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On the dynamics of a nonautonomous rational difference equation

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Abstract

In this paper, we study the following nonautonomous rational difference equation

$$y_{n+1} = \frac{\alpha_n + y_n}{\alpha_n + y_{n-k}}, \quad n = 0, 1, ...,$$

where $\{\alpha_n\}_{n\geq 0}$ is a bounded sequence of positive numbers, k is a positive integer and the initial values $y_{-k}, ..., y_0$ are positive real numbers. We give sufficient conditions under which the unique equilibrium $\bar{y} = 1$ is globally asymptotically stable. Furthermore, we establish an oscillation result for positive solutions about the equilibrium point. Our work generalizes and improves earlier results in the literature.

Keywords: nonautonomous difference equation, global asymptotic stability, oscillation
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1. Introduction

Nonlinear difference equations have been studied intensively in the last few decades. Especially, there has been great interest in the study of the dynamics of rational difference equations, (for example, see [1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 14, 15, 17, 16, 18, 19]).

In [12], Kocic and Ladas studied the $(k + 1)$th order difference equation

$$y_{n+1} = \frac{a + by_n}{A + y_{n-k}}, \quad n \in \mathbb{N}, , \tag{1.1}$$

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a, b, A are nonnegative real numbers and k is a positive integer. They showed that the positive equilibrium point of the Eq. (1.1) is globally asymptotically stable. In addition, they showed that all positive solutions of Eq. (1.1) are oscillatory about the positive equilibrium point. These results were extended by Dekkar *et al.* [4] to the following nonautonomous analogues rational difference equation

$$y_{n+1} = \frac{\alpha_n + y_n}{\alpha_n + y_{n-k}}, \quad n = 0, 1, ... \tag{1.2}$$

They considered Eq. (1.2) in the case where $\{\alpha_n\}_{n\geq 0}$ is a periodic sequence of positive numbers with period T. In addition, they proposed three open problems. In [10], we solved the first open problem (when $\{\alpha_n\}_{n\geq 0}$ is a convergent sequence), and in this work we give an answer to the second one. Precisely, we study the dynamics of the following rational difference equation

$$y_{n+1} = \frac{\alpha_n + y_n}{\alpha_n + y_{n-k}}, \quad n = 0, 1, ..., \tag{1.3}$$

where $\{\alpha_n\}_{n\geq 0}$ is a *bounded* sequence of positive numbers, k is a positive integer and the initial values $y_{-k}, ..., y_0$ are positive real numbers. We give sufficient conditions under which the unique equilibrium $\bar{y} = 1$ is globally asymptotically stable. Furthermore, we show, under some conditions, that every positive solution of (1.3) is oscillatory about the equilibrium point $\bar{y} = 1$.

2. Preliminaries

In this preliminary section, we recall some notions and results about the theory of difference equations. For more details we refer readers to [5, 13].

Let I be an interval of real numbers and let $f : \mathbb{N} \times I^{k+1} \longrightarrow I$ be a continuously differentiable function. Consider the difference equation

$$y_{n+1} = f(n, y_n, y_{n-1}, ..., y_{n-k}), \quad n \geq 0, \tag{2.1}$$

with $y_0, y_{-1}, ..., y_{-k} \in I$.

Definition 2.1. *A point $\bar{y} \in I$ such that $\bar{y} = f(n, \bar{y}, \bar{y}, ..., \bar{y})$ for all $n \geq 0$, is called an equilibrium point of Eq. (2.1).*

Definition 2.2. *An equilibrium point $\bar{y}$ of (2.1) is said to be*

1. *Stable if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that if $y_0, y_{-1}, ..., y_{-k} \in (\bar{y} - \delta, \bar{y} + \delta) \subset I$ then $|y_n - \bar{y}| < \varepsilon$, for all $n \geq -k$. Otherwise, the equilibrium $\bar{y}$ is called unstable.*
2. *Attractive if there exists $\mu > 0$ such that if $y_0, y_{-1}, ..., y_{-k} \in (\bar{y} - \mu, \bar{y} + \mu) \subset I$ then*
$$\lim_{n\to\infty} y_n = \bar{y}.$$
If $\mu = \infty$, $\bar{y}$ is called globally attractive.
3. *Asymptotically stable if it is stable and attractive.*
4. *Globally asymptotically stable if it is stable and globally attractive.*

Definition 2.3. *A solution $\{y_n\}_{n\geq -k}$ of Eq. (2.1) is called nonoscillatory if there exists $p \geq -k$ such that either*

$$y_n > \bar{y}, \quad \forall n \geq p \qquad or \qquad y_n < \bar{y}, \quad \forall n \geq p,$$

and it is called oscillatory if it is not nonoscillatory.

Lemma 2.4 ([11]). *For $i = 1, 2, \dots, m$ assume that*

$$p_i \in (0, \infty) \quad \textit{and} \quad k_i \in \{0, 1, \dots\} \quad \textit{with} \ \sum_{i=1}^{m} (p_i + k_i) \neq 1.$$

Let $\{P_i(n)\}$ be sequences of positive numbers such that

$$\liminf_{n\to\infty} P_i(n) \geq p_i, \quad \textit{for } i = 1, 2, \dots, m.$$

Suppose that the linear difference inequality

$$z_{n+1} - z_n + \sum_{i=1}^{m} P_i(n) z_{n-k_i} \leq 0, \quad n \in \mathbb{N},$$

has an eventually positive solution. Then, the equation

$$\lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0,$$

has a positive root.

3. Global asymptotic stability

In this section, we show that $\bar{y} = 1$ is a globally asymptotically stable equilibrium of all solutions of Eq. (1.3) with positive initial conditions. Throughout this paper, we use the following notations

$$a = \inf_{n\geq 0}\{\alpha_n\} \quad \text{and} \quad A = \sup_{n\geq 0}\{\alpha_n\}.$$

First, we have the following result.

Theorem 3.1. *Assume that $a > 1$. Then, for every positive solution $\{y_n\}_{n\geq -k}$ of (1.3) there exist positive numbers m and M such that*

$$m \leq y_n \leq M, \qquad \forall n \geq 0.$$

Proof . Let $\{y_n\}_{n\geq -k}$ be a positive solution of (1.3). For all $n \geq 0$, we have

$$\begin{aligned} y_{n+1} &= \frac{\alpha_n + y_n}{\alpha_n + y_{n-k}} \\ &< 1 + \frac{1}{\alpha_n} y_n \\ &\leq 1 + \frac{1}{a} y_n, \qquad \forall n \geq 0. \end{aligned} \tag{3.1}$$

Since $a > 1$, the right hand side of (3.1) tends to $a/(a-1)$ as $n \to \infty$, and so, there exists $M > 0$, such that $y_n \leq M$. Hence, Eq. (1.3) yields

$$y_{n+1} \geq \frac{a}{A + M} = m, \qquad \forall n \geq 0.$$

□

Theorem 3.2. *Assume that $a > 1$. Then, $\bar{y} = 1$ is stable.*

Proof . Choose $M > A/(a-1)$ such that

$$y_{-k}, ..., y_0 \in \left(\frac{1}{A+M}, M\right).$$

Therefore, it is readily checked that

$$y_n \in \left(\frac{1}{A+M}, M\right), \qquad \forall n \geq -k. \tag{3.2}$$

Next, setting

$$M(\varepsilon) = \min\left\{1+\varepsilon, \frac{1}{1-\varepsilon} - A\right\}$$

and

$$\delta(\varepsilon) = \min\left\{M(\varepsilon) - 1, 1 - \frac{1}{A + M(\varepsilon)}\right\},$$

for $\varepsilon \in (0,1)$, we obtain

$$(1-\delta, 1+\delta) \subseteq \left(\frac{1}{A+M}, M\right) \subseteq (1-\varepsilon, 1+\varepsilon). \tag{3.3}$$

Now, if we take $y_{-k}, ..., y_0 \in (1-\delta, 1+\delta)$, then (3.2), combined with (3.3), yields

$$y_n \in (1-\varepsilon, 1+\varepsilon), \qquad \forall n \geq -k,$$

and so $\bar{y}$ is stable. □

Theorem 3.3. *Assume that $a > 1$. Then, the unique equilibrium point of (1.3) is globally attractive.*

Remark 3.4. *When $\{\alpha_n\}_n$ is T-periodic, Dekkar et al. [4, Theorem 6.1] established the global attractivity of the equilibrium point of Eq. (1.3), provided that $\alpha_n \geq 2$. In Theorem 3.3, the coefficients α_n just have to be greater than 1.*

Proof . Let $\{y_n\}_{n\geq -k}$ be an arbitrary positive solution of (1.3). Set

$$I = \liminf_{n\to\infty} y_n \qquad \text{and} \qquad S = \limsup_{n\to\infty} y_n$$

which by Theorem 3.1 exist. Let $\{n_p\}$ and $\{n_q\}$ be an infinite increasing sequences of positive integers such that

$$\lim_{q\to\infty} y_{n_q+1} = I \quad \text{and} \quad \lim_{p\to\infty} y_{n_p+1} = S.$$

By taking subsequences we assume that $\{\alpha_{n_p}\}_p$, $\{\alpha_{n_q}\}_q$, $\{y_{n_p}\}_p$, $\{y_{n_q}\}_q$, $\{y_{n_p-k}\}_p$ and $\{y_{n_q-k}\}_q$ converge to A_0, a_0, L_0, l_0, L_k and l_k respectively. Clearly

$$l_0, L_0, l_k, L_k \in [I, S] \quad \text{and} \quad a_0, A_0 \in [a, A].$$

Then, the Eq. (1.3) yields

$$I = \frac{a_0 + l_0}{a_0 + l_k} \geq \frac{a_0 + I}{a_0 + S}$$

and

$$S = \frac{A_0 + L_0}{A_0 + L_k} \leq \frac{A_0 + S}{A_0 + I}.$$

Since the function $(x + I)/(x + S)$ is non-decreasing, we have

$$I \geq \frac{a + I}{a + S}. \tag{3.4}$$

Similarly, since $(x + S)/(x + I)$ is non-increasing, we obtain

$$S \leq \frac{a + S}{a + I}. \tag{3.5}$$

Combining (3.4) with (3.5) gives

$$a + (1 - a)I \leq IS \leq a + (1 - a)S.$$

Consequently, since $a > 1$ we obtain $I \geq S$, and so the sequence $\{y_n\}$ is convergent to the unique limit $l = 1$. □

From Theorems 3.2 and 3.3 we obtain the following result.

Theorem 3.5. *Assume that $a > 1$. Then, the unique equilibrium point of Eq. (1.3) is globally asymptotically stable.*

We conclude this section by two illustrative examples:

Example 3.6. *We consider the following third order difference equation*

$$y_{n+1} = \frac{3 + \cos(n\pi) + 1/(n + 1) + y_n}{3 + \cos(n\pi) + 1/(n + 1) + y_{n-2}}, \tag{3.6}$$

with the initial values $y_{-2} = 6$, $y_{-1} = 7.5$ and $y_0 = 0.8$. From Theorem 3.5, the equilibrium point $\bar{y} = 1$ of Eq. (3.6) is globally asymptotically stable, see Figure 1.

Example 3.7. *We consider the following eighth order difference equation*

$$y_{n+1} = \frac{[\sin(n\pi/2) - 4]^2/8 + y_n}{[\sin(n\pi/2) - 4]^2/8 + y_{n-7}}, \tag{3.7}$$

with the initial values $y_{-7} = 1.9$, $y_{-6} = 0.01$, $y_{-5} = 6$, $y_{-4} = 0.4$, $y_{-3} = 1.2$, $y_{-2} = 0.5$, $y_{-1} = 2.5$ and $y_0 = 0.5$. From Theorem 3.5, the equilibrium point $\bar{y} = 1$ of Eq. (3.7) is globally asymptotically stable and this appears clearly in Figure 2.

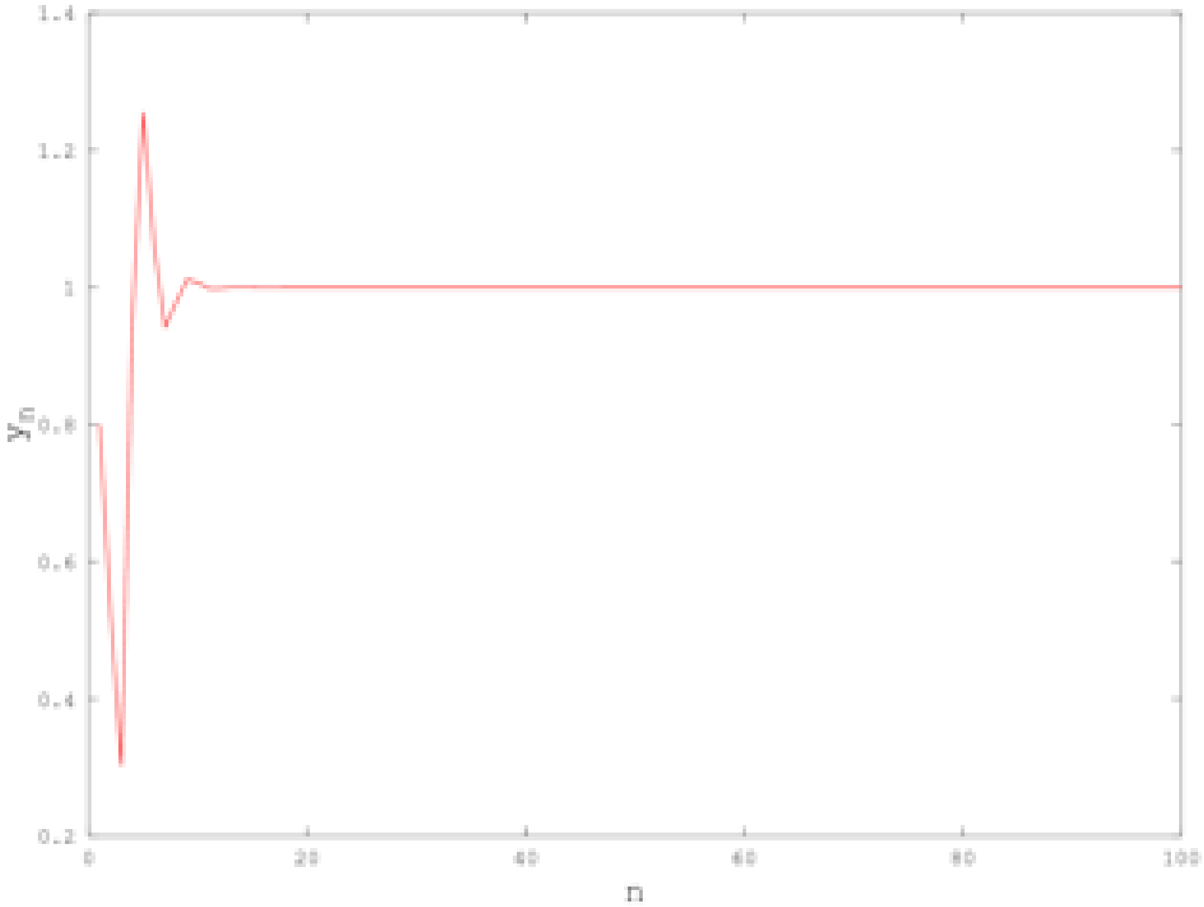

Figure 1: Plot of the solution $\{y_n\}_{n\geq 0}$ of Eq. (3.6) for the initial values $y_{-2} = 6$, $y_{-1} = 7.5$ and $y_0 = 0.8$.

4. Oscillation of positive solutions

To study the oscillation phenomenon we use the following lemma.

Lemma 4.1. *Every positive solution of (1.3) which is not oscillatory about the equilibrium $\bar{y} = 1$, tends to $\bar{y}$ as $n \to \infty$.*

Proof . Let $\{y_n\}_{n\geq -k}$ be a positive solution of Eq. (1.3) which is not oscillatory about 1, that is, there exists $n_0 \geq -k$ such that

$$y_n > 1, \quad \text{for all } n \geq n_0, \tag{4.1}$$

or

$$y_n < 1, \quad \text{for all } n \geq n_0. \tag{4.2}$$

Suppose that (4.1) holds. The case where (4.2) holds is similar and will be omitted. Then for $n \geq n_0 + k$,

$$\frac{y_{n+1}}{y_n} = \frac{\alpha_n / y_n + 1}{\alpha_n + y_{n-k}} < \frac{\alpha_n + 1}{\alpha_n + y_{n-k}} < 1, \tag{4.3}$$

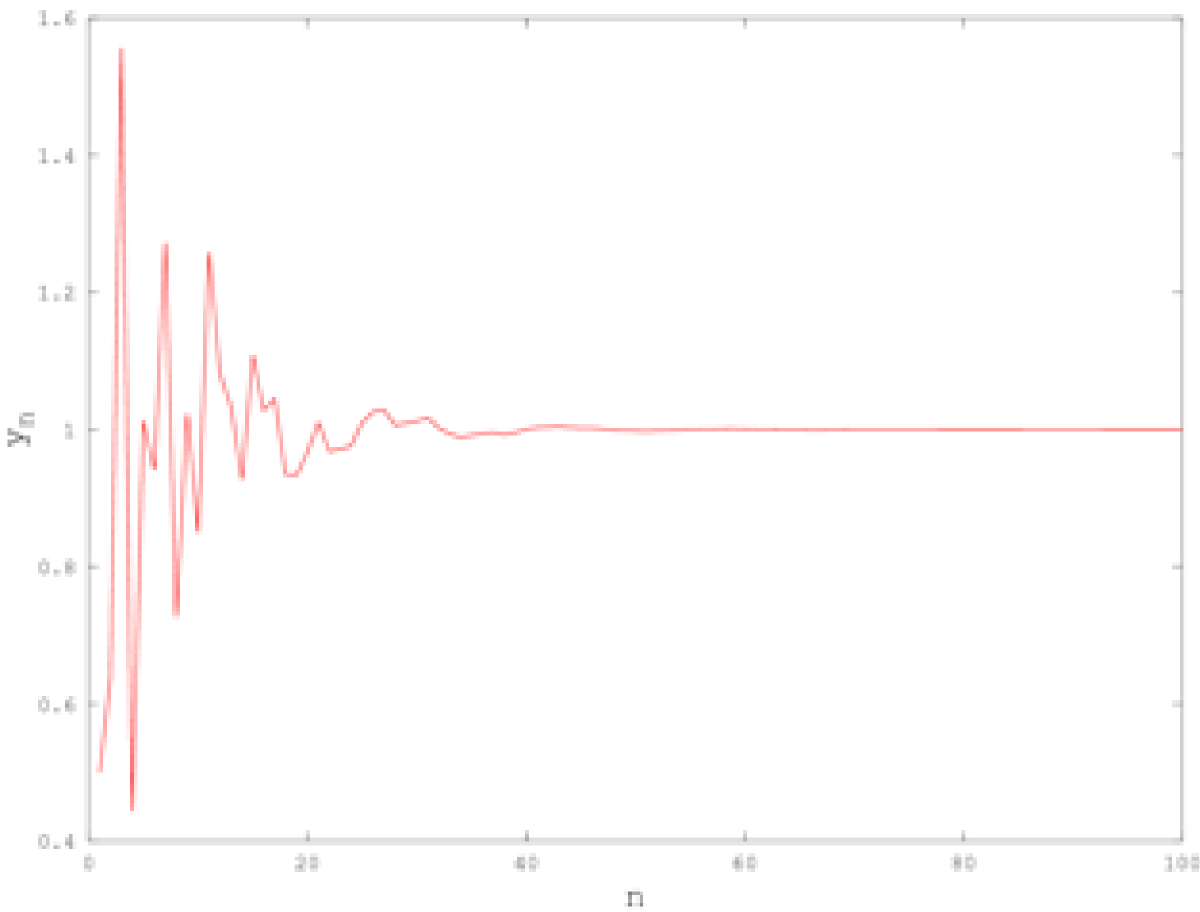

Figure 2: Plot of the solution $\{y_n\}_{n\geq 0}$ of Eq. (3.7) for the initial values $y_{-7} = 1.9$, $y_{-6} = 0.01$, $y_{-5} = 6$, $y_{-4} = 0.4$, $y_{-3} = 1.2$, $y_{-2} = 0.5$, $y_{-1} = 2.5$ and $y_0 = 0.5$.

and this implies that $\{y_n\}_{n\geq n_0+k}$ is decreasing. Thus, the sequence $\{y_n\}_{n\geq -k}$ is convergent to a limit l. The sequence $\{\alpha_n\}_{n\geq 0}$ is bounded, so there exists a subsequence $\{\alpha_{n_i}\}_{i\geq 0}$ which converges to a limit α. Therefore, by taking limits on both sides of Eq. (1.3) we find that

$$l = \lim_{i\to\infty} y_{n_i+1} = \frac{\lim_{i\to\infty} \alpha_{n_i} + \lim_{i\to\infty} y_{n_i}}{\lim_{i\to\infty} \alpha_{n_i} + \lim_{i\to\infty} y_{n_i-k}} = \frac{\alpha + l}{\alpha + l} = 1.$$

□

Theorem 4.2. *Assume that $a > 0$ and*

$$\frac{k^k}{(k+1)^{k+1}} < \frac{(A+1)^k}{(A-a+1)^{k+1}}. \tag{4.4}$$

Then, every positive solution of (1.3) is oscillatory about the equilibrium point $\bar{y} = 1$.

Proof . For the sake of contradiction we assume that Eq. (1.3) has a solution $\{y_n\}_{n\geq -k}$ which is not oscillatory about 1. In this case, we suppose that (4.1) holds, and similarly we prove the case when (4.2) holds. In view of Lemma 4.1, we have

$$\lim_{n\to\infty} y_n = 1.$$

Set $z_n = y_n - 1$. Then, Eq. (1.3) yields

$$z_{n+1} - z_n + P_1(n)z_n + P_2(n)z_{n-k} = 0,$$

where

$$P_1(n) = \frac{\alpha_n}{\alpha_n + y_{n-k}} \qquad \text{and} \qquad P_2(n) = \frac{y_n}{\alpha_n + y_{n-k}}.$$

We have

$$\liminf_{n\to\infty} P_1(n) \geq \frac{a}{A+1} = p_1$$

and

$$\liminf_{n\to\infty} P_2(n) \geq \frac{1}{A+1} = p_2.$$

Hence, by applying Lemma 2.4 we see that the equation

$$\lambda - 1 + p_1 + p_2\lambda^{-k} = 0$$

has a positive root. Let

$$F(\lambda) = \lambda^{k+1} + (p_1 - 1)\lambda^k + p_2.$$

On one hand, the equation $F(\lambda) = 0$ has a positive root. On the other hand, we have

$$F'(\lambda) = \lambda^{k-1}\left[(k+1)\lambda - k\left(1 - p_1\right)\right].$$

F' has then two roots: $\lambda = 0$ and $\lambda_0 = k(A - a + 1)/[(k+1)(A+1)]$. So, one can check that the function F is decreasing on $(0, \lambda_0]$ and increasing on $[\lambda_0, +\infty)$. Furthermore, we have

$$F(0) = p_2 > 0, \quad \lim_{\lambda\to\infty} F(\lambda) = +\infty$$

and

$$\begin{aligned} F(\lambda_0) &= \left(\frac{k}{k+1}\right)^{k+1}\left(\frac{A-a+1}{A+1}\right)^{k+1} - \left(\frac{A-a+1}{A+1}\right)^{k+1}\left(\frac{k}{k+1}\right)^{k} + \frac{1}{1+A} \\ &= \left(\frac{k}{k+1}\right)^{k}\left(\frac{A-a+1}{A+1}\right)^{k+1}\left(\frac{k}{k+1} - 1\right) + \frac{1}{A+1} \\ &= -\frac{k^k}{(k+1)^{k+1}}\left(\frac{A-a+1}{A+1}\right)^{k+1} + \frac{1}{A+1}. \end{aligned}$$

Under the condition (4.4), we see that $F(\lambda_0) > 0$, and so F is positive on $[0, \infty)$, which is a contradiction. □

Corollary 4.3. *Assume that $a > 0$ and $A - a \leq k$. Then, every solution of Eq. (1.3) is oscillatory about $\bar{y} = 1$.*

Proof . Since $A - a \leq k$, then we have

$$\frac{A-a+1}{k+1} \leq 1,$$

and so

$$\left(\frac{A-a+1}{k+1}\right)^{k+1} \leq \left(\frac{A-a+1}{k+1}\right)^{k} < \left(\frac{A+1}{k}\right)^{k}.$$

Hence, condition (4.4) is fulfilled, and the result follows by Theorem 4.2. □

To confirm our result on the oscillatory behavior of the positive solutions of Eq. (1.3), we consider the two following numerical examples.

Example 4.4. *We consider the following fourth order difference equation*

$$y_{n+1} = \frac{[2+\cos(n\pi)]/10 + y_n}{[2+\cos(n\pi)]/10 + y_{n-3}}, \tag{4.5}$$

with the initial values $y_{-3} = 2$, $y_{-2} = 1.2$, $y_{-1} = 1.3$ *and* $y_0 = 2.5$. *We have* $A - a = 4/10 < 3 = k$, *thus the solution of Eq. (4.5) is oscillatory about the equilibrium point* $\bar{y} = 1$, *see Figure 3.*

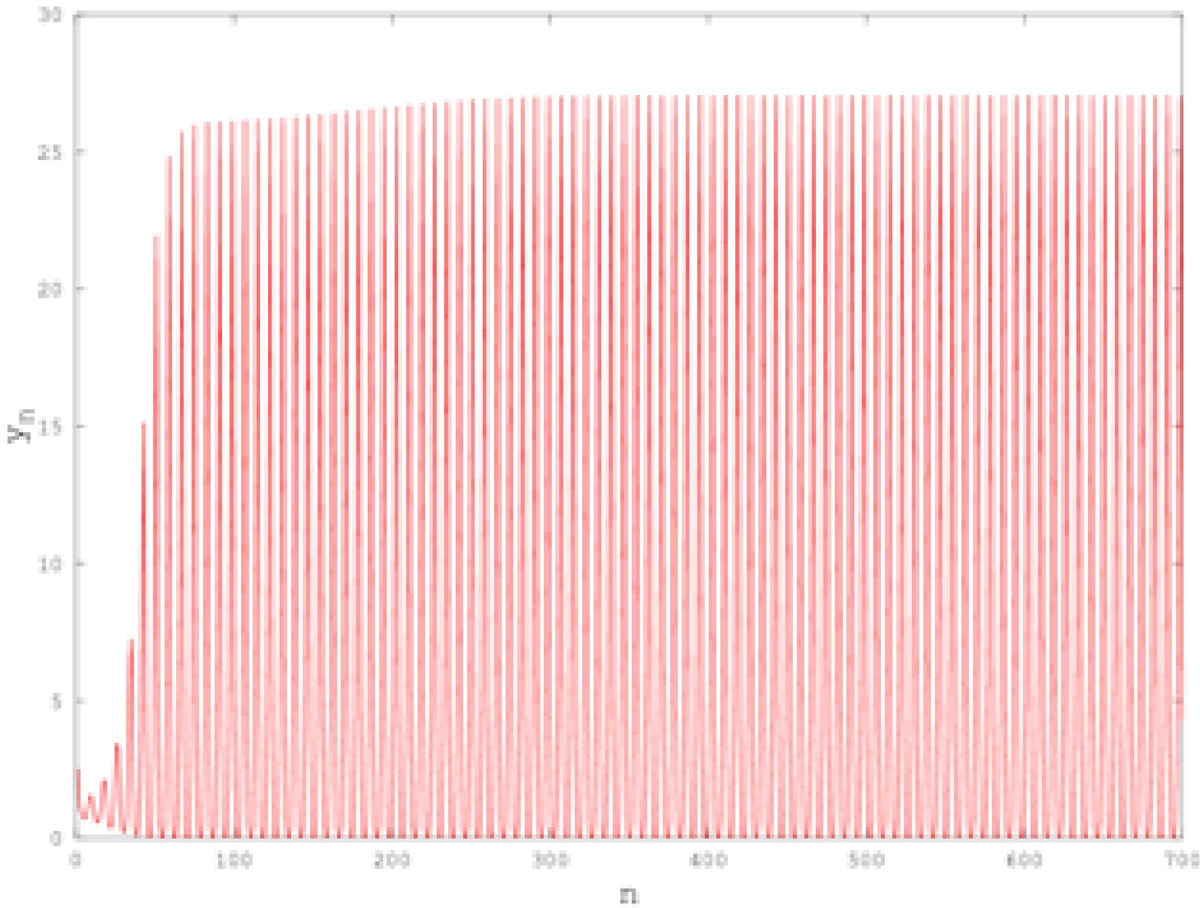

Figure 3: Plot of the solution $\{y_n\}_{n\geq 0}$ of Eq. (4.5) for the initial values $y_{-3} = 2$, $y_{-2} = 1.2$, $y_{-1} = 1.3$ and $y_0 = 2.5$.

Example 4.5. *We consider the following fifth order difference equation*

$$y_{n+1} = \frac{[3+\cos(n\pi)]/5 + 1/(n+1) + y_n}{[3+\cos(n\pi)]/5 + 1/(n+1) + y_{n-4}}, \tag{4.6}$$

with the initial values $y_{-4} = 1.5$, $y_{-3} = 0.8$, $y_{-2} = 1.7$, $y_{-1} = 2.5$ *and* $y_0 = 1.5$. *We have* $A - a = 7/5 < 4 = k$, *thus the solution of Eq. (4.6) is oscillatory about the equilibrium point* $\bar{y} = 1$, *see Figure 4.*

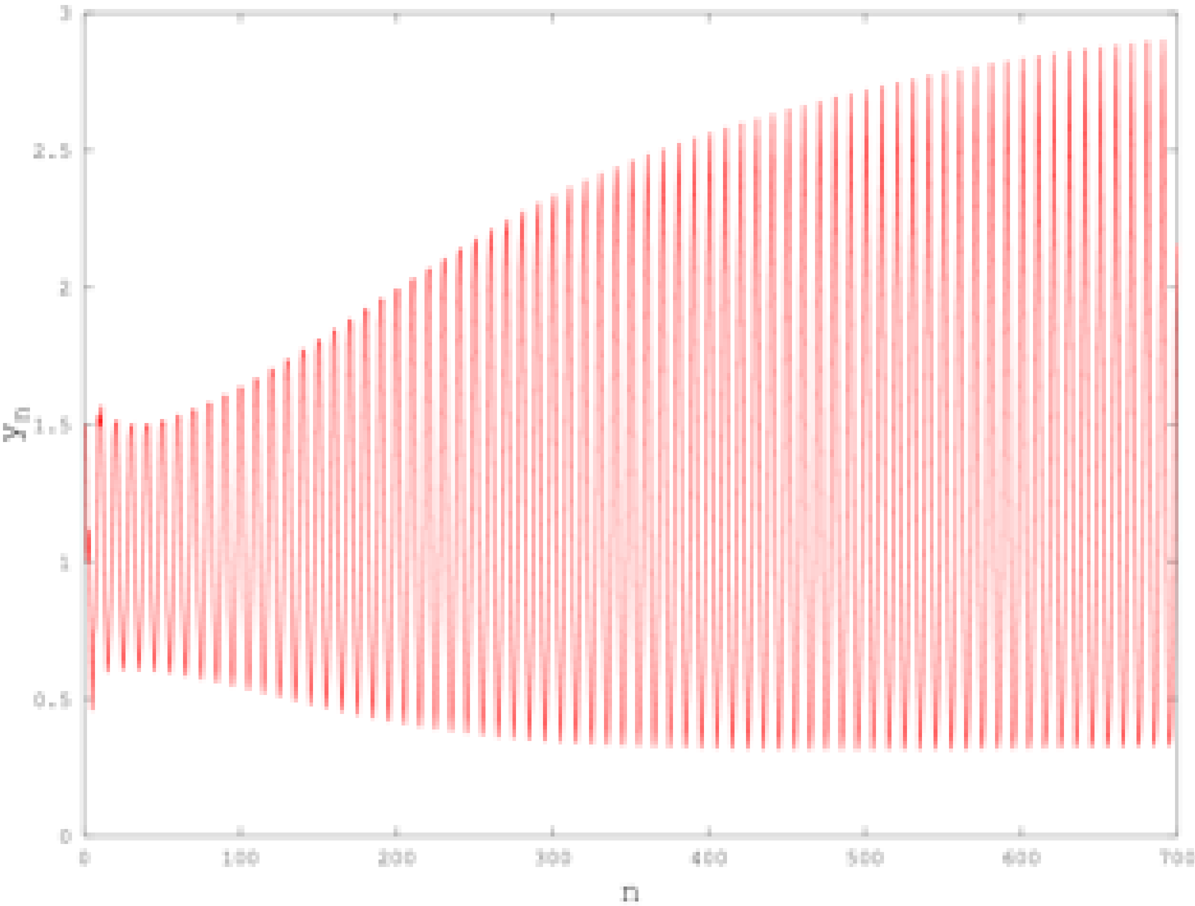

Figure 4: Plot of the solution $\{y_n\}_{n\geq 0}$ of Eq. (4.6) for the initial values $y_{-4} = 1.5$, $y_{-3} = 0.8$, $y_{-2} = 1.7$, $y_{-1} = 2.5$ and $y_0 = 1.5$.

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References

[1] R. Abo-Zeid, *Global behavior of a higher order rational difference equation*, Filomat 30(12) (2016) 3265–3276.

[2] M. Abu Alhalawa and M. Saleh, *Dynamics of higher order rational difference equation* $x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-k}}$, Int. J. Nonlinear Anal. Appl. 8(2) (2017) 363–379.

[3] E. Chatterjee and S. Hassan, *On the asymptotic character of a generalized rational difference equation*, Dis. Cont. Dyn. Sys. A 38 (2018) 1707–1718.

[4] I. Dekkar, N. Touafek and Q. Din, *On the global dynamics of a rational difference equation with periodic coefficients*, J. Appl. Math. Comput. 60(1) (2019) 567–588.

[5] S. Elaydi, *An Introduction to Difference Equations, Undergraduate Texts in Mathematics*, Springer, New York, 2005.

[6] M. Gümüs and R. Abo-Zeid, *On the solutions of a* $(2k+2)$*th order difference equation*, Dyn. Contin. Discrete Impuls Syst. Ser. B Appl. Algorithms 25 (2018) 129–143.

[7] M. Gümüs and R. Abo-Zeid, *Global behavior of a rational second order difference equation*, J. Appl. Math. Comput. 62(1) (2020) 119–133.

[8] T. F. Ibrahim and N. Touafek, *On a third order rational difference equation with variable coeffitients*, Dyn. Contin. Discrete Impuls Syst. Ser. B Appl. Algorithms 20 (2013) 251–264.

[9] A. Jafar and M. Saleh, *Dynamics of nonlinear difference equation* $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + B x_n + C x_{n-k}}$, J. Appl. Math. Comput. 57(1-2) (2018) 493–522.

[10] M.A. Kerker, E. Hadidi, and A. Salmi, *Qualitative behavior of a higher-order nonautonomous rational difference equation*, J. Appl. Math. Comput. 64 (2020) 399–409.

[11] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[12] V. L. Kocic, G. Ladas, and L. W. Rodrigues, *On rational recursive sequences*, J. Math. Anal. Appl. 173 (1993) 127–157.

[13] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations, Numerical Methods and Applications*, Marcel Dekker, Inc., New York, 2002.

[14] P. Liu, C. Wang, Y. Li and R. Li, *Asymptotic behavior of equilibrium point for a system of fourth-order rational difference equations*, *J. Comput. Anal. Appl.* 27 (2019) 947–961.

[15] C. Qian and J. Smith, *On quasi-periodic solutions of forced higher order nonlinear difference equations*, Electron. J. Qual. Theory Differ. Equ. 2020(2020) 20. Id/No 6.

[16] M. Saleh, N. Alkoumi and A. Farhat, *On the dynamics of a rational difference equation* $x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}}$, Chaos Solitons Fract. 96 (2017) 76–84.

[17] M. Saleh and A. Farhat, *Global asymptotic stability of the higher order equation* $x_{n+1} = \frac{a x_n + b x_{n-k}}{A + B x_{n-k}}$, J. Appl. Math. Comput. 55(1-2) (2017) 135–148.

[18] N. Touafek, *On a second order rational difference equation*, Hacet. J. Math. Stat. 41 (2012) 867–874.

[19] C. Wang, S. Wang and W. Wang, *Global asymptotic stability of equilibrium point for a family of rational difference equations*, Appl. Math. Lett. 24 (2011) 714–718.