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New Construction of Triangle by Wilson Angle and $I-$Angle in Morrey Spaces

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Abstract

In this paper we discussed the concept of new triangle in Morrey spaces which is defined by using Wilson angle and $I-$angle. We also discuss about some fundamental properties of a triangle in Morrey spaces.

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Keywords: Morrey Spaces, Wilson Angle, $I-$angle, Triangle.

1. Introduction

In Euclidean spaces, triangle has been known since 2000 years ago. The triangle and its properties also has been written in many Euclidean spaces $\mathbb{R}^2$. Suh as Lang and Murrow [5] had proven that for any isosceles triangle, the opposite angle which induced by two sides is same. Next Aref and Wernick [3] also proved that for any triangle in a plane apply cosine rule, sine rule, and triangle side rule. Then, in [2] Anton and Rorres defined the triangles in inner product spaaces. That triangle is defined by using Phythagoras angle in inner product spaces. In new research Zakir, et al [9] had defined the triangle in normed spaces by using Wilson angle.

According to define a new triangle, we need an angle. There are some angle which defined by some author. It is firstly start from Valentine and Wayment who have covered the Wilson angle in normed spaces [8]. Then Milicic [6], has discussed the $B-$angle and $g-$angle in normed spaces. Furthermore, Gunawan, et al [4] have discussed the $P-$angle $I-$angle, and also $g-$angle. In 2011, again, Milicic had covered new angle in normed spaces named by the $thy-$angle [7].

In this paper, we will define the triangle by using Wilson angle and $I-$angle in Morrey spaces. Let $1 \le p \le q < \infty$, the Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^n)$ is a set of all the measurable function f in $\mathbb{R}^n$ such

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that

$$\|f\|_{\mathcal{M}_q^p} := \sup_{B(a,r)} |B|^{\frac{1}{q}-\frac{1}{p}} \left(\int_B^{\square} |f(x)|dx \right)^{\frac{1}{p}} < \infty$$

where $B(a,r)$ denotes the ball centered at $a \in \mathbb{R}^n$ having radius $r > 0$ and Lebesgue measure $|B|$[1]. Since Morrey spaces is Banach spaces, it is also normed spaces. Then, the Wilson angle and $I-$angle can be applied in Morrey spaces.

Let $(V, \|.\|)$, the Wilson angle between two vectors a and b has properties [9]

$$\angle_W(a,b) := arccos \left(\frac{\|a\|^2 + \|b\|^2 - \|a-b\|^2}{2\|a\|.\|b\|} \right). \tag{1}$$

Next, let $(V, \|.\|)$, the $I-$angle between two nonzero vectors a and b has properties [4]

$$\angle_I(a,b) := arccos \left(\frac{\|a \mp b\|^2 - \|a-b\|^2}{4\|a\|.\|b\|} \right). \tag{2}$$

Our result is discussed in the following section, by defining the triangle with Wilson angle and $I-$angle, and also their properties will show in some theorem.

2. Triangle with Wilson Angle on Morrey Spaces

Before we start to give some properties of triangle on Morrey spaces, let we define the triangle on Morrey spaces as definition below,

Definition 2.1. *Let $1 \leq p < q < \infty$. For any $f, g, h \in \mathcal{M}_q^p$, we define a triangle $\Delta[f, g, h]$ is f, g, h with $f + h = g$ and completed with Wilson angle $\angle_{W_{\mathcal{M}_q^p}}(f,g)$, $\angle_{W_{\mathcal{M}_q^p}}(f,h)$, and $\angle_{W_{\mathcal{M}_q^p}}(-f,h)$.*

In our result, we give three properties of the triangle which described in following theorems. The theorem below gave the properties of cosine rule and sine rule of the triangle which contructed by Wilson angle.

Theorem 2.2. *Let $1 \leq p < q < \infty$. The triangle $\Delta[f, g, h]$ in Morrey spaces $\mathcal{M}_q^p(\mathbb{R}^n)$, cosine rule is applied if only if the sine rules is applied.*

Proof . To prove this theorem, we have to prove by two side.

The proof from the left side, first we let $\cos \angle_{W_{\mathcal{M}_q^p}}(f,g) = \left(\frac{\|f\|^2_{\mathcal{M}_q^p} + \|g\|^2_{\mathcal{M}_q^p} - \|h\|^2_{\mathcal{M}_q^p}}{2\|f\|_{\mathcal{M}_q^p}.\|g\|_{\mathcal{M}_q^p}} \right)$ exist, then

$$\begin{aligned}
\sin^2 \angle_{W_{\mathcal{M}_q^p}}(f,g) &= 1 - \cos^2 \angle_{W_{\mathcal{M}_q^p}}(f,g) \\
&= 1 - \left(\frac{\|f\|^2_{\mathcal{M}_q^p} + \|g\|^2_{\mathcal{M}_q^p} - \|h\|^2_{\mathcal{M}_q^p}}{2\|f\|_{\mathcal{M}_q^p}.\|g\|_{\mathcal{M}_q^p}} \right) \\
&= \frac{16s(s-f)(s-g)(s-h)}{4\|f\|^2_{\mathcal{M}_q^p}\|g\|^2_{\mathcal{M}_q^p}} \\
\|f\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_q^p} \sin \angle_{W_{\mathcal{M}_q^p}}(f,g) &= 2\sqrt{s(s-f)(s-g)(s-h)} = k.
\end{aligned}$$

Therefore, we got the sine rule.

For the proof of right side, first we have noted by sine rule, we obtained:

$$\begin{aligned} & K^2\left(\|f\|^2_{\mathcal{M}^p_q}+\|g\|^2_{\mathcal{M}^p_q}-\|h\|^2_{\mathcal{M}^p_q}\right) \\ & = \|f\|^2_{\mathcal{M}^p_q}\|g\|^2_{\mathcal{M}^p_q}\|h\|^2_{\mathcal{M}^p_q}\left[\sin^2\angle_{W_{\mathcal{M}^p_q}}(g,h)\right. \\ & \left. + \sin^2\angle_{W_{\mathcal{M}^p_q}}(-f,h) - \sin^2\angle_{W_{\mathcal{M}^p_q}}(f,g)\right] \\ & = \|f\|^2_{\mathcal{M}^p_q}\|g\|^2_{\mathcal{M}^p_q}\|h\|^2_{\mathcal{M}^p_q}\left[\sin^2\angle_{W_{\mathcal{M}^p_q}}(g,h) + \sin^2\angle_{W_{\mathcal{M}^p_q}}(-f,h)\right. \\ & \left. - \sin^2\left(\angle_{W_{\mathcal{M}^p_q}}(g,h)+\angle_{W_{\mathcal{M}^p_q}}(-f,h)\right)\right] \\ & = 2K^2\|f\|_{\mathcal{M}^p_q}\|g\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(f,g) \end{aligned}$$

$$2\|f\|_{\mathcal{M}^p_q}\|g\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(f,g) = \|f\|^2_{\mathcal{M}^p_q}+\|g\|^2_{\mathcal{M}^p_q}-\|h\|^2_{\mathcal{M}^p_q}$$

From these, the cosine rule is applied. The proof is completed.□

In the second properties by theorem below, we discussed the triangle side rule on Morrey spaces by cosine rule which we obtained in Theorem 2.2.

Theorem 2.3. *Let $1 \le p < q < \infty$. The triangle $\Delta[f,g,h]$ in Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^n)$, cosine rule is applied if only if triangle side rule is applied.*

Proof . Since the proof is two side, we have to proof this twice. The proof from left to right is discussed below.

$$\|f\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(f,g) = \|f\|_{\mathcal{M}^p_q}\frac{\|f\|^2_{\mathcal{M}^p_q}+\|g\|^2_{\mathcal{M}^p_q}-\|h\|^2_{\mathcal{M}^p_q}}{2\|f\|_{\mathcal{M}^p_q}.\|g\|_{\mathcal{M}^p_q}} \tag{3}$$

$$\|h\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(g,h) = \|h\|_{\mathcal{M}^p_q}\frac{\|g\|^2_{\mathcal{M}^p_q}+\|h\|^2_{\mathcal{M}^p_q}-\|f\|^2_{\mathcal{M}^p_q}}{2\|g\|_{\mathcal{M}^p_q}.\|h\|_{\mathcal{M}^p_q}} \tag{3}$$

by summing (3) and (4), then we obtain the triangle rule:

$$\|f\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(f,g) + \|h\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(g,h) = \|g\|_{\mathcal{M}^p_q} \tag{5}$$

and then it is proven.

Next, we move to proof from right to left as described below.
By equation (5) next we obtain the triangle rule there are:

$$\|f\|^2_{\mathcal{M}^p_q} = \|f\|_{\mathcal{M}^p_q}\|g\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(f,g) + \|f\|_{\mathcal{M}^p_q}\|h\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(-f,h) \tag{6}$$

$$\|g\|^2_{\mathcal{M}^p_q} = \|f\|_{\mathcal{M}^p_q}\|g\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(f,g) + \|g\|_{\mathcal{M}^p_q}\|h\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(g,h) \tag{7}$$

$$\|h\|^2_{\mathcal{M}^p_q} = \|g\|_{\mathcal{M}^p_q}\|h\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(g,h) + \|g\|_{\mathcal{M}^p_q}\|h\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(-f,h) \tag{8}$$

then, by eliminating (6), (7), and (8), then we also obtain the cosine rule:

$$\|f\|^2_{\mathcal{M}^p_q}+\|g\|^2_{\mathcal{M}^p_q}-2\|f\|_{\mathcal{M}^p_q}\|g\|_{\mathcal{M}^p_q}\cos\angle_{W_{\mathcal{M}^p_q}}(f,g) = \|h\|^2_{\mathcal{M}^p_q} \tag{9}$$

Since, the proof from two side is proven, then the proof of this theorem is complete.□

At last, the properties of the following theorem is showed the sum of the angle of triangle on Morrey Spaces by Wilson angle is 180° orπ.

Theorem 2.4. *Let* $1 \leq p < q < \infty$. *If* $\Delta[f,g,h]$ *in Morrey spaces* $\mathcal{M}_q^p(\mathbb{R}^n)$, *then* $\angle_{W_{\mathcal{M}_q^p}}(f,g) + \angle_{W_{\mathcal{M}_q^p}}(f,h) + \angle_{W_{\mathcal{M}_q^p}}(-f,h) = \pi$.

Proof . ($\Rightarrow$) Let we define $f(x) := |x|^{-\dfrac{n}{q}}$, $g(x) := \chi_{(0,1)}(|x|)f(x)$, and $h(x) = f(x) - g(x)$. By changing the variable, we have

$$\|s^{n/q}g(s.)\|_{\mathcal{M}_q^p} = \|g\|_{\mathcal{M}_q^p}$$

and

$$\|s^{n/q}h(s.)\|_{\mathcal{M}_q^p} = \|h\|_{\mathcal{M}_q^p}$$

for all $s > 0$. Since

$$s^{n/q}g(tx) = \chi_{(0,1)}(s|x|)f(x)$$

and

$$s^{n/q}h(tx) = \chi_{(0,1)}(s|x|)f(x) - \chi_{[1,\infty)}(s|x|)f(x).$$

for $s > 0$ and $x \in \mathbb{R}^n$, from convergence monotonicity properties of Morrey spaces, then we have

$$\|f\|_{\mathcal{M}_q^p} = \|g\|_{\mathcal{M}_q^p} = \|h\|_{\mathcal{M}_q^p} \in (0,\infty).$$

since that, we can obtained,

$$\begin{aligned}
\angle_{W_{\mathcal{M}_q^p}}(f,g) &= \arccos\left(\frac{\|f\|^2_{\mathcal{M}_q^p} + \|g\|^2_{\mathcal{M}_q^p} - \|h\|^2_{\mathcal{M}_q^p}}{2\|f\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_q^p}}\right) \\
&= \frac{\pi}{3}. \\
\angle_{W_{\mathcal{M}_q^p}}(g,h) &= \arccos\left(\frac{\|g\|^2_{\mathcal{M}_q^p} + \|h\|^2_{\mathcal{M}_q^p} - \|f\|^2_{\mathcal{M}_q^p}}{2\|g\|_{\mathcal{M}_q^p}\|h\|_{\mathcal{M}_q^p}}\right) \\
&= \frac{\pi}{3}.
\end{aligned}$$

and

$$\begin{aligned}
\angle_{W_{\mathcal{M}_q^p}}(-f,h) &= \arccos\left(\frac{\|f\|^2_{\mathcal{M}_q^p} + \|h\|^2_{\mathcal{M}_q^p} - \|g\|^2_{\mathcal{M}_q^p}}{2\|f\|_{\mathcal{M}_q^p}\|h\|_{\mathcal{M}_q^p}}\right) \\
&= \frac{\pi}{3}.
\end{aligned}$$

Eventually, the sum of these angle $\Delta[f,g,h]$ is

$$\angle_{W_{\mathcal{M}_q^p}}(f,g) + \angle_{W_{\mathcal{M}_q^p}}(g,h) + \angle_{W_{\mathcal{M}_q^p}}(-f,h) = \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} = \pi$$

□

3. Triangle with I-Angle on Morrey Spaces

We also discussed the triangle which constructed by I-angle. The idea for constructing this triangle is same as the triangle by using Wilson angle. We start for the definition of triangle on Morrey spaces by using $I-$angle.

Definition 3.1. *Let $1 \leq p < q < \infty$. For any $f, g, h \in \mathcal{M}_q^p$, we define a triangle denoted by $\Delta[f, g, h]$ is f, g, h which is f+h=g and completed by $I-$angle $\angle_{I_{\mathcal{M}_q^p}}(f, g)$, $\angle_{I_{\mathcal{M}_q^p}}(f, h)$, $\angle_{I_{\mathcal{M}_q^p}}(-f, h)$.*

Such as the the triangle by using Wilson angle, the triangle by using $I-$angle also have three properties which described in the following theorem.

Theorem 3.2. *Let $1 \leq p < q < \infty$. The triangle $\Delta[f, g, h]$ Morrey spaces $\mathcal{M}_q^p(\mathbb{R}^n)$, cosine rule is applied if and only if sine rule is applied.*

Proof . *Such as the Theorem 2.2, we have to suppose that* $\cos \angle_{I_{\mathcal{M}_q^p}}(f, g) = \left(\frac{\|f+g\|^2_{\mathcal{M}_q^p} - \|f-g\|^2_{\mathcal{M}_q^p}}{4\|f\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_q^p}} \right)$ *exists, then we obtained,*

$$\begin{aligned}
\sin^2 \angle_{I_{\mathcal{M}_q^p}}(f, g) &= 1 - \cos^2 \angle_{I_{\mathcal{M}_q^p}}(f, g) \\
&= 1 - \left(\frac{\|f+g\|^2_{\mathcal{M}_q^p} - \|f-g\|^2_{\mathcal{M}_q^p}}{4\|f\|_{\mathcal{M}_q^p}.\|g\|_{\mathcal{M}_q^p}} \right)^2 \\
&= \frac{\left(4\|f\|_{\mathcal{M}_q^p}.\|g\|_{\mathcal{M}_q^p}\right)^2 - \left(\|f+g\|^2_{\mathcal{M}_q^p} - \|f-g\|^2_{\mathcal{M}_q^p}\right)^2}{16\|f\|^2_{\mathcal{M}_q^p}\|g\|^2_{\mathcal{M}_q^p}} \\
\sin \angle_{I_{\mathcal{M}_q^p}}(f, g) &= \frac{\sqrt{\left(4\|f\|_{\mathcal{M}_q^p}.\|g\|_{\mathcal{M}_q^p}\right)^2 - \left(\|f+g\|^2_{\mathcal{M}_q^p} - \|f-g\|^2_{\mathcal{M}_q^p}\right)^2}}{4\|f\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_q^p}} \\
\|f\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_q^p} \sin \angle_{I_{\mathcal{M}_q^p}}(f, g) &= \frac{\sqrt{\left(4\|f\|_{\mathcal{M}_q^p}.\|g\|_{\mathcal{M}_q^p}\right)^2 - \left(\|f+g\|^2_{\mathcal{M}_q^p} - \|f-g\|^2_{\mathcal{M}_q^p}\right)^2}}{4} \\
&= K
\end{aligned}$$

Therefore, we obtained the sine rule. For the proof of right side, we can look at for the sine rule, then we have:

$$\begin{aligned}
& K^2 \left(\|f\|^2_{\mathcal{M}_q^p} + \|g\|^2_{\mathcal{M}_q^p} - \|h\|^2_{\mathcal{M}_q^p} \right) \\
& = \|f\|^2_{\mathcal{M}_q^p}\|g\|^2_{\mathcal{M}_q^p}\|h\|^2_{\mathcal{M}_q^p} \big[\sin^2 \angle_{I_{\mathcal{M}_q^p}}(g, h) \\
& + \sin^2 \angle_{I_{\mathcal{M}_q^p}}(-f, h) - \sin^2 \angle_{I_{\mathcal{M}_q^p}}(f, g) \big] \\
& = \|f\|^2_{\mathcal{M}_q^p}\|g\|^2_{\mathcal{M}_q^p}\|h\|^2_{\mathcal{M}_q^p} \big[\sin^2 \angle_{I_{\mathcal{M}_q^p}}(g, h) + \sin^2 \angle_{I_{\mathcal{M}_q^p}}(-f, h) \\
& - \sin^2 \left(\angle_{I_{\mathcal{M}_q^p}}(g, h) + \angle_{I_{\mathcal{M}_q^p}}(-f, h) \right) \big]
\end{aligned}$$

$$\begin{aligned}
\|f\|^2_{\mathcal{M}_q^p} + \|g\|^2_{\mathcal{M}_q^p} - \|h\|^2_{\mathcal{M}_q^p} &= 2\|f\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(f, g) \\
\|f+g\|^2_{\mathcal{M}_q^p} - \|h\|^2_{\mathcal{M}_q^p} &\leq 2\|f\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(f, g) \\
\|f+g\|^2_{\mathcal{M}_q^p} - \|h\|^2_{\mathcal{M}_q^p} &\leq 4\|f\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(f, g)
\end{aligned}$$

So that, the cosine rule is obtained, and it completed the proof. □

We next to second propperties which discussed the triangle by using I-angle. It will described in the following theorem.

Theorem 3.3. *Let $1 \leq p < q < \infty$. The triangle $\Delta[f, g, h]$ in Morrey spaces $\mathcal{M}_q^p(\mathbb{R}^n)$, the cosin rule is applied if only if the triangle side is applied.*

Proof . We have to prove by two side proof. For the left proof, let we defined $f(x) := |x|^{-\frac{n}{q}}$, $g(x) := \chi_{(0,1)}(|x|)f(x)$, $k(x) := 2g(x) - f(x)$, and $h(x) = f(x) - g(x)$.
By changing the variable, we can get:

$$\|s^{\frac{n}{q}} g(s.)\|_{\mathcal{M}_q^p} = \|g\|_{\mathcal{M}_q^p}$$

and

$$\|s^{\frac{n}{q}} h(s.)\|_{\mathcal{M}_q^p} = \|h\|_{\mathcal{M}_q^p}$$

for all $s > 0$. Since

$$s^{\frac{n}{q}} g(tx) = \chi_{(0,1)}(s|x|)f(x) - \chi_{[1,\infty)}(s|x|)f(x).$$

for $s > 0$ and $x \in \mathbb{R}^n$, by monotonicity convergence properties of Morrey spaces we have,

$$\|f\|_{\mathcal{M}_q^p} = \|g\|_{\mathcal{M}_q^p} = \|h\|_{\mathcal{M}_q^p} = \|h\|_{\mathcal{M}_q^p} \in (0, \infty).$$

Note that,

$$\|f\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(f, g) = \|f\|_{\mathcal{M}_q^p} \frac{\|f+g\|^2_{\mathcal{M}_q^p} - \|f-g\|^2_{\mathcal{M}_q^p}}{4\|f\|_{\mathcal{M}_q^p}.\|g\|_{\mathcal{M}_q^p}} \tag{10}$$

$$\|h\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(g, h) = \|h\|_{\mathcal{M}_q^p} \frac{\|g+h\|^2_{\mathcal{M}_q^p} - \|g-h\|^2_{\mathcal{M}_q^p}}{4\|g\|_{\mathcal{M}_q^p}.\|h\|_{\mathcal{M}_q^p}} \tag{11}$$

By summing (10) and (11), then we obtained the triangle side rule:

$$\|f\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(f, g) + \|h\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(g, h) = \frac{3}{4}\|f\|_{\mathcal{M}_q^p}$$
$$\frac{4}{3}(\|f\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(f, g) + \|h\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(g, h)) = \|g\|_{\mathcal{M}_q^p}(10)$$

Then we move to the right proof. From equation (??) we get the triangle side rule:

$$\|f\|^2_{\mathcal{M}_q^p} = \frac{4}{3}(\|f\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(f, g) + \|f\|_{\mathcal{M}_q^p}\|h\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(-f, h)) \tag{12}$$

$$\|g\|^2_{\mathcal{M}_q^p} = \frac{4}{3}(\|f\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(f, g) + \|g\|_{\mathcal{M}_q^p}\|h\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(g, h)) \tag{13}$$

$$\|h\|^2_{\mathcal{M}_q^p} = \frac{4}{3}(\|g\|_{\mathcal{M}_q^p}\|h\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(g, h) + \|f\|_{\mathcal{M}_q^p}\|h\|_{\mathcal{M}_q^p} \cos \angle_{I_{\mathcal{M}_q^p}}(-f, h)) \tag{14}$$

By eliminating (12), (13), and (14) then we obtained the cosine rule:

$$\|f\|^2_{\mathcal{M}^p_q} + \|g\|^2_{\mathcal{M}^p_q} + \|h\|^2_{\mathcal{M}^p_q} = \frac{8}{2}\|f\|_{\mathcal{M}^p_q}\|g\|_{\mathcal{M}^p_q}\cos\angle_{I_{\mathcal{M}^p_q}}(f,g)$$
$$\|f+g\|^2_{\mathcal{M}^p_q} + \|h\|^2_{\mathcal{M}^p_q} = \frac{8}{2}\|f\|_{\mathcal{M}^p_q}\|g\|_{\mathcal{M}^p_q}\cos\angle_{I_{\mathcal{M}^p_q}}(f,g)$$
$$\|f+g\|^2_{\mathcal{M}^p_q} + \|h\|^2_{\mathcal{M}^p_q} = 4\|f\|_{\mathcal{M}^p_q}\|g\|_{\mathcal{M}^p_q}\cos\angle_{I_{\mathcal{M}^p_q}}(f,g)$$

At last, the proof on this theorem is completed.□

The last properties of the triangle in our result is sum of angles in the triangle. Then, we had proven that the sum of angles in the triangle by using $I-$angle on Morrey spaces is 180° or π. It will described in the following theorem.

Theorem 3.4. *Let* $1 \leq p < q < \infty$. *If* $\Delta[f,g,h]$ *in Morrey spaces* $\mathcal{M}^p_q(\mathbb{R}^n)$, *then* $\angle_{I_{\mathcal{M}^p_q}}(f,g) + \angle_{I_{\mathcal{M}^p_q}}(f,h) + \angle_{I_{\mathcal{M}^p_q}}(-f,h) = \pi$.

Proof . The norms which defined in proof of throrem 3.3, we obtained

$$\|f\|^2_{\mathcal{M}^p_q} = \|g\|^2_{\mathcal{M}^p_q} = \|k\|^2_{\mathcal{M}^p_q} = \|h\|^2_{\mathcal{M}^p_q}$$

Note that:

$$\begin{aligned}
\angle_{I_{\mathcal{M}^p_q}}(f,g) &= \arccos\left(\frac{\|f+g\|^2_{\mathcal{M}^p_q} - \|f-g\|^2_{\mathcal{M}^p_q}}{4\|f\|_{\mathcal{M}^p_q}.\|g\|_{\mathcal{M}^p_q}}\right) \\
&= \arccos\left(\frac{4\|f\|^2_{\mathcal{M}^p_q} - \|h\|^2_{\mathcal{M}^p_q}}{4\|f\|^2_{\mathcal{M}^p_q}}\right) \\
&= \arccos\left(\frac{4\|f\|^2_{\mathcal{M}^p_q} - \|f\|^2_{\mathcal{M}^p_q}}{4\|f\|^2_{\mathcal{M}^p_q}}\right) \\
&= \arccos\left(\frac{3\|f\|^2_{\mathcal{M}^p_q}}{4\|f\|^2_{\mathcal{M}^p_q}}\right) \\
&= 41.4096,
\end{aligned}$$

$$\begin{aligned}
\angle_{I_{\mathcal{M}^p_q}}(g,h) &= \arccos\left(\frac{\|g+h\|^2_{\mathcal{M}^p_q} - \|g-h\|^2_{\mathcal{M}^p_q}}{4\|g\|_{\mathcal{M}^p_q}.\|h\|_{\mathcal{M}^p_q}}\right) \\
&= \arccos\left(\frac{\|f\|^2_{\mathcal{M}^p_q} - 4\|f\|^2_{\mathcal{M}^p_q}}{4\|f\|^2_{\mathcal{M}^p_q}}\right) \\
&= \arccos\left(\frac{-3\|f\|^2_{\mathcal{M}^p_q}}{4\|f\|^2_{\mathcal{M}^p_q}}\right) \\
&= 138.5904,
\end{aligned}$$

and

$$\begin{aligned}
\angle_{I_{\mathcal{M}_q^p}}(-f,h) &= \arccos\left(\frac{\|h+f\|^2_{\mathcal{M}_q^p} - \|h-f\|^2_{\mathcal{M}_q^p}}{4\|h\|_{\mathcal{M}_q^p}.\|f\|_{\mathcal{M}_q^p}}\right)\\
&= \arccos\left(\frac{\|2h+g\|^2_{\mathcal{M}_q^p} - \|-g\|^2_{\mathcal{M}_q^p}}{4\|f\|^2_{\mathcal{M}_q^p}}\right)\\
&\geq \arccos\left(\frac{4\|f\|^2_{\mathcal{M}_q^p}\|f\|^2_{\mathcal{M}_q^p} - \|f\|^2_{\mathcal{M}_q^p}}{4\|f\|^2_{\mathcal{M}_q^p}}\right)\\
&\geq \arccos\left(\frac{4\|f\|^2_{\mathcal{M}_q^p}}{4\|f\|^2_{\mathcal{M}_q^p}}\right)\\
&= 0
\end{aligned}$$

Therefore, the sume of the angles $[f,g,h]$ are

$$\angle_{I_{\mathcal{M}_q^p}}(f,g) + \angle_{I_{\mathcal{M}_q^p}}(g,h) + \angle_{I_{\mathcal{M}_q^p}}(-f,h) = 41.4096 + 138.5904 + 0 = 180 = \pi$$

□

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