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Positive Solutions for Fractional-order Nonlinear Boundary Value Problems on Infinite Interval

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Abstract

In this paper, Avery-Henderson (Double) fixed point theorem and Ren fixed point theorem are used to investigate the existence of positive solutions for fractional-order nonlinear boundary value problems on infinite interval. As applications, some examples are given to illustrate the main results.

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1. Introduction

In applied mathematics and mathematical analysis, fractional derivative is a derivative of any arbitrary order, real or complex. The first appearance of the concept of a fractional derivative is based in a letter written to Guillaume de L'Hospital by Gottfried Wilhelm Leibniz in 1695. Since then, the new theory turned out to be very attractive to mathematicians as well as physicists, biologists, engineers and economists. The first application of fractional calculus was due to Abel in his solution to the Tautocrone problem [12]. It also appears in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of aerodynamics, electrodynamics of complex medium, polymer rheology, physics, chemistry. In [8], Oldham et al. provide an encyclopedic treatment of the subject. Recently, there are many papers dealing with the existence of solutions of nonlinear initial or boundary value problems for fractional differential equations by virtue of techniques of nonlinear analysis, for example, see [4], [5], [9], [10], [11], [14] and the references therein. Also, few existence of positive solutions results are obtained on an infinite interval, see [2], [15], [17]

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and [18]. For general results and backround on the fractional calculus, we refer the reader to [1], [3] and [13].

In [17], Ge and Zhao considered the following fractional boundary value problem on an infinite interval:

$$\begin{cases} D^{\alpha}_{0+}u(t) + a(t)f(t,u(t)) = 0, \quad t \in (0,\infty), \quad \alpha \in (1,2), \\ u(0) = 0, \\ \lim\limits_{t\to+\infty} D^{\alpha-1}_{0+}u(t) = \beta u(\xi), \end{cases}$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative. They obtained the existence of the unique positive solution for the above fractional order boundary value problem by using the Leray-Schauder nonlinear alternative theorem.

In [15], Liang and Zhang considered the following m-point fractional boundary value problem on an infinite interval:

$$\begin{cases} D^{\alpha}_{0+}u(t) + a(t)f(u(t)) = 0, \ 0 < t < +\infty, \\ u(0) = u'(0) = 0, \\ D^{\alpha-1}_{0+}u(+\infty) = \sum\limits_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases}$$

where $2 < \alpha < 3$, D^{α}_{0+} denotes the standard Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < +\infty$, $i = 1, ..., m-2$ satisfies $0 < \sum\limits_{i=1}^{m-2} \beta_i u(\xi_i) < \Gamma(\alpha)$. They obtained the existence of three positive solutions by using the Legget-Williams fixed point theorem.

In [18], Gholami considered the following fractional integral boundary value problem on an infinite interval:

$$\begin{cases} D^{\alpha}_{0+}u(t) + a(t)f(t,u(t),u'(t)) = 0; \quad t \in (0,\infty), \quad \alpha \in (2,3), \\ u(0) = u'(0) = 0, \\ \lim\limits_{t\to+\infty} D^{\alpha-1}_{0+}u(t) = \sum\limits_{i=1}^{m-2} \beta_i D^{\alpha-1}_{0+}u(t)\ |_{t=\xi_i}, \end{cases}$$

where D^{α}_{0+} denotes the standard Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < +\infty$, $i = 1, ..., m-2$, $\beta_i \in \mathbb{R}$. The author obtained the existence of an unbounded solution for a fractional order boundary value problem by using the Leray-Schauder nonlinear alternative theorem.

In [2], Wang considered the following fractional boundary value problem on semi-infinite interval:

$$\begin{cases} D^{\alpha}u(t) + f(t,u(t)) = 0, \ 0 < t < +\infty, \\ u(0) = u'(0) = 0, \\ D^{\alpha-1}u(+\infty) = \xi I^{\beta}u(\eta), \quad \beta > 0. \end{cases}$$

The author obtained the existence of the unique solution by using the monotone iterative technique.

Motivated by the above works, in this paper, we consider the following fractional-order nonlinear boundary value problem (BVP). Our boundary conditions are more complicated than the BVP studied in [2].

$$\begin{cases} D_{0+}^{\alpha}u(t) + a(t)f(t,u(t)) = 0, \quad t \in [0,+\infty) \\ u(0) = u'(0) = u''(0) = ... = u^{(n-2)}(0) = 0 \\ D_{0+}^{\alpha-1}u(+\infty) = \sum_{i=1}^{m-2} \eta_i I_{0+}^{\beta} u^{(n-3)}(\xi_i) \end{cases} \tag{1.1}$$

where $n \in \mathbb{N}$, $n \geq 3$, $n-1 < \alpha \leq n$, D_{0+}^{α} denotes the standard Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < +\infty$, $\eta_i > 0$, $i = 1, ..., m-2$, $\beta > 0$. Throughout this paper we assume that following conditions hold:

$(H1)$ $0 < \sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha+\beta-n+2} < \Gamma(\alpha+\beta-n+3)$;

$(H2)$ $f : [0,+\infty) \times \mathbb{R} \longrightarrow [0,+\infty)$ continuous and $0 < \int_0^{+\infty} a(s)ds < +\infty$;

$(H3)$ $F(t,u) = f(t,(1+t^{\alpha-1})u)$, $f \in C([0,+\infty) \times \mathbb{R}, [0,+\infty))$, $f(0,u) \not\equiv 0$ on any subinterval of $(0,+\infty)$ and when u is bounded $f(t,(1+t^{\alpha-1})u)$ is bounded on $[0,+\infty)$.

In this paper, we will prove the existence of at least two and three positive solutions for the BVP (1.1) using Avery-Henderson (double) fixed point theorem in [6] and Ren's fixed point theorem in [7] respectively.

This paper is organized as follows. In section 2, we provide some definitions and preliminary lemma which are key tools for our main results. Section 3 is devoted to the theoretical discussions concerning the existence of at least two and three positive solutions of the problem under consideration and give some examples to illustrate how the main results can be used in practice. Finally, conclusion part is established in section 4.

2. Preliminaries

In this section, to state the main results of this paper, we need the following lemmas and definitions.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,+\infty) \to \mathbb{R}$ is given by*

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right) \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n-1 \leq \alpha < n$, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Lemma 2.3. *Let $\alpha > 0$; then $I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + ... + c_n t^{\alpha-n}$, where $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$ $(n = [\alpha]+1)$. Here I_{0+}^{α} stands for the standard Riemann-Liouville fractional integral of order $\alpha > 0$ and D_{0+}^{α} denotes the Riemann-Liouville fractional derivative.*

3. Main Results

In this section, we will give the necessary lemma and theorems to demonstrate the existence of solutions.

Lemma 3.1. *Assume that the conditions $(H1)-(H3)$ are satisfied. If $h \in C[0,+\infty)$, fractional boundary value problem*

$$\begin{cases} D^{\alpha}_{0^+}u(t)+h(t)=0, \quad t\in[0,+\infty) \\ u(0)=u'(0)=u''(0)=...=u^{(n-2)}(0)=0 \\ D^{\alpha-1}_{0^+}u(+\infty)=\displaystyle\sum_{i=1}^{m-2}\eta_i I^{\beta}_{0^+}u^{(n-3)}(\xi_i) \end{cases} \tag{3.1}$$

has an integral expression

$$u(t)=\int_0^{+\infty}G(t,s)h(s)ds \tag{3.2}$$

where

$$G(t,s)=G_1(t,s)+G_2(t,s), \tag{3.3}$$

here

$$G_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t<+\infty, \\ t^{\alpha-1}, & 0\le t\le s<+\infty. \end{cases} \tag{3.4}$$

and

$$\begin{aligned} G_2(t,s) &= \frac{\displaystyle\sum_{i=1}^{m-2}\eta_i t^{\alpha-1}}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3)-\displaystyle\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\right]} \\ &\quad \cdot\begin{cases} \xi_i^{\alpha+\beta-n+2}-(\xi_i-s)^{\alpha+\beta-n+2}, & 0\le s\le\xi_i<+\infty, \\ \xi_i^{\alpha+\beta-n+2}, & 0\le\xi_i\le s<+\infty. \end{cases} \end{aligned} \tag{3.5}$$

Proof . According to Lemma 2.3, we can obtain that $u(t)=-I^{\alpha}_{0^+}h(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}+...+c_nt^{\alpha-n}$. By the boundary conditions of (3.1), we have

$$c_1=\frac{\Gamma(\alpha+\beta-n+3)\displaystyle\int_0^{+\infty}h(s)ds-\sum_{i=1}^{m-2}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha+\beta-n+2}h(s)ds}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3)-\displaystyle\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\right]}$$

and $c_2 = 0$, $c_3 = 0$, ..., $c_n = 0$. Therefore, we obtain

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{\Gamma(\alpha+\beta-n+3)t^{\alpha-1}}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3)-\sum\limits_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\right]}\int_0^{+\infty} h(s)ds \\ &\quad - \frac{\sum\limits_{i=1}^{m-2}\eta_i t^{\alpha-1}}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3)-\sum\limits_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\right]}\int_0^{\xi_i}(\xi_i-s)^{\alpha+\beta-n+2}h(s)ds \\ &= \int_0^{+\infty} G(t,s)h(s)ds, \quad t\in[0,+\infty). \end{aligned}$$

□

Lemma 3.2. *[15] The function $G_1(t,s)$ defined by (3.4) satisfies*

- ***i)*** *$G_1(t,s)$ is a continuous and $G_1(t,s) \geq 0$ for $(t,s) \in [0,+\infty)\times[0,+\infty)$,*
- ***ii)*** *$G_1(t,s)$ is strictly increasing in the first variable,*
- ***iii)*** *$G_1(t,s)$ is concave in the first variable for $0 < s < t < +\infty$.*

Lemma 3.3. *[15] If $k > 1$, then $G_1(t,s)$ defined by (3.4) has the following property*

$$\min_{\frac{1}{k}\leq t\leq k}\frac{G_1(t,s)}{1+t^{\alpha-1}} \geq \frac{1}{4k^2(1+k^{\alpha-1})}\max_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}.$$

Lemma 3.4. *From the definition of $G_1(t,s)$, we have*

$$\frac{G_1(t,s)}{1+t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)}, \qquad \frac{G(t,s)}{1+t^{\alpha-1}} \leq L \quad for \quad (t,s)\in[0,+\infty)\times[0,+\infty),$$

where $L = \dfrac{\Gamma(\alpha+\beta-n+3)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3)-\sum\limits_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\right]}$.

Proof . The functions $G(t,s)$, $G_1(t,s)$ and $G_2(t,s)$ are as defined in (3.3), (3.4) and (3.5) respectively. Let $s \leq t$. Using Lemma 3.2, we have

$$\frac{G_1(t,s)}{1+t^{\alpha-1}} = \frac{t^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)(1+t^{\alpha-1})} \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)(1+t^{\alpha-1})} \leq \frac{1}{\Gamma(\alpha)}.$$

Let $t \leq s$. From Lemma 3.2, we get

$$\frac{G_1(t,s)}{1+t^{\alpha-1}} = \frac{t^{\alpha-1}}{\Gamma(\alpha)(1+t^{\alpha-1})} \leq \frac{1}{\Gamma(\alpha)}, \quad \frac{t^{\alpha-1}}{1+t^{\alpha-1}} \to 1 \quad \text{with} \quad t\to+\infty$$

In both cases, we obtain

$$\frac{G_1(t,s)}{1+t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)}.$$

Similarly, we can get an inequality for function $G_2(t,s)$. If $0 \le s \le \xi_i$, then

$$\begin{aligned}\frac{G_2(t,s)}{1+t^{\alpha-1}} &= \frac{\displaystyle\sum_{i=1}^{m-1} \eta_i t^{\alpha-1}\left(\xi_i^{\alpha+\beta-n+2} - (\xi_i - s)^{\alpha+\beta-n+2}\right)}{(1+t^{\alpha-1})\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3) - \displaystyle\sum_{i=1}^{m-2}\eta_i \xi_i^{\alpha+\beta-n+2}\right]} \\ &\le \frac{\displaystyle\sum_{i=1}^{m-2} \eta_i t^{\alpha-1}\xi_i^{\alpha+\beta-n+2}}{(1+t^{\alpha-1})\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3) - \displaystyle\sum_{i=1}^{m-2}\eta_i \xi_i^{\alpha+\beta-n+2}\right]} \\ &\le \frac{\displaystyle\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha+\beta-n+2}}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3) - \displaystyle\sum_{i=1}^{m-2}\eta_i \xi_i^{\alpha+\beta-n+2}\right]}.\end{aligned}$$

On the other hand, if $0 \le \xi_i \le s$, then

$$\begin{aligned}\frac{G_2(t,s)}{1+t^{\alpha-1}} &= \frac{\displaystyle\sum_{i=1}^{m-1} \eta_i t^{\alpha-1}\xi_i^{\alpha+\beta-n+2}}{(1+t^{\alpha-1})\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3) - \displaystyle\sum_{i=1}^{m-2}\eta_i \xi_i^{\alpha+\beta-n+2}\right]} \\ &\le \frac{\displaystyle\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha+\beta-n+2}}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3) - \displaystyle\sum_{i=1}^{m-2}\eta_i \xi_i^{\alpha+\beta-n+2}\right]}.\end{aligned}$$

The following inequality is obtained from both cases:

$$\frac{G_2(t,s)}{1+t^{\alpha-1}} \le \frac{\displaystyle\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha+\beta-n+2}}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3) - \displaystyle\sum_{i=1}^{m-2}\eta_i \xi_i^{\alpha+\beta-n+2}\right]}.$$

Finally, by means of the equation (3.3), we get

$$\frac{G(t,s)}{1+t^{\alpha-1}} \le \frac{\Gamma(\alpha+\beta-n+3)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3) - \displaystyle\sum_{i=1}^{m-2}\eta_i \xi_i^{\alpha+\beta-n+2}\right]}.$$

□

Lemma 3.5. *If $k > 1$, then $G_2(t,s)$ defined by* (3.5) *has the following property*

$$\min_{\frac{1}{k}\le t\le k}\frac{G_2(t,s)}{1+t^{\alpha-1}}\ge\frac{1}{k^{\alpha-1}(1+k^{\alpha-1})}\max_{t\in[0,+\infty)}\frac{G_2(t,s)}{1+t^{\alpha-1}}.$$

Proof . Let $0\le s\le\xi_i$, then by using (3.5), we obtain

$$\begin{aligned}\min_{\frac{1}{k}\le t\le k}\frac{G_2(t,s)}{1+t^{\alpha-1}}&=\min_{\frac{1}{k}\le t\le k}\frac{\sum_{i=1}^{m-2}\eta_i t^{\alpha-1}\left(\xi_i^{\alpha+\beta-n+2}-(\xi_i-s)^{\alpha+\beta-n+2}\right)}{(1+t^{\alpha-1})\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\right]}\\
&\ge\frac{\sum_{i=1}^{m-2}\eta_i\left(\frac{1}{k}\right)^{\alpha-1}\left(\xi_i^{\alpha+\beta-n+2}-(\xi_i-s)^{\alpha+\beta-n+2}\right)}{(1+k^{\alpha-1})\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\right]}\\
&=\frac{1}{k^{\alpha-1}(1+k^{\alpha-1})}\max_{t\in[0,+\infty)}\frac{\sum_{i=1}^{m-2}\eta_i t^{\alpha-1}\left(\xi_i^{\alpha+\beta-n+2}-(\xi_i-s)^{\alpha+\beta-n+2}\right)}{(1+t^{\alpha-1})\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\right]}.\end{aligned}$$

Let $0\le\xi_i\le s$, then the following inequality is obtained:

$$\min_{\frac{1}{k}\le t\le k}\frac{G_2(t,s)}{1+t^{\alpha-1}}\ge\frac{1}{k^{\alpha-1}(1+k^{\alpha-1})}\max_{t\in[0,+\infty)}\frac{G_2(t,s)}{1+t^{\alpha-1}}.$$

□

Lemma 3.6. *For a fixed $k > 1$,*

$$\min_{\frac{1}{k}\le t\le k}\frac{G(t,s)}{1+t^{\alpha-1}}\ge\lambda(k)\max_{t\in[0,+\infty)}\frac{G(t,s)}{1+t^{\alpha-1}}$$

where

$$\lambda(k)=\min\left\{\frac{1}{4k^2(1+k^{\alpha-1})},\frac{1}{k^{\alpha-1}(1+k^{\alpha-1})}\right\}.$$

Proof . This Lemma is obvious from Lemma 3.4 and Lemma 3.5. □

Set

$$E=\left\{u\in C[0,+\infty):\max_{t\ge0}\ \frac{|u(t)|}{1+t^{\alpha-1}}<+\infty\right\}.$$

Clearly, E is a Banach space with the norm

$$\|u\|=\max_{0\le t<+\infty}\frac{|u(t)|}{1+t^{\alpha-1}}<+\infty.$$

Lemma 3.7. *Assume that* $(H1)$-$(H3)$ *hold. Let* $u \in E$ *and* $k > 1$. *Then,* $u(t) \geq 0$ *and* $\min\limits_{\frac{1}{k} \leq t \leq k} \dfrac{|u(t)|}{1+t^{\alpha-1}} \geq \lambda(k)\, \|u\|$.

Proof . Lemma 3.2, positivity of $G_2(t,s)$ and conditions $(H1)$-$(H3)$ imply that $u(t) \geq 0$. For a fixed $k > 1$, by using Lemma 3.6

$$\begin{aligned}
\min_{\frac{1}{k} \leq t \leq k} \frac{u(t)}{1+t^{\alpha-1}} &= \min_{\frac{1}{k} \leq t \leq k} \frac{1}{1+t^{\alpha-1}} \int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds \\
&\geq \int_0^{+\infty} \min_{\frac{1}{k} \leq t \leq k} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)f(s,u(s))ds \\
&\geq \lambda(k) \int_0^{+\infty} \max_{t \geq 0} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)f(s,u(s))ds \\
&\geq \lambda(k) \max_{t \geq 0} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)f(s,u(s))ds \\
&\geq \lambda(k)\, \|u\|
\end{aligned}$$

□

By using Lemma 3.7, we can define the cone $P \subset E$ by

$$P = \left\{ u \in E : u(t) \geq 0, \min_{\frac{1}{k} \leq t \leq k} \frac{|u(t)|}{1+t^{\alpha-1}} \geq \lambda(k)\, \|u\| \right\}.$$

Denote the operator $T : P \to E$ by

$$Tu(t) = \int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds. \tag{3.6}$$

Lemma 3.8. *Assume that* $(H1)$-$(H3)$ *hold. Then* $T : P \to P$ *is completely continuous operator.*

Proof . Firstly, it is easy to check that $T : P \to P$ is well-defined. Now, we will show that T is a completely continuous operator in three steps.
Step 1: $T : P \to P$ is a continuous operator.
Let $u_n \in P$, there exists a sequence $u_n \to u$, $n \to +\infty$ in P. Since the convergent sequences are bounded, there is a real number r_0 such that $\max\limits_{n \in \mathbb{N} \setminus \{0\}} \|u_n\| < r_0$. Denote the set

$$B_{r_0} = \max \left\{ f(t,(1+t^{\alpha}-1)u), \quad (t,u) \in [0,+\infty) \times [0,r_0] \right\}.$$

For all $(t,s) \in [0,+\infty)$, by the Lebesgue Dominated Convergence theorem and Lemma 3.4, we can get

$$\begin{aligned}
\left| \frac{Tu_n(t) - Tu_0(t)}{1+t^{\alpha-1}} \right| &= \left| \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)[f(s,u_n(s)) - f(s,u(s))]ds \right| \\
&\leq \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)\, |f(s,u_n(s)) - f(s,u(s))|\, ds \\
&\leq L \int_0^{+\infty} a(s)\, |f(s,u_n(s)) - f(s,u(s))|\, ds \to 0, (n \to +\infty).
\end{aligned}$$

This implies that

$$\max_{t\geq 0}\left|\frac{Tu_n(t)-Tu_0(t)}{1+t^{\alpha-1}}\right| = \|Tu_n(t)-Tu_0(t)\| \to 0.$$

Thus, $T: P \to P$ is sequential continuous. If T is sequential continuous, then T is continuous.
Step 2: $T: P \to P$ is relatively compact operator.
Let Ω be any bounded subset of P. Then there exists $r > 0$ such that $\|u\| \leq r$ for all $u \in \Omega$. So, from (H2) and Lemma 3.4, for all $x \in \Omega$,

$$\begin{aligned}
\frac{Tu(t)}{1+t^{\alpha-1}} &\leq L\int_0^{+\infty} a(s)f(s,u(s))ds \\
&= L\int_0^{+\infty} a(s)f\left(s,(1+s^{\alpha-1})\frac{u(s)}{1+s^{\alpha-1}}\right)ds \\
&\leq LB_r\int_0^{+\infty} a(s)ds \\
&< +\infty.
\end{aligned}$$

This yields that $\|Tu(t)\| < +\infty$. So $T\Omega$ is uniformly bounded. Next, we show that $T\Omega$ is equicontinuous on $[0,+\infty)$. For any $a > 0$ and $t_1, t_2 \in [0,a]$, without loss of generality, we may assume that $t_2 > t_1$. For all $u \in \Omega$, we have

$$\begin{aligned}
\left|\frac{(Tu)(t_1)}{1+t_1^{\alpha-1}}-\frac{(Tu)(t_2)}{1+t_2^{\alpha-1}}\right| &\leq \int_0^{+\infty}\left|\frac{G(t_1,s)}{1+t_1^{\alpha-1}}-\frac{G(t_2,s)}{1+t_2^{\alpha-1}}\right| a(s)f(s,u(s))ds \\
&\leq \int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}\right| a(s)f(s,u(s))ds \\
&\quad +\int_0^{+\infty}\left|\frac{G_2(t_1,s)}{1+t_1^{\alpha-1}}-\frac{G_2(t_2,s)}{1+t_2^{\alpha-1}}\right| a(s)f(s,u(s))ds \\
&\leq \int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}\right| a(s)f(s,u(s))ds \\
&\quad +\frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\left|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right|}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\right]}\int_0^{+\infty} a(s)f(s,u(s))ds \\
&\leq \int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}+\frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}\right| a(s)f(s,u(s))ds \\
&\quad +\frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\left|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right|}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\right]}\int_0^{+\infty} a(s)f(s,u(s))ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}\right|a(s)f(s,u(s))ds\\
&+\int_0^{+\infty}\left|\frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}\right|a(s)f(s,u(s))ds\\
&+\frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\left|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right|}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta-n+3)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\beta-n+2}\right]}\int_0^{+\infty}a(s)f(s,u(s))ds.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}\right|a(s)f(s,u(s))ds\\
&=\int_0^{t_1}\left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}\right|a(s)f(s,u(s))ds\\
&+\int_{t_1}^{t_2}\left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}\right|a(s)f(s,u(s))ds\\
&+\int_{t_2}^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}\right|a(s)f(s,u(s))ds\\
&\leq B_r\int_0^{t_1}\frac{(t_2^{\alpha-1}-t_1^{\alpha-1})+((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})}{1+t_1^{\alpha-1}}a(s)ds\\
&+B_r\int_{t_1}^{t_2}\frac{(t_2^{\alpha-1}-t_1^{\alpha-1})+(t_2-s)^{\alpha-1}}{1+t_1^{\alpha-1}}a(s)ds+B_r\int_{t_2}^{+\infty}\frac{(t_2^{\alpha-1}-t_1^{\alpha-1})}{1+t_1^{\alpha-1}}a(s)ds\\
&\leq B_r\int_0^{t_1}\frac{(t_2^{\alpha-1}-t_1^{\alpha-1})+((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})}{1+t_1^{\alpha-1}}a(s)ds\\
&+B_r\int_{t_1}^{t_2}\frac{(t_2^{\alpha-1}-t_1^{\alpha-1})+(t_2-t_1)^{\alpha-1}}{1+t_1^{\alpha-1}}a(s)ds\\
&+B_r\int_{t_2}^{+\infty}\frac{(t_2^{\alpha-1}-t_1^{\alpha-1})}{1+t_1^{\alpha-1}}a(s)ds\quad\to 0\quad as\quad t_1\to t_2.
\end{aligned}
$$

In a similar way, one can see that

$$
\int_0^{+\infty}\left|\frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}\right|a(s)f(s,u(s))ds\to 0\quad as\quad t_1\to t_2.
$$

Hence, we obtain

$$
\left|\frac{Tu(t_1)}{1+t_1^{\alpha-1}}-\frac{Tu(t_2)}{1+t_2^{\alpha-1}}\right|\to 0\quad as\quad t_1\to t_2.
$$

That is, $T\Omega$ is equicontinuous on $[0,+\infty)$.

Step 3: $T : P \to P$ is equiconvergent at $+\infty$.
For any $u \in \Omega$, we have

$$\int_0^{+\infty} a(s)f(s,u(s))ds \le B_r \int_0^{+\infty} a(s)ds < +\infty.$$

It follows from Lemma 3.4 that

$$\begin{aligned}\lim_{t\to+\infty}\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right| &= \lim_{t\to+\infty}\left|\int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)f(s,u(s))ds\right| \\ &\le L\lim_{t\to+\infty}\left|\int_0^{+\infty}a(s)f(s,u(s))ds\right| \\ &< +\infty.\end{aligned}$$

Hence, $T\Omega$ is equiconvergent at $+\infty$. Therefore, we conclude that $T : P \to P$ is a completely continuous operator. □

In order to obtain at least two positive solutions of BVP (1.1), we will apply the following theorem.

Theorem 3.9. *(Avery-Henderson (Double) Fixed Point Theorem)[6] Let P be a cone in a real Banach space E. Assume α and γ are increasing, nonnegative continuous on P. Let β be a nonnegative continuous functional on P with $\beta(0) = 0$ such that, for some positive constants q and M,*

$$\gamma(u) \le \beta(u) \le \alpha(u) \quad \text{and} \quad \|u\| \le M\gamma(u)$$

for all $u \in \overline{P(\gamma, q)}$. Suppose that there exist positive numbers $l < r < q$ such that

$$\beta(\lambda u) \le \lambda\beta(u), \quad \text{for all} \quad 0 \le \lambda \le 1 \quad \text{and} \quad u \in \partial P(\beta, r).$$

If $T : \overline{P(\gamma, q)} \to P$ is a completely continuous operator satisfying

(i) $\gamma(Tu) > q$ for all $u \in \partial P(\gamma, q)$,

(ii) $\beta(Tu) < r$ for all $u \in \partial P(\beta, r)$,

(iii) $P(\alpha, l) \neq \emptyset$ and $\alpha(Tu) > l$ for all $u \in \partial P(\alpha, l)$,

then T has at least two fixed points u_1 and u_2 such that

$$l < \alpha(u_1) \quad \text{with} \quad \beta(u_1) < r \quad \text{and} \quad r < \beta(u_2) \quad \text{with} \quad \gamma(u_2) < q.$$

Let $0 < \dfrac{1}{k} \le t \le k$, α, γ be nonnegative, increasing, continuous functionals on P and β be a nonnegative continuous functional on P be defined by

$$\alpha(u) = \max_{t\in[0,k]}\frac{u(t)}{1+t^{\alpha-1}}, \qquad \beta(u) = \max_{t\in[\frac{1}{k},k]}\frac{u(t)}{1+t^{\alpha-1}}, \qquad \gamma(u) = \min_{t\in[\frac{1}{k},k]}\frac{u(t)}{1+t^{\alpha-1}} \tag{3.7}$$

and let $P(\gamma, q) = \{u \in P : \gamma(u) < q\}$.

For convenience, we denote

$$K = \frac{\displaystyle\int_{\frac{1}{k}}^{k} a(s)ds}{\Gamma(\alpha)k^{\alpha-1}(1+k^{\alpha-1})}, \qquad N = L\int_0^{+\infty} a(s)ds \tag{3.8}$$

where L is defined by Lemma 3.4.

We now state growth conditions on F so that (1.1) has at least two positive solutions.

Theorem 3.10. *Assume that* $(H1)-(H3)$ *hold. Suppose there exist positive numbers* $0<l<r<q$ *such that* $0<l<\dfrac{K}{N}r<\dfrac{K}{N}\lambda(k)q$ *and the function* F *satisfies the following conditions:*

(H4) $F(t,u)>\dfrac{q}{K}$ *for* $(t,u)\in\left[\dfrac{1}{k},k\right]\times\left[q,\dfrac{q}{\lambda(k)}\right]$,

(H5) $F(t,u)<\dfrac{r}{N}$ *for* $(t,u)\in[0,+\infty)\times\left[0,\dfrac{r}{\lambda(k)}\right]$,

(H6) $F(t,u)>\dfrac{l}{K}$ *for* $(t,u)\in[0,k]\times[0,l]$.

where K *and* N *are as defined in* (3.8). *Then the boundary value problem* (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$l<\max_{t\in[0,k]}\frac{u_1(t)}{1+t^{\alpha-1}} \quad with \quad \max_{t\in[\frac{1}{k},k]}\frac{u_1(t)}{1+t^{\alpha-1}}<r \quad and \quad r<\max_{t\in[\frac{1}{k},k]}\frac{u_2(t)}{1+t^{\alpha-1}} \quad with \quad \min_{t\in[\frac{1}{k},k]}\frac{u_2(t)}{1+t^{\alpha-1}}<q.$$

Proof . It is obvious that for each $u\in P$, $\gamma(u)\leq\beta(u)\leq\alpha(u)$. In addition, from Lemma 3.6, for each $u\in P$

$$\|u\|\leq\frac{1}{\lambda(k)}\min_{\frac{1}{k}\leq t\leq k}\frac{u(t)}{1+t^{\alpha-1}}=\frac{1}{\lambda(k)}\gamma(u), \quad M=\frac{1}{\lambda(k)}>0.$$

For $0\leq\lambda\leq 1$ and each $u\in P$ we obtain $\beta(\lambda u)=\lambda\beta(u)$. Also, it is clear that $\beta(0)=0$.
We now show that the remaining conditions of Theorem 3.9 are satisfied. We define the completely continuous operator T by (3.6). So, it is easy to check that $T:\overline{P(\gamma,q)}\to P$. We now turn to property (i) of Theorem 3.9. Choose $u\in\partial P(\gamma,q)$. Then $\gamma(u)=\min\limits_{t\in[\frac{1}{k},k]}\dfrac{u(t)}{1+t^{\alpha-1}}=q$. Since $u\in P$, $\|u\|\leq\dfrac{q}{\lambda(k)}$, $t\in\left[\dfrac{1}{k},k\right]$. This implies that

$$q\leq\frac{u(t)}{1+t^{\alpha-1}}\leq\frac{q}{\lambda(k)}, \quad t\in\left[\frac{1}{k},k\right].$$

As a consequence of $(H4)$,

$$F(t,u)>\frac{q}{K}, \quad (t,u)\in\left[\frac{1}{k},k\right]\times\left[q,\frac{q}{\lambda(k)}\right].$$

Also, $Tu\in P$, and so

$$\begin{aligned}
\gamma(Tu) &= \min_{t\in[\frac{1}{k},k]}\int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)f(s,u(s))ds \\
&\geq \int_0^{+\infty}\min_{t\in[\frac{1}{k},k]}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)f(s,u(s))ds \\
&\geq \int_0^{+\infty}\min_{t\in[\frac{1}{k},k]}\left(\frac{G_1(t,s)}{1+t^{\alpha-1}}+\frac{G_2(t,s)}{1+t^{\alpha-1}}\right)a(s)f(s,u(s))ds \\
&\geq \int_0^{+\infty}\left(\min_{t\in[\frac{1}{k},k]}\frac{G_1(t,s)}{1+t^{\alpha-1}}+\min_{t\in[\frac{1}{k},k]}\frac{G_2(t,s)}{1+t^{\alpha-1}}\right)a(s)f(s,u(s))ds \\
&\geq \int_0^{+\infty}\min_{t\in[\frac{1}{k},k]}\frac{G_1(t,s)}{1+t^{\alpha-1}}a(s)f(s,u(s))ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_0^{+\infty} \min_{t\in[\frac{1}{k},k]} \frac{G_1(t,s)}{1+t^{\alpha-1}} a(s) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&\geq \int_0^{+\infty} \frac{G_1(\frac{1}{k},s)}{1+k^{\alpha-1}} a(s) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&\geq \int_{\frac{1}{k}}^{k} \frac{G_1(\frac{1}{k},s)}{1+k^{\alpha-1}} a(s) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{k^{\alpha-1}} \frac{1}{(1+k^{\alpha-1})} \int_{\frac{1}{k}}^{k} a(s) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&> \frac{q}{K} \frac{1}{\Gamma(\alpha)} \frac{1}{k^{\alpha-1}} \frac{1}{(1+k^{\alpha-1})} \int_{\frac{1}{k}}^{k} a(s) ds \\
&= q.
\end{aligned}
$$

We conclude that (i) of Theorem 3.9 is satisfied. We next adress (ii) of Theorem 3.9. So, let us choose $u \in \partial P(\beta, r)$. Then $\beta(u) = \max_{t\in[\frac{1}{k},k]} \frac{u(t)}{1+t^{\alpha-1}} = r$. This implies $0 \leq \frac{u(t)}{1+t^{\alpha-1}} \leq r, \quad t \in \left[\frac{1}{k}, k\right]$. Noticing that $\|u\| \leq \frac{1}{\lambda(k)}\gamma(u) \leq \frac{1}{\lambda(k)}\beta(u) = \frac{1}{\lambda(k)} r$, we get

$$
0 \leq \frac{u(t)}{1+t^{\alpha-1}} \leq \frac{r}{\lambda(k)} \quad for \quad t \in [0, +\infty).
$$

Using $(H5)$,

$$
F(t,u) < \frac{r}{N}, \quad (t,u) \in [0,+\infty) \times \left[0, \frac{r}{\lambda(k)}\right].
$$

$Tu \in P$, and so, for $t \in \left[\frac{1}{k}, k\right]$,

$$
\begin{aligned}
\frac{Tu(t)}{1+t^{\alpha-1}} &= \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f(s,u(s)) ds \\
&\leq L \int_0^{+\infty} a(s) f(s,u(s)) ds \\
&= L \int_0^{+\infty} a(s) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&< \frac{r}{N} L \int_0^{+\infty} a(s) ds \\
&< r.
\end{aligned}
$$

This yields that $\beta(Tu) < r$. Hence, condition (ii) is satisfied. For the final part, we turn to (iii) of Theorem 3.9. For this part, if we first define $u(t) = \frac{l}{2}$, for all $t \in [0, k]$, then $\alpha(\frac{l}{2}) < l$, and

$P(\alpha, l) \neq \emptyset$. Now, let us choose $u \in \partial P(\alpha, l)$. Then $\alpha(u) = \max_{t\in[0,k]} \frac{u(t)}{1+t^{\alpha-1}} = l$. This implies

$$0 \leq \frac{u(t)}{1+t^{\alpha-1}} \leq l, \quad t \in [0,k].$$

Using assumption $(H6)$,

$$F(t,u) > \frac{l}{K}, \quad (t,u) \in (t,u) \in [0,k] \times [0,l].$$

As before $Tu \in P$, and so

$$\begin{aligned}
\alpha(Tu) &= \max_{t\in[0,k]} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f(s,u(s)) ds \\
&\geq \min_{t\in[\frac{1}{k},k]} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f(s,u(s)) ds \\
&\geq \int_0^{+\infty} \min_{t\in[\frac{1}{k},k]} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f(s,u(s)) ds \\
&\geq \int_0^{+\infty} \min_{t\in[\frac{1}{k},k]} \left[\frac{G_1(t,s)}{1+t^{\alpha-1}} + \frac{G_2(t,s)}{1+t^{\alpha-1}}\right] a(s) f(s,u(s)) ds \\
&\geq \int_0^{+\infty} \min_{t\in[\frac{1}{k},k]} \frac{G_1(t,s)}{1+t^{\alpha-1}} a(s) f(s,u(s)) ds \\
&\geq \int_0^{+\infty} \frac{G_1(\frac{1}{k},s)}{1+k^{\alpha-1}} a(s) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&\geq \frac{1}{1+k^{\alpha-1}} \int_0^k G_1(\frac{1}{k},s) a(s) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&> \frac{l}{K} \frac{1}{1+k^{\alpha-1}} \left[\int_0^{\frac{1}{k}} G_1(\frac{1}{k},s) a(s) ds + \int_{\frac{1}{k}}^k G_1(\frac{1}{k},s) a(s) ds\right] \\
&\geq \frac{l}{K(1+k^{\alpha-1})} \int_{\frac{1}{k}}^k G_1(\frac{1}{k},s) a(s) ds \\
&= \frac{l}{K(1+k^{\alpha-1})\Gamma(\alpha)k^{\alpha-1}} \int_{\frac{1}{k}}^k a(s) ds \\
&= l.
\end{aligned}$$

Thus, (iii) of Theorem 3.9 is satisfied. Therefore, there exist at least two positive solutions u_1 and u_2, belonging to $\overline{P(\gamma,q)}$, of the boundary value problem (1.1) such that

$$l < \max_{t\in[0,k]} \frac{u_1(t)}{1+t^{\alpha-1}} \quad with \quad \max_{t\in[\frac{1}{k},k]} \frac{u_1(t)}{1+t^{\alpha-1}} < r \quad and \quad r < \max_{t\in[\frac{1}{k},k]} \frac{u_2(t)}{1+t^{\alpha-1}} \quad with \quad \min_{t\in[\frac{1}{k},k]} \frac{u_2(t)}{1+t^{\alpha-1}} < q.$$

□

Example 3.11. *Consider the following boundary value problem:*

$$\begin{cases} D_{0^+}^{\frac{7}{2}}u(t) + e^{-t}f(t,u(t)) = 0, \quad t \in [0,+\infty), \\ u(0) = u'(0) = u''(0) = 0, \\ D_{0^+}^{\frac{5}{2}}u(+\infty) = \sum_{i=1}^{1} \eta_i I_{0^+}^{\frac{1}{2}} u(\xi_i). \end{cases} \tag{3.9}$$

where

$$F(t,u(t)) = \begin{cases} \dfrac{9u+740}{14800}, & 0 \leq u \leq 740, \\ \dfrac{10^7-1}{520}(u-740) + \dfrac{1}{2}, & 740 \leq u \leq 10^3, \\ 5.10^6, & 10^3 \leq u \leq 2112.10^3. \end{cases}$$

By simple calculations, we have $N = 0,60180$, $K = 0,0002166994$ and $\lambda(k) = 0,00047348$. If we choose $\ell = 10^{-5}$, $r = \frac{1}{4}$ and $q = 10^3$, then we get $\ell = 10^{-5} < \frac{K}{N}r = (1,200026).10^{-4} < \frac{K}{N}\lambda(k)q = (1,705).10^{-4}$. It can be easily seen that the conditions $(H1)-(H3)$, $(H4)$-$(H6)$ are provided. Then all conditions of Theorem 3.10 hold. Hence, BVP (3.9) has at least two positive solutions u_1 and u_2 such that

$$10^{-5} < \max_{t\in[0,4]} \frac{u_1(t)}{1+t^{\frac{5}{2}}} \quad \textit{with} \quad \max_{t\in[\frac{1}{4},4]} \frac{u_1(t)}{1+t^{\frac{5}{2}}} < \frac{1}{4} \quad \textit{and} \quad \frac{1}{4} < \max_{t\in[\frac{1}{4},4]} \frac{u_2(t)}{1+t^{\frac{5}{2}}} \quad \textit{with} \quad \min_{t\in[\frac{1}{4},4]} \frac{u_2(t)}{1+t^{\frac{5}{2}}} < 10^3.$$

We will apply the following theorem so that (1.1) has at least three positive solutions.

Theorem 3.12. *(Ren Fixed Point Theorem)[7] Let P be a cone in a real Banach space E. Let φ, θ and ψ are increasing, nonnegative continuous functionals on P. There are constants $v > 0$, $\tilde{M} > 0$ such that*

$$\psi(u) \leq \theta(u) \leq \varphi(u) \quad \textit{and} \quad \|u\| \leq \tilde{M}\psi(u)$$

for all $u \in \overline{P(\psi, v)}$. Suppose there exist a completely continuous operator $T : \overline{P(\psi, v)} \to P$ and constants $0 < h < p < v$ such that
(i) *$\psi(Tu) < v$ for $\forall u \in \partial P(\psi, v)$;*
(ii) *$\theta(Tu) > p$ for $\forall u \in \partial P(\theta, p)$;*
(iii) *$P(\varphi, h) \neq \emptyset$ and $\varphi(Tu) < h$ for $\forall u \in \partial P(\varphi, h)$.*
Then T has at least three fixed points u_1, u_2 and u_3 belonging to $P(\psi, v)$ such that

$$0 \leq \varphi(u_1) < h < \varphi(u_2), \quad \theta(u_2) < p < \theta(u_3), \quad \psi(u_3) < v.$$

Let $0 < \frac{1}{k} < \mu < k$ and define the nonnegative, increasing, continuous functionals ψ , θ and φ, by

$$\psi(u) = \min_{t\in[\frac{1}{k},k]} \frac{u(t)}{1+t^{\alpha-1}}, \qquad \theta(u) = \min_{t\in[\frac{1}{k},\mu]} \frac{u(t)}{1+t^{\alpha-1}}, \qquad \varphi(u) = \max_{t\in[0,k]} \frac{u(t)}{1+t^{\alpha-1}}. \tag{3.10}$$

and let $P(\psi, v) = \{u \in P : \psi(u) < v\}$.

For convenience, we define

$$\Omega_1 = \frac{\int_{\frac{1}{k}}^{\mu} a(s)ds}{\Gamma(\alpha)k^{\alpha-1}(1+k^{\alpha-1})}, \quad \Omega_2 = L\int_0^{+\infty} a(s)ds \tag{3.11}$$

where L is defined by Lemma 3.4.

In order to established at least three positive solutions of our boundary value problem, we give growth conditions on F.

Theorem 3.13. *Assume that $(H1)-(H3)$ hold. Suppose there exist positive numbers $h < p < v$ such that $\frac{h}{\lambda(k)} < p < \frac{\Omega_1}{\Omega_2}v$ and the function F satisfies the following conditions:*

(H7) $F(t,u) < \frac{v}{\Omega_2}$ *for* $\forall(t,u) \in [0,+\infty) \times \left[0, \frac{v}{\lambda(k)}\right]$,

(H8) $F(t,u) > \frac{p}{\Omega_1}$ *for* $\forall(t,u) \in \left[\frac{1}{k}, \mu\right] \times \left[p, \frac{p}{\lambda(k)}\right]$,

(H9) $F(t,u) < \frac{h}{\Omega_2}$ *for* $\forall(t,u) \in [0,+\infty) \times \left[0, \frac{h}{\lambda(k)}\right]$.

where Ω_1 and Ω_2 are as defined in (3.11)*. Then the boundary value problem* (1.1) *has at least three positive solutions u_1, u_2 and u_3 such that*

$$0 \le \max_{t\in[0,k]} \frac{u_1(t)}{1+t^{\alpha-1}} < h < \max_{t\in[0,k]} \frac{u_2(t)}{1+t^{\alpha-1}}, \quad \min_{t\in[\frac{1}{k},\mu]} \frac{u_2(t)}{1+t^{\alpha-1}} < p < \min_{t\in[\frac{1}{k},\mu]} \frac{u_3(t)}{1+t^{\alpha-1}}, \quad \min_{t\in[\frac{1}{k},k]} \frac{u_3(t)}{1+t^{\alpha-1}} < v.$$

Proof . It is clear that for each $u \in P$, $\psi(u) \le \theta(u) \le \varphi(u)$. Also, from Lemma 3.6, for each $u \in P$

$$\|u\| \le \frac{1}{\lambda(k)} \min_{\frac{1}{k}\le t\le k} \frac{u(t)}{1+t^{\alpha-1}} = \frac{1}{\lambda(k)}\psi(u), \quad \tilde{M} = \frac{1}{\lambda(k)} > 0.$$

The proof of this theorem will proceed in a similar to the proof of Theorem 3.10. By Lemma 3.8, we know that $T : \overline{P(\psi, v)} \to P$ is completely continuous operator. In order to show that (i) of Theorem 3.12, we choose $u \in \partial P(\psi, v)$. Using $(H7)$, for $t \in \left[\frac{1}{k}, k\right]$

$$\begin{aligned}
\frac{Tu(t)}{1+t^{\alpha-1}} &= \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f(s,u(s))ds \\
&\le L\int_0^{+\infty} a(s)F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right)ds \\
&< L\frac{v}{\Omega_2}\int_0^{+\infty} a(s)ds \\
&= v.
\end{aligned}$$

This yields that $\psi(Tu) < v$. Hence, condition (i) is satisfied. We now turn to property (ii) of Theorem 3.12. Choose $u \in \partial P(\psi, v)$. Using assumption $(H8)$,

$$\begin{aligned}
\theta(Tu) &= \min_{t\in\left[\frac{1}{k},\mu\right]} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f(s,u(s))ds \\
&\geq \min_{t\in\left[\frac{1}{k},k\right]} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f(s,u(s))ds \\
&\geq \int_0^{+\infty} \min_{t\in\left[\frac{1}{k},k\right]} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f(s,u(s))ds \\
&\geq \int_0^{+\infty} \frac{G_1(\frac{1}{k},s)}{1+k^{\alpha-1}} a(s) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&\geq \int_{\frac{1}{k}}^{\mu} \frac{G_1(\frac{1}{k},s)}{1+k^{\alpha-1}} a(s) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&= \frac{1}{\Gamma(\alpha)k^{\alpha-1}(1+k^{\alpha-1})} \int_{\frac{1}{k}}^{\mu} a(s) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&> \frac{p}{\Omega_1\Gamma(\alpha)k^{\alpha-1}(1+k^{\alpha-1})} \int_{\frac{1}{k}}^{\mu} a(s)ds \\
&= p.
\end{aligned}$$

Thus, the condition (ii) Theorem 3.12 is satisfied. Finally, we turn to (iii) of Theorem 3.12. We note that $u(t) = \dfrac{h}{3}$, $t \in [0,k]$ is a member of $P(\varphi,h)$, and so $P(\varphi,h) \neq \emptyset$. Now, let $u \in \partial P(\varphi,h)$. Using $(H9)$, for $t \in [0,k]$,

$$\begin{aligned}
\frac{Tu(t)}{1+t^{\alpha-1}} &= \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f(s,u(s))ds \\
&\leq L \int_0^{+\infty} a(s) f(s,u(s))ds \\
&= L \int_0^{+\infty} a(s) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&< \frac{h}{\lambda(k)} L \int_0^{+\infty} a(s)ds \\
&= h
\end{aligned}$$

This implies that $\varphi(Tu) < h$. So, the condition (iii) Theorem 3.12 is satisfied.
As a result, all conditions of Theorem 3.12 are satisfied. Therefore, the boundary value problem (1.1) has at least three positive solutions u_1, u_2 and u_3, belonging to $\overline{P(\psi,v)}$, of the boundary value problem 1.1 such that

$$0 \leq \max_{t\in[0,k]} \frac{u_1(t)}{1+t^{\alpha-1}} < h < \max_{t\in[0,k]} \frac{u_2(t)}{1+t^{\alpha-1}}, \quad \min_{t\in[\frac{1}{k},\mu]} \frac{u_2(t)}{1+t^{\alpha-1}} < p < \min_{t\in[\frac{1}{k},\mu]} \frac{u_3(t)}{1+t^{\alpha-1}}, \quad \min_{t\in[\frac{1}{k},k]} \frac{u_3(t)}{1+t^{\alpha-1}} < v.$$

□

Example 3.14. *Consider the following boundary value problem:*

$$\begin{cases} D_{0^+}^{\frac{9}{2}}u(t) + 16^{-t}\ln 16 f(t,u(t)) = 0, \quad t \in [0,+\infty)\,, \\ u(0) = u'(0) = u''(0) = u'''(0) = 0, \\ D_{0^+}^{\frac{7}{2}}u(+\infty) = \sum_{i=1}^{1} \eta_i I_{0^+}^{\frac{1}{2}} u(\xi_i). \end{cases} \tag{3.12}$$

where

$$F(t,u(t)) = \begin{cases} 0, & 0 \le u \le 8256.10^{-3}, \\ \dfrac{10^7}{1744}\,(500u - 4128)\,, & 8256.10^{-3} \le u \le 10, \\ 5.10^6, & 10 \le u \le 8256.10^6. \end{cases}$$

After some calculations, we obtain $\Omega_1 = (2,2779).10^{-6}$, $\Omega_2 = 0,1719434922$ and $\lambda(k) = (1,21124).10^{-4}$. If we choose $h = 10^{-3}$, $p = 10$ and $v = 10^6$, then this inequality $\dfrac{h}{\lambda(k)} = \dfrac{10^{-3}}{(1,21124).10^{-4}} < p = 10 < \dfrac{\Omega_1}{\Omega_2} v = (1,3247957).10^{-5}10^6$ is satisfied. It is easy to see that the conditions of $(H1)-(H3)$, $(H7)$-$(H9)$ are satisfied. So, all conditions of Theorem 3.13 hold. Therefore, BVP (3.12) has at least three positive solutions u_1, u_2 and u_3 such that

$$0 \le \max_{t\in[0,4]} \frac{u_1(t)}{1+t^{\frac{7}{2}}} < 10^{-3} < \max_{t\in[0,4]} \frac{u_2(t)}{1+t^{\frac{7}{2}}}, \quad \min_{t\in[\frac{1}{4},1]} \frac{u_2(t)}{1+t^{\frac{7}{2}}} < 10 < \min_{t\in[\frac{1}{4},1]} \frac{u_3(t)}{1+t^{\frac{7}{2}}}, \quad \min_{t\in[\frac{1}{4},4]} \frac{u_3(t)}{1+t^{\frac{7}{2}}} < 10^6.$$

4. Conclusions

In this paper, we consider BVP (1.1) on an infinite interval. We theoretically prove using the double fixed point and Ren's fixed point theorems the existence of at least two or three positive solutions. Some appropriate examples that support the theoretical results are provided. In the future, methods dealing with the existence of positive solutions of boundary value problems can be developed. Furthermore, the existence of exactly two or three positive solutions can also be investigated.

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