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Continuity in fundamental locally multiplicative topological algebras

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Abstract

In this paper, we first derive specific results concerning the continuity and upper semi-continuity of the spectral radius and spectrum functions on fundamental locally multiplicative topological algebras. We continue our investigation by further determining the automatic continuity of linear mappings and homomorphisms in these algebras.

Keywords: FLM algebra, continuity, spectral radius, spectrum function, homomorphism.
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1. Introduction and Preliminaries

Non-normed topological algebras were initially introduced around the year 1950 for the investigation of certain classes of these algebras that appeared naturally in mathematics and physics. Some results concerning such topological algebras had been published earlier in 1947 by R. Arens [14]. It was in 1952 that Arens and Michael [14] independently published the first systematic study on locally m-convex algebras, which constitutes an important class of non-normed topological algebras. Here, we would like to mention about the predictions made by the famous Soviet mathematician M.A. Naimark, an expert in the area of Banach algebras, in 1950 regarding the importance of non-normed algebras and the development of their related theory. During his study concerning cosmology, G. Lassner [14] realized that the theory of normed topological algebras was insufficient for his study purposes.

Ansari [3] introduced the notion of fundamental topological spaces and algebras and proved the Cohen's factorization theorem for these algebras. Fundamental locally multiplicative (FLM)

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topological algebras with a property similar to normed algebras were introduced later by Ansari [4]. Some celebrated theorems of Banach algebras have been generalized for FLM algebras in the past studies [4, 5]. Newburgh [18] introduced the concept of spectral continuity and proved that the spectrum function is upper semi-continuous on any Banach algebra. He gave a first sufficient condition for continuity of the spectrum function at a point of a Banach algebra. Since then, this topic has been studied widely by many researchers and mathematicians. The most outstanding results in this direction are due to Aupetit, Burlando and Daoultzi-Malamou who have generalized the results of Newburgh in certain Banach algebras (see [7, 10, 12, 13]).

Continuity of the spectrum and spectral radius functions play a crucial role in automatic continuity. Automatic continuity of linear mappings and homomorphisms are very important in advanced studies on topological algebras and mathematical analysis. The starting point for automatic continuity theory is the easily proved fact that every homomorphism from a Banach algebra onto the complex field is automaticly continuous [7, 20]. It follows easily from the continuity of multiplicative linear functionals that every homomorphism from a Banach algebra into a commutative semi-simple Banach algebra is continuous. A famous theorem due to Johnson [17] extends this result to arbitrary semi-simple Banach algebras. Some results for automatic continuity in the area of Banach and Frechet algebras have also been obtained by Aupetit [7], and Ghasemi-Honary [15].

Ansari [4] showed in 2001 that every multiplicative linear functional on a complete metrizable FLM algebra is continuous, which leads easily to the continuity of all homomorphisms from a complete metrizable FLM algebra into a semi-simple commutative complete metrizable FLM algebra, but it remains an intriguing open question, commonly known as Michael's problem, whether all multiplicative linear functionals on complete metrizable topological algebras are continuous.

The aim of this paper is twofold. Firstly, we obtain some results concerning the continuity and upper semi-continuity of the spectral radius and spectrum functions in FLM algebras. Several examples of spectral continuity are discussed as well. Secondly, the automatic continuity of linear mappings and homomorphisms in these algebras are investigated.

This paper is divided into the following sections. In section 2, we have gathered a collection of definitions and known results, and in section 3, we derive some results concerning the continuity and upper semi-continuity of the spectral radius and spectrum functions in FLM algebras. In section 4, we investigate the automatic continuity of linear mappings and algebra homomorphisms. Finally, we close the paper with a conclusion.

In the present paper, all theorems are proved in different ways in FLM algebras without using the concept of the local boundedness and local convexity.

2. Definitions and known results

In this section, we present a collection of definitions and known results, which are included in the list of our references.

Definition 2.1. *A topological linear space A is said to be locally bounded if there exists a bounded neighborhood U of zero. A locally bounded algebra is an algebra whose underlying topological linear space is locally bounded.*

It is well known that a topological linear space A is locally bounded if its topology may be given by means of a p-homogeneous norm $\|.\|_p$, $0 < p \leqslant 1$, i.e., a non-negative function $x \mapsto \|x\|_p$ satisfying

(i) $\|x\|_p \geqslant 0$, *and* $\|x\|_p = 0$ *if and only if* $x = 0$;

(ii) $\|x + y\|_p \leqslant \|x\|_p + \|y\|_p$ *for all* $x, y \in A$;

(iii) $\|\lambda x\|_p = |\lambda|^p \|x\|_p$ *for all* $x \in A$ *and* $\lambda \in \mathbb{F}$.

By a p-normed algebra $(A, \|.\|_p)$*, we mean an algebra* A *endowed with a p-homogeneous norm* $\|.\|_p$ *such that* $\|xy\|_p \leqslant \|x\|_p \|y\|_p$ *for all* $x, y \in A$*. For further details one can refer to [8].*

Definition 2.2. *[3, 2.1] A topological linear space* A *is said to be a fundamental if there exists* $b > 1$ *such that for every sequence* $(x_n)_n$ *of* A*, the convergence of* $b^n(x_n - x_{n-1})$ *to zero in* A *implies that* $(x_n)_n$ *is Cauchy.*

Definition 2.3. *[3, 2.3] A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental.*

Definition 2.4. *[4, 4.2] A fundamental topological algebra is said to be locally multiplicative if there exists a neighborhood* U_0 *of zero such that for every neighborhood* V *of zero, the sufficiently large powers of* U_0 *lie in* V*. Such an algebra is known as an FLM algebra.*

It is easy to see that every locally bounded algebra A *is an FLM algebra but the converse do not hold in general. For instance, the FLM algebra* $A \oplus B$ *of Example 3.8 in Section 3 is not locally bounded. However, if* A *is unital, the converse is true. For further details one can refer to [5].*

As pointed out earlier, our proofs for main results will be in different ways without using the notion of the local boundedness.

Theorem 2.5. *[4, 4.5] Let* A *be a unital complete metrizable FLM algebra. Then every multiplicative linear functional on* A *is automatically continuous.*

Definition 2.6. *Let* A *be a unital algebra. The set of all invertible elements of* A *is denoted by* $Inv(A)$.

Definition 2.7. *For a unital algebra* A*, the spectrum* $sp_A(x)$ *of an element* $x \in A$ *is the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda e - x$ *is not invertible in* A*. The spectral radius* $r_A(x)$ *of an element* $x \in A$ *is defined by* $r_A(x) = \sup\{|\lambda| : \lambda \in sp_A(x)\}$.

For a unital topological algebra A, we take $r_A(x) = +\infty$ if $sp_A(x)$ is unbounded and $r_A(x) = 0$ if $sp_A(x) = \emptyset$.

Definition 2.8. *Given elements* x, y *of* A*, the quasi-product of* x, y *is the element* $x \circ y$ *of* A *defined by*

$$x \circ y = x + y - xy.$$

Also, we say that an element x *in* A *is quasi-invertible, if*

$$x \circ y = y \circ x = 0, \text{ for some } y \in A.$$

The quasi-inverse of a quasi-invertible element is denoted by x^0*, the set of all quasi-invertible elements of* A *by* $q - \mathrm{Inv}(A)$*. If* A *does not have a unit element, the spectrum* $sp_A(x)$ *of* $x \in A$ *is defined by*

$$sp_A(x) = \left\{\lambda \in \mathbb{C} - \{0\} : \frac{x}{\lambda} \text{is not quasi} - \text{invertible}\right\} \cup \{0\}.$$

For further information one can refer to [9], [14].

Theorem 2.9. *[4, 4.4] Let A be a unital complete metrizable FLM algebra and $a \in A$. Then the $sp_A(a)$ is compact.*

Definition 2.10. *[22, 3.1] Let (A, d_A) be a metrizable topological algebra. We say that A is a sub-multiplicative metrizable topological algebra if*

$$d_A(0, xy) \leq d_A(0, x)d_A(0, y)$$

for all $x, y \in A$, where d_A is a translation invariant metric on A.

Definition 2.11. *Let A and B be metrizable topological linear spaces and let $T : A \to B$ be a linear mapping. The separating space of T is defined by*

$$G(T) = \{y \in B : \text{there exists}(x_n)_n \text{ in } A \; s.t. \; x_n \to 0 \text{ and } Tx_n \to y\}.$$

The separating space $G(T)$ is a closed linear subspace of B. Moreover, by the Closed Graph Theorem, T is continuous if and only if $G(T) = \{0\}$ [11, 5.1.2].

Definition 2.12. *[1, 2.13] Let x be an element of a topological algebra A. We say that x is bounded if there exists some $r > 0$ such that the sequence $(\frac{x^n}{r^n})_n$ converges to zero. The radius of boundedness of x with respect to A is denoted by $\beta_A(x)$ and defined by*

$$\beta_A(x) = \inf \left\{ r > 0 : \left(\frac{x^n}{r^n} \right) \to 0 \right\},$$

with the convention : $\inf \varnothing = +\infty$.

Lemma 2.13. *[11, 1.5.32] If A is a unital algebra, then*

$$\text{rad}A = \{x \in A : r_A(xy) = 0; \text{ for every} y \in A\},$$

where $\text{rad}A$ is the Jacobson radical of A.

Definition 2.14. *Let A and B be two topological spaces. The set-valued mapping $\varphi : A \longrightarrow 2^B$ is said to be upper (resp. lower) semi-continuous at $a \in A$ if for every open set U in B with $\varphi(a) \subseteq U$ (resp. $U \cap \varphi(a) \neq \emptyset$), there exists a neighborhood V of a in A such that $\varphi(x) \subseteq U$ (resp. $U \cap \varphi(x) \neq \emptyset$) for every $x \in V$. Notice that φ is continuous at a if and only if φ is both upper and lower semi-continuous at a.*
Let A be a complete metrizable topological space and $K_{\mathbb{C}}$ be the set of compact nonempty subsets of complex plane $\mathbb{C}$, endowed with Hausdorff metric. It is well known that $\varphi : A \longrightarrow K_{\mathbb{C}} \cup \{\emptyset\}$ is upper (resp. lower) semi-continuous at $x \in A$ if and only if

$$\lim_{n \to \infty} \sup \varphi(x_n) \subseteq \varphi(x), \; (\varphi(x) \subseteq \lim_{n \to \infty} \inf \varphi(x_n))$$

for every sequence $(x_n)_n$ of elements of A which converges to x (see [10]).

3. New results for the spectral radius and spectrum functions

In this section, we obtain some results concerning the continuity of the spectral radius function at zero and upper semi-continuity of the spectral radius and spectrum functions on complete metrizable FLM algebras.

Lemma 3.1. *Let A be a complete metrizable fundamental topological algebra and $x \in A$. Then*
(i) $\beta_A(x) < 1$ implies that x is quasi-invertible and $x^0 = -\sum_{n=1}^{\infty} x^n$;
(ii) $r_A(x) \leq \beta_A(x)$.

Proof . (i) Let $\lambda > 1$ and $\beta_A(x) < \frac{1}{\lambda} < 1$. Then $\lambda^n x^n \to 0$ as $n \to \infty$. Put $s_n = \sum_{k=1}^{n} x^k$. We have $\lambda^n(s_n - s_{n-1}) = \lambda^n x^n \to 0$ as $n \to \infty$. Fundamentality of A implies that s_n is a Cauchy sequence. Let $s_n \to y$ as $n \to \infty$. Then

$$x \circ s_n = x + s_n - xs_n = x + x - x^{n+1}.$$

Since the multiplication is continuous on A, we get

$$x + y - xy = x + x.$$

Hence,

$$x \circ (-y) = 0.$$

Thus,

$$x^0 = -y = -\sum_{n=1}^{\infty} x^n.$$

(ii) Let $0 \neq \lambda \in \mathbb{C}$ such that $\beta_A(x) < |\lambda|$. Then $\beta_A(\frac{x}{\lambda}) < 1$. By the first part, we get $\frac{x}{\lambda} \in q{-}Inv(A)$ and so $\lambda \notin sp_A(x)$. Hence, $r_A(x) \leq \beta_A(x)$. □

Theorem 3.2. *Let A be a complete metrizable FLM algebra. Then r_A is continuous at zero.*

Proof . Let $(x_k)_k \subseteq A$ be a sequence such that $x_k \to 0$ and let U_0 be a neighborhood of zero in A satisfying Definition 2.4. Let $\varepsilon > 0$. there exists $k_0 \in N$ such that $2\varepsilon^{-1}x_k \in U_0$ for all $k \geq k_0$. Assume that V is any neighborhood of zero. Then there exists $n_0 \in N$ such that $n \geq n_0$ implies that $U_0^n \subseteq V$ and so $(2\varepsilon^{-1}x_k)^n \to 0$ as $n \to \infty$, for every $k \geq k_0$. Therefore, $\beta_A(x_k) \leq \frac{\varepsilon}{2} < \varepsilon$, which implies that β_A is continuous at zero. From this and Lemma 3.1, we conclude that the spectral radius $x \longmapsto r_A(x)$ is continuous at zero.
□

Remark 3.3. *In Theorem 3.2, the spectral radius function may be discontinuous at other points. The example of P.G. Dixon discussed in [11, 2.3.15] shows that the spectral radius function may be discontinuous at other points in a Banach algebra, also in a complete metrizable FLM algebra.*

Theorem 3.4. *Let A be a complete metrizable FLM algebra. If A is commutative, then r_A is continuous on A.*

Proof . Let $Z = A \oplus \mathbb{C}$ be the unitization of the algebra A. Then the algebra multiplication is defined by

$$(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$$

for all $x, y \in A$ and $\alpha, \beta \in \mathbb{C}$. By [4, p. 61], Z is a complete metrizable fundamental topological algebra. Assume that Z is not an FLM algebra. Let $(U_n)_n = (V_n \times B_n)_n$ be a sequence of a base of neighborhoods of 0 in Z, where $B_n = \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{n}\}$. Since Z is not an FLM algebra, there is a neighborhood $W = V \times B$ of 0 (depending on n) such that

$$(V_n \times \{0\})^k \subseteq U_n^k \nsubseteq W \text{ as } k \longrightarrow \infty, \text{ for every} n \geqslant 1.$$

This implies that

$$(V_n \times \{0\})^k \nsubseteq V \times \{0\} \text{ as } k \longrightarrow \infty, \text{ for every } n \geqslant 1.$$

Since we identify $(a, 0)$ with $a \in A$, then we have

$$V_n^k \nsubseteq V \text{ as } k \longrightarrow \infty, \text{ for every } n \geqslant 1,$$

violating the assumption that A is an FLM algebra. Hence, Z is a complete metrizable FLM algebra. Since the unitization of a commutative algebra is commutative, we may assume that A has a unit element. We define the Gelfand transform $\hat{a}$ of $a \in A$ by $\hat{a}(\varphi) = \varphi(a)$ for all $\varphi \in \phi_A$, where ϕ_A is the Gelfand spectrum of A. By [5, 5.5], we have

$$sp_A(a) = \{\varphi(a) : \varphi \in \phi_A\} = \hat{a}(\phi_A),$$

Hence, we obtain

$$sp_A(x + y) = \mathrm{Im}(x + y)^\wedge = \mathrm{Im}(\hat{x} + \hat{y}) \subseteq \mathrm{Im}\hat{x} + \mathrm{Im}\hat{y} = sp_A(x) + sp_A(y);$$

consequently,

$$r_A(x + y) \leq r_A(x) + r_A(y), \text{ for all } x, y \in A.$$

$$|r_A(x) - r_A(y)| \leq r_A(x - y).$$

From this and the continuity of r_A at zero, we conclude that the spectral radius function is continuous on A. □

Remark 3.5. *The example of P.G. Dixon mentioned in Remark 3.3 shows that the assumption of commutativity for A is essential in the previous Theorem.*

Example 3.6. *The algebra $C(\mathbb{R})$ of all continuous complex-valued functions on the real line $\mathbb{R}$ with the sequence $(p_n)_n$ of seminorms defined by $p_n(f) = \sup_{|x| \leq n} |f(x)|$ is a complete metrizable fundamental topological algebra, but not a complete metrizable FLM algebra. It can be readily concluded that the spectral radius function is not continuous at zero. This example shows that the spectral radius function may be discontinuous at zero in general.*

Example 3.7. *Let (A, d_A) and (B, d_B) be complete metrizable FLM algebras with metrics d_A and d_B respectively. Then $A \oplus B$ with product topology and pointwise defined algebraic operations is a complete metrizable topological algebra. By the definition of FLM algebras, $A \oplus B$ is a complete metrizable FLM algebra with the following metric*

$$d((x_1, y_1), (x_2, y_2)) = d_A(x_1, x_2) + d_B(y_1, y_2)$$

for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$. By Theorem 3.2, $r_{A\oplus B}$ is continuous at zero. Moreover, if A and B are commutative, then $r_{A\oplus B}$ is continuous on $A \oplus B$.

Example 3.8. *Let A be a complete locally bounded topological algebra with metric d_A which is not locally convex, and B be a complete metrizable locally convex vector space which is not locally bounded. Setting $xy = 0$ for all $x, y \in B$, we get a complete metrizable locally convex algebra with metric d_B. By the usual pointwise defined algebraic operations, $A \oplus B$ becomes a complete metrizable FLM algebra with metric d defined in Example 3.7. By Theorem 3.2, $r_{A\oplus B}$ is continuous at zero.*

Lemma 3.9. *Let A be a complete p-normed algebra and $a \in A$. Then*

$$\beta_A(a) = \lim_{n\to\infty} (\|a^n\|_p)^{\frac{1}{np}}.$$

Proof . If in [8, 3.3.6], we replace the function p by $\|\cdot\|_p$, then $\lim_{n\to\infty} \|a^n\|_p^{\frac{1}{n}}$ exists. Since

$$\lim_{n\to\infty} \|a^n\|_p^{\frac{1}{np}} = \left(\lim_{n\to\infty} \|a^n\|_p^{\frac{1}{n}}\right)^{\frac{1}{p}},$$

it follows that $\lim_{n\to\infty} \|a^n\|_p^{\frac{1}{np}}$ also exists. The result now follows from [19, Proposition1]. □

The following example shows that the converse of Theorem 3.2 may be false in general.

Example 3.10. *Let A be a commutative complete locally bounded topological algebra with metric d_A which is not locally convex and, X be a complete metrizable locally convex topological vector space with metric d_X which is not locally bounded. Denote by e the unit element of A. Suppose $(a, x) \to xa$ is a bilinear and continuous mapping from $A \oplus X$ into X satisfying $x(a_1a_2) = (xa_1)a_2$ and $xe = x$ for all $a_1, a_2 \in A$ and $x \in X$. Then X is a topological unit linked right A-module with module multiplication defined by $(a, x) \to xa$ and $Z = X \oplus A$ is a non-locally bounded, non-locally convex, fundamental topological vector space with pointwise defined algebraic operations and metric d_Z such that*

$$d_Z((x_1, a_1), (x_2, a_2)) = d_A(a_1, a_2) + d_X(x_1, x_2).$$

Define the multiplication on Z by

$$(x_1, a_1)(x_2, a_2) = (x_1a_2 + x_2a_1, a_1a_2)$$

for all $a_1, a_2 \in A$ and $x_1, x_2 \in X$. Now, Z is an algebra and since the module multiplication is continuous, Z is a unital complete metrizable fundamental topological algebra. Clearly, $(0, e)$ is the unit element of Z (see [6]). Suppose $z = (x, a) \in Z$, then $(x, a)^n = (nxa^{n-1}, a^n)$ for all $n \in N$. Now, we show that if $\beta_A(a) < r$ and $\lambda > 0$, then $\beta_Z(z) \le (1 + \lambda)r$, $z \in Z$. Since $\beta_A(a) < r$, then $\left(\frac{a}{r}\right)^n \to 0$ and so $\frac{1}{r^n}xa^{n-1} \to 0$ in X. Hence, $\frac{n}{r^n(1+\lambda)^n}xa^{n-1} \to 0$ in X and $\frac{1}{r^n(1+\lambda)^n}a^n \to 0$ in A. Therefore, $\frac{z^n}{r^n(1+\lambda)^n} \to 0$ in Z. This gives $\beta_Z(z) \le r(1 + \lambda)$. Since the complete locally bounded topological algebra A is a p-normed algebra, by Lemma 3.9, we have

$$\beta_A(a) = \lim_{n\to\infty} (\|a^n\|_p)^{\frac{1}{np}} \le (\|a\|_p)^{\frac{1}{p}} = (d_A(a, 0))^{\frac{1}{p}},\ 0 < p \le 1.$$

Set $r = (\|a\|_p)^{\frac{1}{p}} + \lambda$. Then

$$\beta_A(a) < (\|a\|_p)^{\frac{1}{p}} + \lambda.$$

Hence,

$$\beta_Z(z) \le (1 + \lambda)\left((\|a\|_p)^{\frac{1}{p}} + \lambda\right).$$

By Lemma 3.1 and since λ is arbitrary, we have

$$r_Z(z) \leq \beta_Z(z) \leq (\|a\|_p)^{\frac{1}{p}} \leq (d_Z(z,0))^{\frac{1}{p}}.$$

This implies that r_Z is continuous at zero. However, Z is not an FLM algebra.

Lemma 3.11. *Let A be a unital topological algebra whose the set of invertible elements is open and $a \in A$. Then*

$$sp_A(x) \subseteq B(0,r), \text{ for all} x \in a + W,$$

where W is a symmetric neighborhood of zero in A and $B(0,r)$ is a closed disk with radius r.

Proof . Let $U \subseteq A$ be a symmetric neighborhood of zero such that $e + U \subseteq \mathrm{Inv}(A)$. The continuity of scalar multiplication implies that for all $a \in A$, there exists a neighborhood W of zero in A and $\delta > 0$ such that $\alpha x \in U$ whenever $x \in a + W$ and $|\alpha| < \delta$. Hence, $e - \alpha^{-1}x \in \mathrm{Inv}(A)$, whenever $|\alpha| > \frac{1}{\delta} = r$ and $x \in a + W$. This implies that

$$sp_A(x) \subseteq B(0,r), \text{ for all} x \in a + W.$$

The proof is now complete. □

Theorem 3.12. *Let A be a unital complete metrizable topological algebra whose the set of invertible elements is open. Then the spectrum function $x \longmapsto sp_A(x)$ is upper semi-continuous on A.*

Proof . Suppose that the spectrum function is not upper semi-continuous at $a \in A$. Then there exists a neighborhood U of $sp_A(a)$ and a sequence $(x_n)_n$ with $x_n \to a$ such that

$$\forall n \exists \lambda_n \in sp_A(x_n) \cap (\mathbb{C} - U).$$

By Lemma 3.11, there exists $N \in \mathrm{N}$ such that for every $n > N$, $sp_A(x_n) \subseteq B(0,r)$. Consequently $(\lambda_n)_n$ is a bounded sequence. By the Bolzano-Weierstrass Theorem, we may suppose without loss of generality that it converges to λ. But $\lambda \notin U$ because $\mathbb{C} \backslash U$ is closed, so $\lambda e - a \in \mathrm{Inv}(A)$. Since $\lambda_n e - x_n \to \lambda e - a$ and $\mathrm{Inv}(A)$ is an open set in A, we obtain $\lambda_n e - x_n \in \mathrm{Inv}(A)$ for n large, which is a contradiction. □

Corollary 3.13. *If A is a unital complete metrizable FLM algebra, then the spectrum function $x \longmapsto sp_A(x)$ is upper semi-continuous on A.*

Proof . By [4, 4.3], $\mathrm{Inv}(A)$ is an open set in FLM algebra A. The result now follows from Theorem 3.12. □

Corollary 3.14. *Let A be a unital complete metrizable topological algebra whose the set of invertible elements is open (in particular, a unital complete metrizable FLM algebra). Then the spectral radius function $x \longmapsto r_A(x)$ is upper semi-continuous on A.*

Example 3.15. *Let A be a unital complete metrizable FLM algebra and $C(\mathbb{C},\mathbb{R})$ be the separable topological algebra of real continuous functions on $\mathbb{C}$. So we may assume that t_n is a dense sequence of functions in $C(\mathbb{C},\mathbb{R})$. Define*

$$z_n(x) = sup\{|t_n(\lambda)| : \lambda \in sp_A(x)\}, x \in A.$$

By Corollary 3.13, the z_n are upper semi-continuous.

Let A be a topological space and let $K_{\mathbb{C}} \cup \{\emptyset\}$ be as in Definition 2.14. A set-valued mapping $\varphi : A \longrightarrow K_{\mathbb{C}} \cup \{\emptyset\}$ assigns to each point x of A, a compact subset $\varphi(x)$ of $\mathbb{C}$. For any mapping φ, we denote by $\partial\varphi : A \longrightarrow K_{\mathbb{C}} \cup \{\emptyset\}$ the function which assings to each point x of A, the boundary of $\varphi(x)$.
The relationship between continuity of a set-valued mapping and continuity of its boundary is very important in spectral theory, especially in connection with continuity of spectrum and its boundary in certain topological algebras (see for instance, [10], [21]).

Theorem 3.16. *[21, Theorem 4] Let A be a first countable topological space and let $x \in A$. If $\varphi : A \longrightarrow K_{\mathbb{C}} \cup \{\emptyset\}$ is continuous at x, then $\partial\varphi$ is lower semi-continuous at x.*

Corollary 3.17. *Let A be a unital complete metrizable FlM algebra and $x \in A$. If the spectrum function is continuous at x, then ∂sp_A is lower semi-continuous at x.*

Proof . Since every element of a unital complete metrizable FlM algebra has a compact spectrum, the corollary is an immediate consequence of Theorem 3.16. □

The following theorem shows that lower semi-continuity of ∂sp_A implies continuity of r_A.

Theorem 3.18. *Let A be a unital complete metrizable topological algebra whose the set of invertible elements is open and $x \in A$. If ∂sp_A is lower semi-continuous at x, then the spectral radius function is continuous at x.*

Proof . Let $(x_n)_n$ be a sequence in A such that $x_n \to x$ as $n \to \infty$. Since ∂sp_A is lower semi-continuous at x, it follows that

$$\partial sp_A(x) \subseteq \lim_{n\to\infty} \inf \partial sp_A(x_n).$$

Thus, if $\lambda \in sp_A(x)$ such that $|\lambda| = r_A(x)$, then there exists a sequence $(\lambda_n)_n$ such that $\lambda_k \in \partial sp_A(x_k)$ for any $k \in N$ and λ_n converges to λ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \inf r_A(x_n) \geq \lim_{n\to\infty} |\lambda_n| = |\lambda| = r_A(x).$$

Thus, r_A is lower semi-continuous at x. By Corollary 3.14, it is upper semi-continuous at x. Hence, r_A is continuous at x. □

Corollary 3.19. *If A is a unital complete metrizable FLM algebra and $x \in A$ such that ∂sp_A is lower semi-continuous at x, then the spectral radius function is continuous at x.*

4. Continuity of linear mappings and homomorphisms

In this section, we investigate the automatic continuity of linear mappings and homomorphisms on certain complete metrizable FLM algebras.

Theorem 4.1. *Let A be a unital complete metrizable FLM algebra and B be a unital semi-simple complete metrizable FLM algebra with a sub-multiplicative metric d_B such that r_B is continuous on B. If $T : A \to B$ is a surjective linear mapping satisfying*

$$r_B(Tx) \leq r_A(x), \text{ for all } x \in A,$$

then T is continuous.

Proof . Let $a \in G(T)$. Then there exists a sequence $(x_n)_n$ in A such that $x_n \to 0$ and $Tx_n \to a$. Since T is surjective, there exists $y \in A$ with $Ty = a$. Since $r_B(Tx) \leq r_A(x)$ for all $x \in A$ and $r_A(x_n) \to 0$, we have $r_B(Tx_n) \to 0$. On the other hand, the continuity of r_B on B implies that $r_B(Tx_n) \to r_B(a) = r_B(Ty)$. Hence, $r_B(Ty) = 0$. By [22, 3.6], we have

$$r_B(Tx + Ty) \leq r_B(Tx) + r_B(Ty),$$

so,

$$r_B(Tx + Ty) \leq r_B(Tx).$$

Hence, $r_B(a + q) = 0$ for all quasi-nilpotent elements of q in B. By Zemanek's theorem for FLM algebras [22, 3.4], we have $a \in \text{rad}B = \{0\}$ and so $a = 0$. Therefore, T is continuous. □

Lemma 4.2. *Let A and B be complete metrizable topological algebras, and T be a dense range homomorphism from A to B. Then $G(T)$ is a closed two-sided ideal in B.*

Proof . Since $\overline{T(A)} = B$, it is easy to verify that $G(T)$ is a closed ideal in B. □

Remark 4.3. *If we suppose that $T : A \longrightarrow B$ is a surjective homomorphism, the use of Zemanek's theorem [22, 3.4] is not necessary, and in this case, we can apply Lemma 4.2. Since T is surjective, it satisfies $sp_B(Tx) \subseteq sp_A(x)$, so $r_B(Tx) \leq r_A(x)$ for all x in A. The same argument used in the proof of Theorem 4.1 implies that $r_B(Ty+q) = 0$ for all quasi-nilpotent q in B. Thus, $r_B(Ty) = r_B(a) = 0$. In conclusion, $G(T) \subseteq \text{rad}B = \{0\}$.*

Theorem 4.4. *Let A and B be unital complete metrizable topological algebras such that B is semi-simple, r_B is continuous on $G(T)$, and r_A is continuous at zero. If $T : A \to B$ is a dense range homomorphism, then T is continuous.*

Proof . By Lemma 4.2, $G(T)$ is an ideal in B. Let $a \in G(T)$. There exists a sequence $(x_n)_n$ in A such that $x_n \to 0$ and $Tx_n \to a$. Since $r_B(Tx) \leq r_A(x)$ for all $x \in A$ and $r_A(x_n) \to 0$, we obtain $r_B(Tx_n) \to 0$. On the other hand, $r_B(Tx_n) \to r_B(a)$. So, $r_B(a) = 0$. This implies that $G(T)$ is contained in the set of quasi-nilpotent elements of B and so, in the Jacobson radical of B as well. Hence, $G(T) \subseteq \text{rad}B = \{0\}$ and consequently, T is continuous. □

Corollary 4.5. *If A is a unital complete metrizable topological algebra such that r_A is continuous at zero, then every homomorphism $T : A \to \mathbb{C}$ is continuous.*

Proof . Since $\mathbb{C}$ is commutative, by Theorem 3.4, $r_{\mathbb{C}}$ is continuous on $\mathbb{C}$. The result follows from Theorem 4.4. □

Remark 4.6. *Considering the condition of the corollary, we have an affirmative answer to Michael's problem.*

Corollary 4.7. *Let A be a unital complete metrizable FLM algebra and B be a unital complete metrizable topological algebra such that B is semi-simple and r_B is continuous on $G(T)$. If $T : A \longrightarrow B$ is a dense range homomorphism, then T is continuous.*

Proof . By Theorem 3.2, r_A is continuous at zero. The result follows from Theorem 4.4. □

Theorem 4.8. *Let $T : A \longrightarrow B$ be a homomorphism between complete metrizable FLM algebras. If B is commutative and semi-simple, then T is continuous.*

Proof . Since the unitization of a semi-simple algebra is semi-simple, we may assume that B has a unit element. For any multiplicative linear functional $F : B \longrightarrow \mathbb{C}$, FoT is a multiplicative linear functional on A, so based on [4, 4.5] it is continuous. Hence, from the Closed Graph Theorem, T is continuous. □

Lemma 4.9. *Let A be a complete metrizable fundamental topological algebra and φ be a multiplicative linear functional on A. If for some $b > 1$, $b^n x^n \to 0$ in A, $x \in A$, then $|\varphi(x)| < 1$.*

Proof . See the proof of the theorem in [4, 4.5]. □

Theorem 4.10. *Let A be a complete metrizable fundamental topological algebra and B be a complete metrizable topological algebra. If A and B satisfy the following properties (i) and (ii), respectively, then every homomorphism $T : A \to B$ is continuous.*

(i) *For every sequence $(x_n)_n$, $x_n \to 0$, there exists $x_m \in (x_n)_n$ such that $b^k x_m^k \to 0$ as $k \to \infty$, for some $b > 1$.*

(ii) *For every sequence $(y_n)_n \subseteq B$, $y_n \neq 0$ and $y_n \not\to 0$, there is a sequence $(\varphi_m)_m$ of multiplicative linear functionals on B such that $\inf\limits_{m,n} |\varphi_m(y_n)| = \varepsilon > 0$.*

Proof . Suppose that T is not continuous. Let $(x_n)_n \subseteq A$ be a sequence such that $x_n \to 0$, but $T(x_n) \not\to 0$. Put $y_n = T(x_n)$. We may assume that $y_n \neq 0$ for all $n \geq 1$ (otherwise choose a subsequence). By hypothesis, $\inf\limits_{m,n} |\varphi_m(y_n)| = \varepsilon > 0$. Thus, we have $|\varphi_m(T(\varepsilon^{-1}x_n))| = |\varepsilon^{-1}\varphi_m(T(x_n))| \geq 1$ for all $m, n \geq 1$. Set $z_n = \varepsilon^{-1}x_n$. Then $z_n \to 0$. By property (i), there exists $z \in \{z_n\}$, $z = \varepsilon^{-1}x_n$ (for some n) such that $b^k z^k \to 0$ for some $b > 1$. Since $\varphi_m \circ T$ is a multiplicative linear functional on A, then by Lemma 4.9, $|\varphi_m \circ T(z)| = |\varphi_m(\varepsilon^{-1}T(x_n))| < 1$. This gives a contradiction and so, T is continuous. □

If we suppose that A is also a complete metrizable FLM algebra, then property (i) is not necessary and is therefore omitted. The resulting theorem is given below:

Theorem 4.11. *Let A be a complete metrizable FLM algebra and B be a complete metrizable topological algebra. If B satisfies property (ii), then every homomorphism $T : A \longrightarrow B$ is continuous.*

Proof . By the same argument used in the proof of Theorem 4.10, we have $z_n = \varepsilon^{-1}x_n \to 0$. Since $\varphi_m \circ T$ is a multiplicative linear functional on A, by [4, 4.5], it is continuous and so, $\varphi_m \circ T(z_n) \to 0$. On the other hand,

$$|\varphi_m \circ T(\varepsilon^{-1}x_n)| = |\varphi_m T(z_n)| \geq 1.$$

This contradiction implies that T is continuous. □

Example 4.12. *Let $C(\mathbb{R})$ be as defined in Example 3.6. For $(f_n)_n \subseteq C(\mathbb{R})$, we define*

$$f_n(x) = \begin{cases} x - n & \text{if } x > n \\ 0 & \text{if } x \in [-n, n] \\ -(x + n) & \text{if } x < -n; \end{cases}$$

then $f_n \to 0$ in the compact-open topology of $C(\mathbb{R})$ but for no f_n, $f_n^k \to 0$ as $k \to \infty$ [16, 3.29]. On the other hand,

$$\alpha f_n(x) \geq f_n(x), \quad \text{for every } \alpha > 1.$$

This implies that for no f_n and no $b > 1$, $b^k f_n^k \to 0$ as $k \to \infty$. Hence, $C(\mathbb{R})$ cannot satisfy property (i).

Remark 4.13. *A complete metrizable FLM algebra A (in particular, a Banach algebra) satisfies property (i). To see this, let Let $(x_n)_n \subseteq A$ be a sequence such that $x_n \to 0$ and let U_0 be a neighborhood of zero in A satisfying Definition 2.4, and $b > 1$. Then there exists $n_0 \in N$ such that $bx_n \in U_0$ for all $n \geq n_0$. As in the proof of Theorem 3.2, we obtain $b^k x_n^k \to 0$ as $k \to \infty$, for all $n \geq n_0$. Also, we can apply this method for a class of topological algebras [6].*

Remark 4.14. *T. Husain [16] introduced property (ii) for a class of topological algebras (in particular, Frechet algebras). He also proved that if a Frechet algebra A satisfies property (ii), then*

$$r_A(x) = \sup\{|\varphi(x)| : \varphi \in \phi_A\} = \infty,$$

[16, p.77]. This implies that a Frechet algebra A whose the set of invertible elements is open (in particular, a Banach algebra) cannot satisfy property (ii) because the spectrum $sp_A(x)$ of every $x \in A$ is compact and so $r_A(x) < \infty$.

5. Conclusion

(i) We proved that in complete metrizable FLM algebras, the spectral radius function is always continuous at zero but it may be discontinuous at other points. However, if the aforementioned algebras are commutative, then the spectral radius function is continuous at all points of these algebras.

(ii) We also obtained some automatic continuity results for linear mappings and homomorphisms on certain FLM algebras.

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