{A}) \subseteq \mathcal{I}$ . Now, suppose that  $\dim\{d(a) + \mathcal{I} \mid a \in \mathcal{A}\} = 1$ . Hence, there exists an element  $x$  of  $\mathcal{A}$  such that  $\{d(a) + \mathcal{I} \mid a \in \mathcal{A}\} = \{\alpha(x + \mathcal{I}) \mid \alpha \in \mathbb{C}\} =$



$\{\alpha x + \mathcal{I} \mid \alpha \in \mathbb{C}\}$ . It is evident that  $x \notin \mathcal{I}$ . For convenience, we denote  $a + \mathcal{I}$  by  $\widehat{a}$  for any  $a \in \mathcal{A}$ . We define  $D : \frac{\mathcal{A}}{\mathcal{I}} \rightarrow \frac{\mathcal{A}}{\mathcal{I}}$  by  $D(a + \mathcal{I}) = d(a) + \mathcal{I}$  for all  $a \in \mathcal{A}$ , i.e.  $D(\widehat{a}) = \widehat{d(a)}$ . Clearly,  $D$  is linear. We show that  $D$  is well-defined. Suppose that  $\widehat{a} = \widehat{b}$ . It implies that  $a - b \in \mathcal{I}$  and so  $d(a - b) \in \mathcal{I}$ , since  $d(\mathcal{I}) \subseteq \mathcal{I}$ . Consequently,  $D(\widehat{a}) = D(\widehat{b})$ . Moreover, we have

$$\begin{aligned} D(\widehat{a^2}) &= D(\widehat{a^2}) = \widehat{d(a^2)} = d(a^2) + \mathcal{I} \\ &= d(a)a + ad(a) + \mathcal{I} \\ &= (d(a) + \mathcal{I})(a + \mathcal{I}) + (a + \mathcal{I})(d(a) + \mathcal{I}) \\ &= D(\widehat{a})\widehat{a} + \widehat{a}D(\widehat{a}), \end{aligned}$$

which means that  $D$  is a Jordan derivation. Supposed that  $d(\mathcal{A}) \not\subseteq \mathcal{I}$ . So, there exists an element  $a_0$  in  $\mathcal{A}$  such that  $d(a_0) \notin \mathcal{I}$ . Thus,  $D(\widehat{a_0}) = d(a_0) + \mathcal{I} \neq \mathcal{I}$ . Since  $\dim\{D(\widehat{a}) \mid a \in \mathcal{A}\} = 1$ , we can consider the functional  $\psi : \frac{\mathcal{A}}{\mathcal{I}} \rightarrow \mathbb{C}$  such that  $D(\widehat{a}) = \psi(\widehat{a})\widehat{x}$  for all  $a \in \mathcal{A}$ . Since  $\psi(\widehat{a_0})\widehat{x} = D(\widehat{a_0}) \neq \mathcal{I}$ ,  $\psi(\widehat{a_0}) \neq 0$ . Having put  $\widehat{b_0} = \frac{1}{\psi(\widehat{a_0})}\widehat{a_0}$ , we have  $D(\widehat{b_0}) = D(\frac{1}{\psi(\widehat{a_0})}\widehat{a_0}) = \frac{1}{\psi(\widehat{a_0})}\psi(\widehat{a_0})\widehat{x} = \widehat{x}$  and this implies that  $\psi(\widehat{b_0}) = 1$ . First, we show that  $\widehat{a}\widehat{x} + \widehat{x}\widehat{a}$  is a scalar multiple of  $\widehat{x}$  for any  $a$  in  $\mathcal{A}$ . For an arbitrary element  $a$  in  $\mathcal{A}$ , we have

$$D(\widehat{a^2}) = \psi(\widehat{a^2})\widehat{x}. \tag{2.1}$$

Using the fact that  $D$  is a Jordan derivation, we have  $D(\widehat{a}\widehat{b} + \widehat{b}\widehat{a}) = D(\widehat{a})\widehat{b} + \widehat{a}D(\widehat{b}) + D(\widehat{b})\widehat{a} + \widehat{b}D(\widehat{a})$  for all  $a, b \in \mathcal{A}$ . Since  $D$  is a Jordan derivation and  $\dim\{d(a) + \mathcal{I} \mid a \in \mathcal{A}\} = 1$ , we have

$$D(\widehat{a^2}) = D(\widehat{a})\widehat{a} + \widehat{a}D(\widehat{a}) = \psi(\widehat{a})\widehat{x}\widehat{a} + \widehat{a}\psi(\widehat{a})\widehat{x} = \psi(\widehat{a})(\widehat{x}\widehat{a} + \widehat{a}\widehat{x}). \tag{2.2}$$

Comparing Equations (2.1) and (2.2), we find that  $\psi(\widehat{a^2})\widehat{x} = \psi(\widehat{a})(\widehat{a}\widehat{x} + \widehat{x}\widehat{a})$ . If  $\psi(\widehat{a}) \neq 0$ , then  $\widehat{a}\widehat{x} + \widehat{x}\widehat{a} = \frac{\psi(\widehat{a^2})}{\psi(\widehat{a})}\widehat{x}$  and this shows that  $\widehat{a}\widehat{x} + \widehat{x}\widehat{a}$  is a scalar multiple of  $\widehat{x}$ . Now, suppose that  $\psi(\widehat{a}) = 0$ . Then

$$\begin{aligned} \psi(\widehat{a}\widehat{b_0} + \widehat{b_0}\widehat{a})\widehat{x} &= D(\widehat{a}\widehat{b_0} + \widehat{b_0}\widehat{a}) \\ &= D(\widehat{a})\widehat{b_0} + \widehat{a}D(\widehat{b_0}) + D(\widehat{b_0})\widehat{a} + \widehat{b_0}D(\widehat{a}) \\ &= \psi(\widehat{a})\widehat{x}\widehat{b_0} + \widehat{a}\psi(\widehat{b_0})\widehat{x} + \psi(\widehat{b_0})\widehat{x}\widehat{a} + \widehat{b_0}\psi(\widehat{a})\widehat{x} \\ &= \widehat{a}\widehat{x} + \widehat{x}\widehat{a}. \end{aligned}$$

Therefore,  $\widehat{a}\widehat{x} + \widehat{x}\widehat{a}$  is a scalar multiple of  $\widehat{x}$  for any  $a$  in  $\mathcal{A}$ . Our next task is to show that  $\widehat{x^2} = 0$ . Note that

$$\begin{aligned} \psi(\widehat{b_0^2})\widehat{x} &= D(\widehat{b_0^2}) \\ &= D(\widehat{b_0})\widehat{b_0} + \widehat{b_0}D(\widehat{b_0}) \\ &= \psi(\widehat{b_0})\widehat{x}\widehat{b_0} + \widehat{b_0}\psi(\widehat{b_0})\widehat{x} \\ &= \widehat{x}\widehat{b_0} + \widehat{b_0}\widehat{x}. \end{aligned}$$

Hence,  $D(\widehat{b_0}\widehat{x} + \widehat{x}\widehat{b_0}) = D(\psi(\widehat{b_0^2})\widehat{x}) = \psi(\widehat{b_0^2})D(\widehat{x}) = \psi(\widehat{b_0^2})\psi(\widehat{x})\widehat{x}$ . In this case, if we suppose that  $\psi(\widehat{x}) = 0$ , then  $D(\widehat{b_0}\widehat{x} + \widehat{x}\widehat{b_0}) = 0$  and so we have the following expressions:

$$\begin{aligned} 0 &= \psi(\widehat{b_0^2})\psi(\widehat{x})\widehat{x} = \psi(\widehat{b_0^2})D(\widehat{x}) = D(\widehat{b_0}\widehat{x} + \widehat{x}\widehat{b_0}) \\ &= D(\widehat{b_0})\widehat{x} + \widehat{b_0}D(\widehat{x}) + D(\widehat{x})\widehat{b_0} + \widehat{x}D(\widehat{b_0}) \\ &= \psi(\widehat{b_0})\widehat{x^2} + 0 + 0 + \psi(\widehat{b_0})\widehat{x^2} \\ &= 2\widehat{x^2}, \end{aligned}$$

which implies that  $\widehat{x}^2 = 0$ . Now, suppose  $\psi(\widehat{x}) \neq 0$ . Thus,

$$\psi(\widehat{x}^2)\widehat{x} = D(\widehat{x}^2) = D(\widehat{x})\widehat{x} + \widehat{x}D(\widehat{x}) = \psi(\widehat{x})\widehat{x}^2 + \psi(\widehat{x})\widehat{x}^2 = 2\psi(\widehat{x})\widehat{x}^2,$$

which means that

$$\psi(\widehat{x}^2)\widehat{x} = 2\psi(\widehat{x})\widehat{x}^2. \tag{2.3}$$

If  $\psi(\widehat{x}^2) = 0$ , then it follows from Equation (2.3) that  $\widehat{x}^2 = 0$ . Now, assume that  $\psi(\widehat{x}^2) \neq 0$ ; so  $\widehat{x}^2 = \frac{\psi(\widehat{x}^2)}{2\psi(\widehat{x})}\widehat{x}$ . Simplifying the notation, we put  $\lambda = \frac{\psi(\widehat{x}^2)}{2\psi(\widehat{x})}$ . From  $D(\alpha\widehat{a}) = \alpha D(\widehat{a})$ , we get that  $\psi(\alpha\widehat{a})\widehat{x} = \alpha\psi(\widehat{a})\widehat{x}$  and since we are assuming that  $\widehat{x}$  is nonzero, it is concluded that  $\psi(\alpha\widehat{a}) = \alpha\psi(\widehat{a})$  for all  $a \in \mathcal{A}$  and  $\alpha \in \mathbb{C}$ . Replacing  $\widehat{x}^2$  by  $\lambda\widehat{x}$  in Equation (2.3) and then using Equation (2.3), we obtain that  $\lambda\psi(\widehat{x})\widehat{x} = \psi(\lambda\widehat{x})\widehat{x} = \psi(\widehat{x}^2)\widehat{x} = 2\psi(\widehat{x})\widehat{x}^2 = 2\psi(\widehat{x})\lambda\widehat{x}$ . This equation forces that  $\widehat{x} = 0$ , which is a contradiction. Therefore,  $\psi(\widehat{x}^2)$  must be zero. This conclusion along with Equation (2.3) imply that  $\widehat{x}^2 = 0$ . In the above, we have shown that  $\widehat{x}\widehat{a} + \widehat{a}\widehat{x}$  is a scalar multiple of  $\widehat{x}$ , i.e.  $\widehat{x}\widehat{a} + \widehat{a}\widehat{x} = \alpha\widehat{x}$ , where  $\alpha \in \mathbb{C}$ . Multiplying the previous equation by  $\widehat{x}$  and using the fact that  $\widehat{x}^2 = 0$ , we see that  $\widehat{x}\widehat{a}\widehat{x} = 0$  for any  $a$  in  $\mathcal{A}$  which means that  $xax \in \mathcal{I}$  for all  $a \in \mathcal{A}$ . Since  $\mathcal{I}$  is a semiprime ideal of  $\mathcal{A}$ ,  $x \in \mathcal{I}$ . It implies that  $\widehat{x} = x + \mathcal{I} = 0$ , a contradiction. From this contradiction we deduce that there is no element  $a_0$  of  $\mathcal{A}$  such that  $d(a_0) \notin \mathcal{I}$ . Therefore,  $d(\mathcal{A}) \subseteq \mathcal{I}$ .  $\square$

In the following, there are some immediate consequences of the above theorem.

**Corollary 2.7.** *Let  $\mathcal{A}$  be an algebra such that each primitive ideal of  $\mathcal{A}$  has codimension 1. Suppose that  $d : \mathcal{A} \rightarrow \mathcal{A}$  is a Jordan derivation such that  $d(\mathcal{P}) \subseteq \mathcal{P}$  for every primitive ideal  $\mathcal{P}$  of  $\mathcal{A}$ . Then  $d(\mathcal{A}) \subseteq rad(\mathcal{A})$ .*

**Proof .** According to the aforementioned assumption,  $d(\mathcal{P}) \subseteq \mathcal{P}$  for each primitive ideal  $\mathcal{P}$  of  $\mathcal{A}$ . Since each primitive ideal of  $\mathcal{A}$  has codimension 1 (i.e.,  $\dim(\frac{\mathcal{A}}{\mathcal{P}}) = 1$ ),  $\dim\{d(a) + \mathcal{P} \mid a \in \mathcal{A}\} \leq 1$ . According to [4, Proposition 1.4.34 (iii)],  $\mathcal{P}$  is a prime (and obviously is a semiprime) ideal of  $\mathcal{A}$ . Now, Theorem 2.6 implies that  $d(\mathcal{A}) \subseteq \mathcal{P}$ . Since we are assuming  $\mathcal{P}$  is an arbitrary primitive ideal of  $\mathcal{A}$ ,  $d(\mathcal{A}) \subseteq rad(\mathcal{A})$ .  $\square$

**Corollary 2.8.** *Suppose that  $\mathcal{A}$  is a Banach algebra and  $d : \mathcal{A} \rightarrow \mathcal{A}$  is a derivation satisfying  $\|Q_{\mathcal{P}}d^n\| \leq C^n$  ( $n \in \mathbb{N}$ ) for some positive real constant  $C$ , where  $Q_{\mathcal{P}} : \mathcal{A} \rightarrow \frac{\mathcal{A}}{\mathcal{P}}$  is the quotient map and  $\mathcal{P}$  is a primitive ideal of  $\mathcal{A}$ . If  $\dim\{Q_{\mathcal{P}}d(a) \mid a \in \mathcal{A}\} \leq 1$ , then  $d(\mathcal{A}) \subseteq \mathcal{P}$ .*

**Proof .** It follows from [19, Lemma 1.2] that  $d(\mathcal{P}) \subseteq \mathcal{P}$ . Now, Theorem 2.6 is exactly what we need to complete the proof.  $\square$

In the next theorem, we present a necessary and sufficient condition for the commutativity of Banach algebras.

**Theorem 2.9.** *Let  $\mathcal{A}$  be a semisimple Banach algebra. Then  $\mathcal{A}$  is commutative if and only if each primitive ideal of  $\mathcal{A}$  is of codimension 1.*

**Proof .** Let  $\mathcal{A}$  be a commutative Banach algebra. It follows from [4, Proposition 1.4.36] that each primitive ideal of  $\mathcal{A}$  is of codimension 1. Conversely, suppose that each primitive ideal of  $\mathcal{A}$  is of codimension 1. We define  $d_x(a) = [a, x] = ax - xa$ , where  $x$  is a nonzero arbitrary fixed element of  $\mathcal{A}$ . Evidently,  $d_x$  is a continuous (Jordan) derivation and it follows from [16, Lemma 3.2] that



$d_x(\mathcal{P}) \subseteq \mathcal{P}$  for any primitive ideal  $\mathcal{P}$  of  $\mathcal{A}$ . Now, Corollary 2.7 implies that  $d_x(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$  and semi-simplicity of  $\mathcal{A}$  forces that  $d_x(a) = 0$  for all  $a \in \mathcal{A}$ . Since  $x$  is considered an arbitrary element,  $\mathcal{A}$  is commutative.  $\square$

In the following theorem, we show that every continuous Jordan derivation on a unital finite-dimensional Banach algebra is identically zero under certain conditions.

**Theorem 2.10.** *Let  $n$  be a positive integer and let  $\mathcal{A}$  be an  $n$ -dimensional unital Banach algebra with a basis  $\mathfrak{B} = \{b_1, b_2, \dots, b_n\}$ . If for any integer  $k$ , where  $1 \leq k \leq n$ , an ideal  $\mathcal{I}_k$  generated by  $\mathfrak{B} - \{b_k\}$  is a proper subset of  $\mathcal{A}$ , then every continuous Jordan derivation on  $\mathcal{A}$  is identically zero.*

**Proof .** Evidently,  $\dim(\frac{\mathcal{A}}{\mathcal{I}_k}) = 1$  for every  $k \in \{1, 2, \dots, n\}$ . We claim that  $\mathcal{I}_k$  is a maximal ideal of  $\mathcal{A}$  for each  $k \in \{1, 2, \dots, n\}$ . To obtain a contradiction, suppose that  $\mathcal{I}_{k_0}$  is not a maximal ideal of  $\mathcal{A}$  for some  $k_0$ . Then there exists a maximal ideal  $M_{k_0}$  of  $\mathcal{A}$  such that  $\mathcal{I}_{k_0} \subset M_{k_0} \subset \mathcal{A}$ . So, we have  $n - 1 = \dim(\mathcal{I}_{k_0}) < \dim(M_{k_0}) < n$ , which is a contradiction. Hence, every  $\mathcal{I}_k$  is a maximal ideal of  $\mathcal{A}$ . It follows from [4, Proposition 1.4.34 (iv)] that every  $\mathcal{I}_k$  is a primitive ideal of  $\mathcal{A}$ . Since  $d$  is a continuous Jordan derivation, [16, Lemma 3.2] implies that  $d(\mathcal{I}_k) \subseteq \mathcal{I}_k$  and we conclude from Theorem 2.6 that  $d(\mathcal{A}) \subseteq \mathcal{I}_k$  for any  $k \in \{1, 2, \dots, n\}$ . Hence,  $d(\mathcal{A}) \subseteq \bigcap_{k=1}^n \mathcal{I}_k$ . Suppose that there exists an element  $a_0$  of  $\mathcal{A}$  such that  $d(a_0) \neq 0$ . Since  $\mathfrak{B} = \{b_1, b_2, \dots, b_n\}$  is a basis of  $\mathcal{A}$ , there exist the complex numbers  $\alpha_{i_j}$  and the elements  $b_{i_j}$  of  $\mathfrak{B}$  such that

$$d(a_0) = \sum_{j=1}^m \alpha_{i_j} b_{i_j} = \alpha_{i_1} b_{i_1} + \alpha_{i_2} b_{i_2} + \dots + \alpha_{i_m} b_{i_m}, \quad (m \leq n) \tag{2.4}$$

Since  $d(\mathcal{A}) \subseteq \mathcal{I}_k$  for any  $k \in \{1, 2, \dots, n\}$ , we may assume that  $d(\mathcal{A}) \subseteq \mathcal{I}_{i_1}$ . So, we have

$$d(a_0) = \alpha_{i_1} b_{i_1} + \alpha_{i_2} b_{i_2} + \dots + \alpha_{i_m} b_{i_m} \in \mathcal{I}_{i_1}.$$

It follows from the previous relation that  $b_{i_1} \in \mathcal{I}_{i_1}$ , which is a contradiction of the assumption that every  $\mathcal{I}_k$  is a proper subset of  $\mathcal{A}$ . This contradiction proves our assertion that  $d$  is identically zero on  $\mathcal{A}$ .  $\square$

As a consequence of Theorem 2.10, we have the following result.

**Corollary 2.11.** *Let  $\mathcal{A}$  and  $\mathfrak{B}$  be as above. Then  $\mathcal{A}$  is commutative.*

**Proof .** First, we define a liner mapping  $d_c : \mathcal{A} \rightarrow \mathcal{A}$  by  $d_c(a) = [a, c] = ac - ca$ , where  $c$  is an arbitrary fixed element of  $\mathcal{A}$ . Clearly,  $d_c$  is a continuous derivation. It follows from Theorem 2.10 that  $d_c$  is identically zero. Since  $c$  is an arbitrary,  $\mathcal{A}$  is a commutative algebra.  $\square$

If  $\mathcal{Y}$  and  $\mathcal{Z}$  are Banach spaces and  $T : \mathcal{Y} \rightarrow \mathcal{Z}$  is a linear mapping, then the set

$$S(T) = \{z \in \mathcal{Z} : \exists \{y_n\} \subseteq \mathcal{Y} \text{ such that } y_n \rightarrow 0, T(y_n) \rightarrow z\}$$

is called the separating space of  $T$ . By the closed graph Theorem,  $T$  is continuous if and only if  $S(T) = \{0\}$ . For additional information about separating spaces, the reader is referred to [4].

Let  $\mathcal{S}$  be a subset of a ring  $\mathcal{R}$ . The left annihilator of  $\mathcal{S}$  is  $\text{lann}(\mathcal{S}) := \{x \in \mathcal{R} \mid x\mathcal{S} = \{0\}\}$ . Similarly, the right annihilator of  $\mathcal{S}$  is  $\text{rann}(\mathcal{S}) := \{x \in \mathcal{R} \mid \mathcal{S}x = \{0\}\}$ . The annihilator of  $\mathcal{S}$  is defined as  $\text{ann}(\mathcal{S}) := \text{lann}(\mathcal{S}) \cap \text{rann}(\mathcal{S})$ .

**Theorem 2.12.** *Let  $\mathcal{A}$  be a Banach algebra and let  $d : \mathcal{A} \rightarrow \mathcal{A}$  be a derivation such that  $S(d) \subseteq \text{lann}(\mathcal{A})$  or  $S(d) \subseteq \text{rann}(\mathcal{A})$ . If any primitive ideal of  $\mathcal{A}$  has codimension 1, then  $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$ .*

**Proof .** Let  $S(d) \subseteq \text{lann}(\mathcal{A})$ . We will now proceed with the proof based on the argument of [3, Lemma 3.1]. It is clear that  $\text{lann}(\mathcal{A})$  is a closed bi-ideal of  $\mathcal{A}$  and so, the mapping  $d_1 : \mathcal{A} \rightarrow \frac{\mathcal{A}}{\text{lann}(\mathcal{A})}$  defined by  $d_1(a) = d(a) + \text{lann}(\mathcal{A})$  is continuous. Now, we show that  $d(\text{lann}(\mathcal{A})) \subseteq \text{lann}(\mathcal{A})$ . Let  $x$  be an arbitrary element of  $\text{lann}(\mathcal{A})$ . For all  $a \in \mathcal{A}$ , we have

$$0 = d(0) = d(xa) = d(x)a + xd(a) = d(x)a,$$

which means that  $d(\text{lann}(\mathcal{A})) \subseteq \text{lann}(\mathcal{A})$ . Hence, we can define the mapping  $\tilde{d} : \frac{\mathcal{A}}{\text{lann}(\mathcal{A})} \rightarrow \frac{\mathcal{A}}{\text{lann}(\mathcal{A})}$  by  $\tilde{d}(a + \text{lann}(\mathcal{A})) = d(a) + \text{lann}(\mathcal{A})$ . Note that  $\tilde{d}$  is a continuous derivation and it follows from [4, Proposition 2.7.22] that  $\tilde{d}(\tilde{\mathcal{P}}) \subseteq \tilde{\mathcal{P}}$  for every primitive ideal  $\tilde{\mathcal{P}}$  of  $\frac{\mathcal{A}}{\text{lann}(\mathcal{A})}$ . Obviously, if  $\mathcal{P}$  is a primitive ideal of  $\mathcal{A}$ , then  $\frac{\mathcal{P}}{\text{lann}(\mathcal{A})}$  is a primitive ideal of  $\frac{\mathcal{A}}{\text{lann}(\mathcal{A})}$ . Thus,  $\tilde{d}(\frac{\mathcal{P}}{\text{lann}(\mathcal{A})}) \subseteq \frac{\mathcal{P}}{\text{lann}(\mathcal{A})}$  for every primitive ideal  $\mathcal{P}$  of  $\mathcal{A}$  and one can easily deduce that  $d(\mathcal{P}) \subseteq \mathcal{P} + \text{lann}(\mathcal{A}) = \mathcal{P}$ . Since  $\dim(\frac{\mathcal{A}}{\mathcal{P}}) = 1$ ,  $\dim\{d(a) + \mathcal{P} \mid a \in \mathcal{A}\} \leq \dim(\frac{\mathcal{A}}{\mathcal{P}}) = 1$ . It follows from [4, Proposition 1.4.34] that each primitive ideal in  $\mathcal{A}$  is a prime ideal and also is a semiprime ideal of  $\mathcal{A}$ . In view of Theorem 2.6,  $d(\mathcal{A}) \subseteq \mathcal{P}$ . Since  $\mathcal{P}$  is an arbitrary primitive ideal of  $\mathcal{A}$ ,  $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$ , as desired. If  $S(d) \subseteq \text{rann}(\mathcal{A})$ , then reasoning like above gives the required result and we leave it to the reader.  $\square$

In the following example, we give a discontinuous derivation  $d$  on a Banach algebra  $\mathfrak{B}$  such that  $S(d) \subseteq \text{lann}(\mathfrak{B})$ .

**Example 2.13.** *Let  $\mathcal{A}$  be a Banach algebra and let*

$$\mathfrak{B} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathcal{A} \right\}$$

*It is clear that  $\mathfrak{B}$  with the norm below is a Banach algebra.*

$$\left\| \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right\| = \|a\| + \|b\| + \|c\|.$$

*Let  $T : \mathcal{A} \rightarrow \mathcal{A}$  be a discontinuous linear mapping. Define the linear mappings  $f, d : \mathfrak{B} \rightarrow \mathfrak{B}$  by*

$$f\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$d\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & T(a) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

*It is routine to see that*

$$f(AB) = d(A)B + Af(B),$$

$$d(AB) = d(A)B + Ad(B)$$

for all  $A, B \in \mathfrak{B}$ . It is clear that the derivation  $d$  is discontinuous, because so is  $T$ . Also, note that  $f$  is a continuous linear mapping on  $\mathfrak{B}$ . Now, we show that  $S(d) \subseteq \text{lann}(\mathfrak{B})$ . Let  $A_0 \in S(d)$ . Then there exists a sequence  $\{A_n\} \subset \mathfrak{B}$  such that  $A_n \rightarrow 0$  and  $d(A_n) \rightarrow A_0$ . Since  $f$  is continuous, we have

$$0 = \lim_{n \rightarrow \infty} f(A_n A) = \lim_{n \rightarrow \infty} (d(A_n)A + A_n f(A)) = A_0 A,$$

for all  $A \in \mathfrak{B}$ . It mean that  $A_0 \in \text{lann}(\mathfrak{B})$ .

### 3. Automatic continuity of $\{g, h\}$ -derivations

Throughout this section,  $\mathcal{A}$  denotes an associative algebra over the complex field  $\mathbb{C}$  with center  $Z(\mathcal{A})$  and the identity element  $\mathbf{e}$ . For any  $a \in \mathcal{A}$ , we define the linear mappings  $L_a, R_a : \mathcal{A} \rightarrow \mathcal{A}$  by  $L_a(b) = ab$  and  $R_a(b) = ba$  for all  $b \in \mathcal{A}$ . A straightforward verification shows that  $R_b L_a = L_a R_b$ ,  $\lambda L_a = L_{\lambda a}$  and  $\lambda R_a = R_{\lambda a}$  for all  $a, b \in \mathcal{A}$  and  $\lambda \in \mathbb{C}$ . Clearly,  $L_a = R_a$  if and only if  $a \in Z(\mathcal{A})$ .

Let  $f, g, h : \mathcal{A} \rightarrow \mathcal{A}$  be linear mappings. Recall that  $f$  is a  $\{g, h\}$ -derivation if

$$f(ab) = g(a)b + ah(b) = h(a)b + ag(b), \quad a, b \in \mathcal{A}. \tag{3.1}$$

Using a straightforward verification, Brešar [1] showed that  $g(\mathbf{e}), h(\mathbf{e}) \in Z(\mathcal{A})$ . Taking  $b = \mathbf{e}$  in (3.1), we obtain

$$f(a) = g(a) + ah(\mathbf{e}) = h(a) + ag(\mathbf{e}),$$

and taking  $a = \mathbf{e}$  in (3.1), we get

$$f(b) = g(\mathbf{e})b + h(b) = h(\mathbf{e})b + g(b).$$

Comparing the above-mentioned expressions, we deduce that  $g(\mathbf{e}), h(\mathbf{e}) \in Z(\mathcal{A})$ . For an arbitrary  $\{g, h\}$ -derivation  $f : \mathcal{A} \rightarrow \mathcal{A}$ , we have

$$\begin{aligned} f(ab) &= \frac{f(ab)}{2} + \frac{f(ab)}{2} \\ &= \frac{1}{2} (g(a)b + ah(b)) + \frac{1}{2} (h(a)b + ag(b)) \\ &= \left( \frac{g(a) + h(a)}{2} \right) b + a \left( \frac{g(b) + h(b)}{2} \right). \end{aligned}$$

Letting  $G = \frac{g+h}{2}$ , we see that  $f$  is a  $\{G, G\}$ -derivation. So, we deduce that a linear map  $f$  is a  $\{g, h\}$ -derivation if and only if it is a  $\{G, G\}$ -derivation, where  $g, h, G : \mathcal{A} \rightarrow \mathcal{A}$  are linear maps. Similarly,  $f$  is a Jordan  $\{g, h\}$ -derivation if and only if it is a Jordan  $\{G, G\}$ -derivation. Brešar [1] in his article has written that ” $\{g, h\}$ -derivations are special examples of what is known in the literature as generalized derivations.” In the following theorem, we show that if  $f$  is a  $\{g, h\}$ -derivation, then  $f, g, h$  are generalized derivations.

Here, we establish the first theorem of this section.

**Theorem 3.1.** *If  $f : \mathcal{A} \rightarrow \mathcal{A}$  is a  $\{g, h\}$ -derivation, then  $f, g$  and  $h$  are generalized derivations associated with the derivation  $\delta$  on  $\mathcal{A}$ .*

**Proof .** According to the above discussion,  $g(\mathbf{e}), h(\mathbf{e}) \in Z(\mathcal{A})$  and also,

$$f(a) = g(a) + ah(\mathbf{e}) = h(a) + ag(\mathbf{e}).$$

So, we have

$$h(a) = g(a) + (h(\mathbf{e}) - g(\mathbf{e}))a, \quad a \in \mathcal{A}. \quad (3.2)$$

Consequently,  $h = g + L_{h(\mathbf{e})-g(\mathbf{e})}$ . This equation explains the exact relationship between the mappings  $g$  and  $h$ . We know that  $f(a) = g(a) + h(\mathbf{e})a$  and  $h(a) = g(a) + (h(\mathbf{e}) - g(\mathbf{e}))a$  for all  $a \in \mathcal{A}$ . Since  $g(a)b + ah(b) = f(ab) = g(ab) + h(\mathbf{e})ab$ , we obtain that

$$\begin{aligned} g(ab) &= g(a)b + ah(b) - h(\mathbf{e})ab \\ &= g(a)b + a(g(b) + (h(\mathbf{e}) - g(\mathbf{e}))b) - h(\mathbf{e})ab \\ &= g(a)b + ag(b) + h(\mathbf{e})ab - g(\mathbf{e})ab - h(\mathbf{e})ab \\ &= g(a)b + ag(b) - g(\mathbf{e})ab, \end{aligned}$$

which means that

$$g(ab) = g(a)b + ag(b) - g(\mathbf{e})ab, \quad a, b \in \mathcal{A}. \quad (3.3)$$

Letting  $\delta = g - L_{g(\mathbf{e})}$ , we see that

$$\begin{aligned} \delta(ab) &= g(ab) - g(\mathbf{e})ab \\ &= g(a)b + ag(b) - g(\mathbf{e})ab - g(\mathbf{e})ab \\ &= (g(a) - g(\mathbf{e})a)b + a(g(b) - g(\mathbf{e})b) \\ &= \delta(a)b + a\delta(b), \end{aligned}$$

which means that  $\delta$  is a derivation on  $\mathcal{A}$ . We see that

$$g(ab) = g(a)b + a\delta(b) = \delta(a)b + ag(b), \quad a, b \in \mathcal{A}.$$

Thus,  $g$  is a generalized derivation associated with the derivation  $\delta$ . By a similar argument, we can obtain that

$$h(ab) = h(a)b + ah(b) - h(\mathbf{e})ab, \quad a, b \in \mathcal{A}. \quad (3.4)$$

Considering  $\Delta = h - L_{h(\mathbf{e})}$ , it is observed that  $\Delta$  is a derivation. So,  $h$  is a generalized derivation associated with the derivation  $\Delta$ . In the following, we show that  $\delta = \Delta$ . We know that  $f(a) = g(a) + ah(\mathbf{e}) = h(a) + ag(\mathbf{e})$  for all  $a \in \mathcal{A}$ . So, we have  $h(a)b + ag(b) = f(ab) = h(ab) + abg(\mathbf{e})$  and consequently

$$h(ab) = h(a)b + ag(b) - abg(\mathbf{e}) = h(a)b + a\delta(b) \quad (3.5)$$

In view of Equation (3.5), we get that

$$\begin{aligned} \Delta(ab) &= h(ab) - abh(\mathbf{e}) \\ &= h(a)b + a\delta(b) - abh(\mathbf{e}) \\ &= (h(a) - ah(\mathbf{e}))b + a\delta(b) \\ &= \Delta(a)b + a\delta(b), \end{aligned}$$

for all  $a, b \in \mathcal{A}$ . We can thus see that  $\Delta(b) = \Delta(\mathbf{e})b + \mathbf{e}\delta(b) = \delta(b)$  for all  $b \in \mathcal{A}$ , which means that  $\Delta = \delta$ . Therefore, we have  $g = \delta + L_{g(\mathbf{e})}$  and  $h = \delta + L_{h(\mathbf{e})}$ . Our next task is to show that  $f$  is also a generalized derivation associated with the derivation  $\delta$ . We know that  $f(a) = g(a) + h(\mathbf{e})a$  and  $g(a) = \delta(a) + g(\mathbf{e})a$  for all  $a \in \mathcal{A}$ . Hence, we have

$$f(a) = \delta(a) + g(\mathbf{e})a + h(\mathbf{e})a = \delta(a) + (g(\mathbf{e}) + h(\mathbf{e}))a,$$

which means that  $f = \delta + L_{g(\mathbf{e})+h(\mathbf{e})} = \delta + L_{f(\mathbf{e})}$ . Therefore,  $f, g$  and  $h$  are generalized derivations associated with derivation  $\delta$ . This completes the proof of our theorem.  $\square$

It is an immediate conclusion from equation (3.2) that if  $f$  is a  $\{g, h\}$ -derivation on an algebra  $\mathcal{A}$  such that  $g(\mathbf{e}) = h(\mathbf{e})$ , then  $g = h$ .

**Remark 3.2.** Let  $f$  be a  $\{g, h\}$ -derivation on a complex algebra  $\mathcal{A}$ . As previously stated,  $f(a) = g(a) + ah(\mathbf{e}) = h(a) + ag(\mathbf{e})$  for all  $a \in \mathcal{A}$ . If  $h(\mathbf{e}) = 0$ , then clearly  $f = g$  and  $h$  is a derivation. Similarly, it is observed that if  $g(\mathbf{e}) = 0$ , then  $f = h$  and  $g$  is a derivation. Therefore, if  $h(\mathbf{e}) = g(\mathbf{e}) = 0$ , then  $f = g = h$ , and consequently,  $f$  is a derivation. Besides, if  $f$  is a  $\{g, h\}$ -derivation on a unital algebra  $\mathcal{A}$ , then a simple argument shows that  $f - g$  and  $f - h$  are centralizers on  $\mathcal{A}$ . Recall that a linear mapping  $T : \mathcal{A} \rightarrow \mathcal{A}$  is called a left (resp. right) centralizer of  $\mathcal{A}$  if  $T(ab) = T(a)b$  (resp.  $T(ab) = aT(b)$ ) holds for all  $a, b \in \mathcal{A}$ . The map  $T$  is said to be a centralizer if it is both a left and a right centralizer on  $\mathcal{A}$ . Before Theorem 3.1, we have shown that if  $f$  is a  $\{g, h\}$ -derivation, then it is a  $\{G, G\}$ -derivation, where  $G = \frac{g+h}{2}$ . So, for each  $a \in \mathcal{A}$ , we have  $f(a) = G(a) + aG(\mathbf{e})$  and  $G(a) = D(a) + G(\mathbf{e})a$ , where  $D$  is a derivation. If  $G(\mathbf{e}) = 0$ , then  $f = G$  and  $G$  is a derivation. It means that if  $f$  is a  $\{g, h\}$ -derivation such that  $h(\mathbf{e}) + g(\mathbf{e}) = 0$ , then  $f = \frac{g+h}{2}$  and it is a derivation on  $\mathcal{A}$ .

Below, we present a result concerning the automatic continuity of  $\{g, h\}$ -derivations. Before that, we give the following definition.

**Definition 3.3.** Let  $\mathcal{A}$  be a Banach algebra. We say that  $\mathcal{A}$  has the Cohen factorization property if for any sequence  $\{a_n\} \subset \mathcal{A}$  with  $\lim_{n \rightarrow \infty} a_n = 0$ , there exist an element  $c \in \mathcal{A}$  and a sequence  $\{b_n\} \subset \mathcal{A}$  such that  $\lim_{n \rightarrow \infty} b_n = 0$  and  $a_n = cb_n$  for any  $n \in \mathbb{N}$ .

It follows from Corollary 11.12 of [2] that if a Banach algebra  $\mathcal{A}$  has a bounded approximate identity, then  $\mathcal{A}$  has the Cohen factorization property. Recall that if  $\mathcal{Y}$  and  $\mathcal{Z}$  are Banach spaces and  $T : \mathcal{Y} \rightarrow \mathcal{Z}$  is a linear mapping, then the set

$$S(T) = \{z \in \mathcal{Z} : \exists \{y_n\} \subseteq \mathcal{Y} \text{ such that } y_n \rightarrow 0, T(y_n) \rightarrow z\}$$

is called the separating space of  $T$ . By the closed graph Theorem,  $T$  is continuous if and only if  $S(T) = \{0\}$ . For additional information about separating spaces, the reader is referred to [4].

Let  $\mathcal{S}$  be a subset of a ring  $\mathcal{R}$ . The left annihilator of  $\mathcal{S}$  is  $lann(\mathcal{S}) := \{x \in \mathcal{R} \mid x\mathcal{S} = \{0\}\}$ . Similarly, the right annihilator of  $\mathcal{S}$  is  $rann(\mathcal{S}) := \{x \in \mathcal{R} \mid \mathcal{S}x = \{0\}\}$ . The annihilator of  $\mathcal{S}$  is defined as  $ann(\mathcal{S}) := lann(\mathcal{S}) \cap rann(\mathcal{S})$ .

**Theorem 3.4.** Let  $\mathcal{A}$  be a complex Banach algebra and let  $f : \mathcal{A} \rightarrow \mathcal{A}$  be a  $\{g, h\}$ -derivation.

- (i) Let  $ann(\mathcal{A}) = \{0\}$ . If  $f$  is continuous, then both  $g$  and  $h$  are continuous.
- (ii) If  $\mathcal{A}$  has the Cohen factorization property and at least one of  $g$  or  $h$  is continuous, then  $f$  is continuous.
- (iii) If  $\mathcal{A}$  is unital, then the continuity of one of these mappings (, i.e.  $f$  or  $g$  or  $h$ ) implies the continuity of the other two.

**Proof .** (i) Let  $f$  be continuous and  $c \in S(h)$ . So, there exists a sequence  $\{c_n\} \subseteq \mathcal{A}$  such that  $\lim_{n \rightarrow \infty} c_n = 0$  and  $\lim_{n \rightarrow \infty} h(c_n) = c$ . We will show that  $c = 0$ . For any  $a \in \mathcal{A}$ , we have

$$0 = \lim_{n \rightarrow \infty} f(ac_n) = \lim_{n \rightarrow \infty} (g(a)c_n + ah(c_n)) = ac.$$

As well as, we have

$$0 = \lim_{n \rightarrow \infty} f(c_na) = \lim_{n \rightarrow \infty} (h(c_n)a + c_ng(a)) = ca.$$

Since  $a$  is an arbitrary element of  $\mathcal{A}$ ,  $c \in \text{ann}(\mathcal{A}) = \{0\}$ . It means that  $h$  is a continuous linear mapping. Similarly, we can show that  $g$  is a continuous linear mapping.

(ii) Let  $h$  be a continuous linear mapping and let  $\{a_n\}$  be a sequence of  $\mathcal{A}$  converging to zero. Since  $\mathcal{A}$  has the Cohen factorization property, there exist  $c \in \mathcal{A}$  and  $\{b_n\} \subset \mathcal{A}$  such that  $a_n = cb_n$  ( $n \in \mathbb{N}$ ) and  $\lim_{n \rightarrow \infty} b_n = 0$ . Therefore, we have

$$\lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} f(cb_n) = \lim_{n \rightarrow \infty} (g(c)b_n + ch(b_n)) = 0,$$

which means that  $f$  is a continuous linear mapping. Also, the continuity of  $g$  implies the continuity of  $f$ .

(iii) According to Theorem 3.1, the mappings  $f, g$  and  $h$  are generalized derivations associated with the same derivation  $\delta$ . Indeed, we have  $g = \delta + L_{g(\mathbf{e})}$ ,  $h = \delta + L_{h(\mathbf{e})}$  and  $f = \delta + L_{g(\mathbf{e})+h(\mathbf{e})}$ . It is clear that if  $f$  is continuous, then the derivation  $\delta$  is so. Consequently, both  $g$  and  $h$  are continuous linear mappings. Similarly, we can complete the proof.  $\square$

**Remark 3.5.** We know that every  $C^*$ -algebra is semisimple and recall that every semisimple algebra is semiprime. It follows from [18, Corollary 7.5 ] that any  $C^*$ -algebra  $\mathcal{A}$  admits an approximate identity. According to page 26 of [18], if  $\mathcal{A}$  is a  $C^*$ -algebra, then we require two more properties for an approximate identity  $\{u_i\}$  as follows:

- $0 \leq u_i \leq u_j$ , if  $i \leq j$ ;
- $\|u_i\| \leq 1$ .

**Corollary 3.6.** Let  $\mathcal{A}$  be a complex Banach algebra having a bounded approximate identity and let  $f : \mathcal{A} \rightarrow \mathcal{A}$  be a  $\{g, h\}$ -derivation. Then  $f$  is continuous if and only if  $h$  or  $g$  is continuous.

**Proof .**It follows from [2, Corollary 11.12] that if a Banach algebra  $\mathcal{A}$  has a bounded approximate identity, then  $\mathcal{A}$  has the Cohen factorization property. Since  $\mathcal{A}$  has an approximate identity,  $\text{ann}(\mathcal{A}) = \{0\}$ . Now, Theorem 3.4 completes the proof.  $\square$

**Remark 3.7.** If  $\mathcal{A}$  is a semiprime algebra (or ring), then it is evident that  $\text{ann}(\mathcal{A}) = \{0\}$ . But, the converse is in general not true. For instance, if  $\mathcal{A}$  is a unital algebra (or ring), then  $\text{ann}(\mathcal{A}) = \{0\}$ . Moreover, if  $\mathcal{A}$  is a normed algebra having an approximate identity, then  $\text{ann}(\mathcal{A}) = \{0\}$ . Hence, if  $\mathcal{A}$  is a  $C^*$ -algebra, then  $\text{ann}(\mathcal{A}) = \{0\}$ .



In the next theorem, we present conditions under which any  $\{g, h\}$ -derivation is a centralizer. We denote the set of all primitive ideals of an algebra  $\mathcal{A}$  having codimension 1 in  $\mathcal{A}$  by  $\Pi_1(\mathcal{A})$ . It then follows from Propositions 1.3.37 (i) and 1.4.34 (iv) of [4] that

$$\{ker\varphi : \varphi \in \Phi_{\mathcal{A}}\} \subseteq \Pi_1(\mathcal{A}).$$

So, we have

$$\bigcap_{\mathcal{P} \in \Pi_1(\mathcal{A})} \mathcal{P} \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker\varphi.$$

**Theorem 3.8.** *Let  $\mathcal{A}$  be a unital, complex Banach algebra and let  $f : \mathcal{A} \rightarrow \mathcal{A}$  be a continuous  $\{g, h\}$ -derivation. If  $\bigcap_{\mathcal{P} \in \Pi_1(\mathcal{A})} \mathcal{P} = \{0\}$ , then  $f, g$  and  $h$  are centralizers on  $\mathcal{A}$ .*

**Proof .** It follows from Theorem 3.1 that  $f, g$  and  $h$  are generalized derivations associated with the same derivation  $\delta$ . Also, it follows from Theorem 3.1 that  $f = \delta + L_{f(\mathbf{e})}$  and obviously, the continuity of  $f$  forces the continuity of  $\delta$ . According to Sinclair’s Theorem,  $\delta(\mathcal{P}) \subseteq \mathcal{P}$  for any primitive ideal  $\mathcal{P}$  of  $\mathcal{A}$  (see [4, Proposition 2.7.22]). Let  $\mathcal{P}$  be an arbitrary element of  $\Pi_1(\mathcal{A})$ . It follows from part (iii) of [4, Proposition 1.4.34] that  $\mathcal{P}$  is a prime and of course is a semiprime ideal of  $\mathcal{A}$ . Thus, all the conditions of Theorem 2.6 are fulfilled and this yields that  $\delta(\mathcal{A}) \subseteq \mathcal{P}$ . Since we are assuming that  $\mathcal{P}$  is an arbitrary element of  $\Pi_1(\mathcal{A})$  and  $\bigcap_{\mathcal{P} \in \Pi_1(\mathcal{A})} \mathcal{P} = \{0\}$ ,  $\delta$  is identically zero. This fact along with Theorem 3.1 imply that  $f = L_{f(\mathbf{e})}, g = L_{g(\mathbf{e})}$  and  $h = L_{h(\mathbf{e})}$  and since  $f(\mathbf{e}), g(\mathbf{e}), h(\mathbf{e}) \in Z(\mathcal{A})$ , we deduce that  $f, g$  and  $h$  are centralizers on  $\mathcal{A}$ .  $\square$

#### 4. Characterization of $\{g, h\}$ -homomorphisms on algebras

In this section,  $\mathcal{A}$  denotes an associative algebra over the complex field  $\mathbb{C}$ . If  $\mathcal{A}$  is unital, then  $\mathbf{e}$  stands for the identity element of  $\mathcal{A}$ . In this section, we introduce the concept of a  $\{g, h\}$ -homomorphism on an algebra and characterize it under certain conditions.

**Definition 4.1.** *Let  $f, g, h : \mathcal{A} \rightarrow \mathcal{A}$  be linear maps. We say that  $f$  is a  $\{g, h\}$ -homomorphism if  $f(ab) = g(a)h(b) = h(a)g(b)$  for all  $a, b \in \mathcal{A}$ . If a  $\{g, h\}$ -homomorphism is onto and one-to-one, then it is called a  $\{g, h\}$ -automorphism.*

In the following proposition, we provide an example of a  $\{g, h\}$ -automorphism on an algebra  $\mathcal{A}$ . We denote by  $B(\mathcal{A})$  the set of all bounded (equivalently continuous) linear maps from  $\mathcal{A}$  into itself.

**Proposition 4.2.** *Let  $f$  be a continuous  $\{g, h\}$ -derivation on a Banach algebra  $\mathcal{A}$  such that  $g$  and  $h$  are also continuous linear maps. Then  $e^f$  is a continuous  $\{e^g, e^h\}$ -automorphism on  $\mathcal{A}$ .*

**Proof .** If  $f$  is a  $\{g, h\}$ -derivation on an algebra, then a straightforward induction shows that

$$f^n(ab) = \sum_{k=0}^n \binom{n}{k} g^{n-k}(a)h^k(b) = \sum_{k=0}^n \binom{n}{k} h^{n-k}(a)g^k(b), \quad (a, b \in \mathcal{A})$$

For  $a, b \in \mathcal{A}$ , we have

$$\begin{aligned}
 e^f(ab) &= \sum_{n=0}^{\infty} \frac{f^n}{n!}(ab) \\
 &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} g^{n-k}(a) h^k(b) \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^n \left( \frac{1}{(n-k)!} g^{n-k}(a) \right) \left( \frac{1}{k!} h^k(b) \right) \\
 &= \left( \sum_{n=0}^{\infty} \frac{1}{n!} g^n(a) \right) \left( \sum_{n=0}^{\infty} \frac{1}{n!} h^n(b) \right) \\
 &= e^g(a) e^h(b).
 \end{aligned}$$

Similar to above, one can easily prove that

$$e^f(ab) = e^h(a) e^g(b).$$

Also,  $e^f$  is an invertible element of  $B(\mathcal{A})$  and is therefore a bijective mapping of  $\mathcal{A}$  onto  $\mathcal{A}$ .  $\square$

In what follows,  $\mathcal{A}$  is a unital algebra and  $f$  is a  $\{g, h\}$ -homomorphism. We know that

$$f(ab) = g(a)h(b) = h(a)g(b) \text{ for all } a, b \in \mathcal{A}. \quad (4.1)$$

Namely, taking  $b = \mathbf{e}$  in (4.1), we obtain

$$f(a) = g(a)h(\mathbf{e}) = h(a)g(\mathbf{e}), \quad (4.2)$$

and taking  $a = \mathbf{e}$ , we obtain

$$f(b) = g(\mathbf{e})h(b) = h(\mathbf{e})g(b). \quad (4.3)$$

Comparing (4.2) and (4.3), we see that

$$g(a)h(\mathbf{e}) = h(\mathbf{e})g(a), \quad (4.4)$$

$$h(a)g(\mathbf{e}) = g(\mathbf{e})h(a), \quad (4.5)$$

for all  $a \in \mathcal{A}$ . Suppose that  $g(\mathbf{e})$  and  $h(\mathbf{e})$  are invertible elements of  $\mathcal{A}$ . Using (4.5), we have

$$g(\mathbf{e})^{-1}h(a) = h(a)g(\mathbf{e})^{-1} \text{ for all } a \in \mathcal{A}. \quad (4.6)$$

Using (4.6), one can easily see that

$$g(\mathbf{e})^{-1}h(\mathbf{e})^{-1} = h(\mathbf{e})^{-1}g(\mathbf{e})^{-1}.$$

Using the above way, we get

$$h(\mathbf{e})^{-1}g(a) = g(a)h(\mathbf{e})^{-1} \text{ for all } a \in \mathcal{A}. \quad (4.7)$$

In the following, we show that  $g(a)g(\mathbf{e}) = g(\mathbf{e})g(a)$  for all  $a \in \mathcal{A}$ . We have

$$\begin{aligned} g(a)g(\mathbf{e}) &= h(a)g(\mathbf{e})h(\mathbf{e})^{-1}g(\mathbf{e}) \quad (\text{see (4.2)}) \\ &= g(\mathbf{e})h(a)h(\mathbf{e})^{-1}g(\mathbf{e}) \quad (\text{see (4.5)}) \\ &= g(\mathbf{e})h(a)g(\mathbf{e})h(\mathbf{e})^{-1} \quad (\text{see (4.7)}) \\ &= g(\mathbf{e})g(a) \quad (\text{see (4.2)}), \end{aligned}$$

which means that

$$g(a)g(\mathbf{e}) = g(\mathbf{e})g(a) \text{ for all } a \in \mathcal{A}. \tag{4.8}$$

Using (4.8), we arrive at

$$g(\mathbf{e})^{-1}g(a) = g(a)g(\mathbf{e})^{-1} \text{ for all } a \in \mathcal{A}. \tag{4.9}$$

Using the above equations, we can obtain the following expressions:

$$\begin{aligned} f(a)g(\mathbf{e}) &= g(\mathbf{e})f(a), & f(a)g(\mathbf{e})^{-1} &= g(\mathbf{e})^{-1}f(a) \\ f(a)h(\mathbf{e}) &= h(\mathbf{e})f(a), & f(a)h(\mathbf{e})^{-1} &= h(\mathbf{e})^{-1}f(a) \end{aligned}$$

Now, we are in a position to present the main result of this section.

**Theorem 4.3.** *Suppose that  $f : \mathcal{A} \rightarrow \mathcal{A}$  is a  $\{g, h\}$ -homomorphism such that  $g(\mathbf{e})$  and  $h(\mathbf{e})$  are invertible elements of  $\mathcal{A}$ . Then there exists a homomorphism  $\theta : \mathcal{A} \rightarrow \mathcal{A}$  such that*

- $f(ab) = f(a)\theta(b) = \theta(a)f(b)$ ;
- $g(ab) = g(a)\theta(b) = \theta(a)g(b)$ ;
- $h(ab) = h(a)\theta(b) = \theta(a)h(b)$ .

for all  $a, b \in \mathcal{A}$ .

**Proof .** We define a linear mapping  $\theta : \mathcal{A} \rightarrow \mathcal{A}$  by  $\theta(a) = g(a)g(\mathbf{e})^{-1}$  for all  $a \in \mathcal{A}$ . So, we have  $\theta(a) = g(a)g(\mathbf{e})^{-1} = g(\mathbf{e})^{-1}g(a)$  for all  $a \in \mathcal{A}$  (see (4.9)). We know that  $h(a)g(b) = f(ab) = g(ab)h(\mathbf{e})$  for all  $a, b \in \mathcal{A}$  (see (4.2)). Putting  $h(a) = g(\mathbf{e})^{-1}h(\mathbf{e})g(a)$  (see (4.3)) in the former equation and using (4.4), we have

$$\begin{aligned} h(\mathbf{e})g(ab) &= g(ab)h(\mathbf{e}) \\ &= h(a)g(b) \\ &= g(\mathbf{e})^{-1}h(\mathbf{e})g(a)g(b) \\ &= h(\mathbf{e})g(\mathbf{e})^{-1}g(a)g(b) \quad (\text{see(4.6)}). \end{aligned}$$

So, we get that

$$g(ab) = g(\mathbf{e})^{-1}g(a)g(b) = \theta(a)g(b), \quad a, b \in \mathcal{A}.$$

In view of the previous equation and (4.9), we deduce that  $g(ab) = g(a)g(\mathbf{e})^{-1}g(b) = g(a)\theta(b)$  for all  $a, b \in \mathcal{A}$ . Therefore, we have

$$g(ab) = g(a)\theta(b) = \theta(a)g(b), \quad a, b \in \mathcal{A}. \tag{4.10}$$

Similar to the above argument, we can prove that  $\psi : \mathcal{A} \rightarrow \mathcal{A}$  defined by  $\psi(a) = h(\mathbf{e})^{-1}h(a) = h(a)h(\mathbf{e})^{-1}$  is a homomorphism and  $h(ab) = h(a)\psi(b) = \psi(a)h(b)$  for all  $a, b \in \mathcal{A}$ . Now, we show that  $\theta = \psi$ . We know that  $f(a) = g(a)h(\mathbf{e}) = h(a)g(\mathbf{e})$  for all  $a \in \mathcal{A}$ . So,  $h(a)g(b) = f(ab) = h(ab)g(\mathbf{e})$ , which means that  $h(ab) = h(a)g(b)g(\mathbf{e})^{-1} = h(a)\theta(b)$  for all  $a, b \in \mathcal{A}$ . We can thus deduce that

$$\psi(ab) = h(\mathbf{e})^{-1}h(ab) = h(\mathbf{e})^{-1}h(a)\theta(b) = \psi(a)\theta(b) \text{ for all } a, b \in \mathcal{A}. \tag{4.11}$$

Replacing  $a$  by  $\mathbf{e}$  in (4.11) and using the fact that  $\psi(\mathbf{e}) = \mathbf{e}$ , we get  $\psi(b) = \theta(b)$  for all  $b \in \mathcal{A}$  and this yields that  $\psi = \theta$ . Therefore,  $g(ab) = g(a)\theta(b) = \theta(a)g(b)$  and  $h(ab) = h(a)\theta(b) = \theta(a)h(b)$  for all  $a, b \in \mathcal{A}$ . In the rest of the proof, we show that  $f(ab) = f(a)\theta(b) = \theta(a)f(b)$  for all  $a, b \in \mathcal{A}$ . In view of (4.3), we have

$$\theta(a)f(b) = g(a)g(\mathbf{e})^{-1}g(\mathbf{e})h(b) = g(a)h(b) = f(ab). \tag{4.12}$$

Considering (4.2), we obtain

$$f(a)\theta(b) = h(a)g(\mathbf{e})g(\mathbf{e})^{-1}g(b) = h(a)g(b) = f(ab). \tag{4.13}$$

Comparing (4.12) and (4.13), we find that  $f(ab) = \theta(a)f(b) = f(a)\theta(b)$  for all  $a, b \in \mathcal{A}$ . This completes the proof of the theorem.  $\square$

**Remark 4.4.** Let  $f : \mathcal{A} \rightarrow \mathcal{A}$  be a  $\{g, h\}$ -homomorphism such that  $g(\mathbf{e})$  and  $h(\mathbf{e})$  are invertible elements of  $\mathcal{A}$ . According to Theorem 4.3, we know that  $f(ab) = f(a)\theta(b) = \theta(a)f(b)$ ,  $g(ab) = g(a)\theta(b) = \theta(a)g(b)$  and  $h(ab) = h(a)\theta(b) = \theta(a)h(b)$  for all  $a, b \in \mathcal{A}$ , where  $\theta : \mathcal{A} \rightarrow \mathcal{A}$  is a homomorphism. Considering  $\Theta = \frac{\theta}{2}$ , we have

$$\begin{aligned} f(ab) &= f(a)\Theta(b) + \Theta(a)f(b), \\ g(ab) &= g(a)\Theta(b) + \Theta(a)g(b), \\ h(ab) &= h(a)\Theta(b) + \Theta(a)h(b), \end{aligned}$$

for all  $a, b \in \mathcal{A}$ . Here, we show that  $f(ab) = f(a)\Theta(b) + \Theta(a)f(b)$ .

$$f(ab) = \frac{f(ab)}{2} + \frac{f(ab)}{2} = \frac{f(a)\theta(b)}{2} + \frac{\theta(a)f(b)}{2} = f(a)\Theta(b) + \Theta(a)f(b).$$

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