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New Ostrowski type conformable fractional inequalities concerning differentiable generalized relative semi–$(r; m, p, q, h_1, h_2)$–preinvex mappings

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Abstract

In this article, we first presented a new integral identity concerning differentiable mappings defined on m–invex set. By using the notion of generalized relative semi–$(r; m, p, q, h_1, h_2)$–preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Ostrowski type conformable fractional integral inequalities are established. It is pointed out that some new special cases can be deduced from main results of the article.

Keywords: Ostrowski type inequality, Hölder's inequality, Minkowski's inequality, power mean inequality, m–invex
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1. Introduction and preliminaries

The subsequent inequality is known as Ostrowski inequality which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

Theorem 1.1. *Let $f : I \longrightarrow \mathbb{R}$ be a mapping differentiable on I° and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a, b]. \tag{1.1}$$

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Ostrowski inequality is playing a very important role in all the fields of mathematics, especially in the theory of approximations. Thus such inequalities were studied extensively by many researches and numerous generalizations, extensions and variants of them for various kind of functions like bounded variation, synchronous, Lipschitzian, monotonic, absolutely, continuous and n-times differentiable mappings etc. appeared in a number of papers, see [2]-[4],[6]-[11],[13, 14],[16]-[22],[25],[28]-[30],[35]-[41],[43, 46, 49]. In numerical analysis many quadrature rules have been established to approximate the definite integrals, see [15, 24, 26, 27, 31, 34, 42, 44, 48]. Ostrowski inequality provides the bounds for many numerical quadrature rules. Fractional calculus [23], was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.2. *Let $f \in L[a,b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du$. *Here* $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Ostrowski type inequalities for functions of different classes, see [23].

Let us recall some special functions and evoke some basic definitions as follows.

Definition 1.3. *The Euler beta function is defined for $a, b > 0$ as*

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Definition 1.4. *The incomplete beta function is defined for $a, b > 0$ as*

$$\beta_t(a,b) = \int_0^t x^{a-1}(1-x)^{b-1}dx, \quad 0 < t \leq 1.$$

For $t = 1$, the incomplete beta function coincides with the complete beta function.

Recently, some authors, started to study on conformable fractional integrals. In [19], Khalil et al. defined the fractional integral of order $0 < \alpha \leq 1$ only. In [1], Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha > 0$.

Definition 1.5. *Let $\alpha \in (n, n+1]$ and set $\beta = \alpha - n$, then the left conformable fractional integral starting at a is defined by*

$$\left(I^a_{\alpha}f\right)(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx.$$

Analogously, the right conformable fractional integral is defined by

$$\left(I_{\alpha}^{b} f\right)(t)=\frac{1}{n!}\int_{t}^{b}(x-t)^{n}(b-x)^{\beta-1}f(x)dx.$$

Notice that if $\alpha=n+1$ then $\beta=\alpha-n=n+1-n=1$ where $n=0,1,2,\ldots$, and hence $\left(I_{\alpha}^{a} f\right)(t)=\left(J_{n+1}^{a} f\right)(t)$. Set et al. established some results for some kind of inequalities via conformable fractional integrals, see [37]-[40].

Definition 1.6. *[47] A set $M_{\varphi}\subseteq\mathbb{R}^{n}$ is said to be a relative convex (φ-convex) set, if and only if, there exists a function $\varphi:\mathbb{R}^{n}\longrightarrow\mathbb{R}^{n}$ such that,*

$$t\varphi(x)+(1-t)\varphi(y)\in M_{\varphi},\ \forall\, x,y\in\mathbb{R}^{n}:\varphi(x),\varphi(y)\in M_{\varphi},t\in[0,1]. \tag{1.2}$$

Definition 1.7. *[47] A function f is said to be a relative convex (φ-convex) on a relative convex (φ-convex) set M_{φ}, if and only if, there exists a function $\varphi:\mathbb{R}^{n}\longrightarrow\mathbb{R}^{n}$ such that,*

$$f(t\varphi(x)+(1-t)\varphi(y))\leq tf(\varphi(x))+(1-t)f(\varphi(y)), \tag{1.3}$$

$\forall\ x,y\in\mathbb{R}^{n}:\varphi(x),\varphi(y)\in M_{\varphi},t\in[0,1]$.

Definition 1.8. *[5] A set $K\subseteq\mathbb{R}^{n}$ is said to be invex with respect to the mapping $\eta:K\times K\longrightarrow\mathbb{R}^{n}$, if $x+t\eta(y,x)\in K$ for every $x,y\in K$ and $t\in[0,1]$.*

Notice that every convex set is invex with respect to the mapping $\eta(y,x)=y-x$, but the converse is not necessarily true, see [5, 45].

Definition 1.9. *[33] The function f defined on the invex set $K\subseteq\mathbb{R}^{n}$ is said to be preinvex with respect η, if for every $x,y\in K$ and $t\in[0,1]$, we have that*

$$f\left(x+t\eta(y,x)\right)\leq(1-t)f(x)+tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x)=y-x$, but the converse is not true.

Definition 1.10. *[24] Let $h:[0,1]\longrightarrow\mathbb{R}$ be a non-negative function and $h\neq0$. The function f on the invex set K is said to be h-preinvex with respect to η, if*

$$f\big(x+t\eta(y,x)\big)\leq h(1-t)f(x)+h(t)f(y) \tag{1.4}$$

for each $x,y\in K$ and $t\in[0,1]$ where $f(\cdot)>0$.

Definition 1.11. *[44] Let $h:J\subseteq\mathbb{R}\rightarrow\mathbb{R}$ be a positive function, $h\not\equiv0$. We say that $f:I\subseteq\mathbb{R}\rightarrow\mathbb{R}$ is h-convex, if f is non-negative and for all $x,y\in I$ and $t\in(0,1)$, one has*

$$f\big(tx+(1-t)y\big)\leq h(t)f(x)+h(1-t)f(y). \tag{1.5}$$

Definition 1.12. *[42] Let $f:K\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a non-negative function. We say that $f:K\longrightarrow\mathbb{R}$ is a tgs-convex function on K, if the inequality*

$$f\big((1-t)x+ty\big)\leq t(1-t)[f(x)+f(y)] \tag{1.6}$$

holds for all $x,y\in K$ and $t\in(0,1)$. We say that f is tgs-concave if $(-f)$ is tgs-convex.

Definition 1.13. *[27] A function: $I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall\, x, y \in I$, and $t \in (0,1)$, with $m \in [0,1]$, satisfies the following inequality*

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.7}$$

Definition 1.14. *[12] A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ grips for each $x, y \in K$ and any $t \in [0,1]$.*

Remark 1.15. *In definition 1.14, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.*

Definition 1.16. *[32] Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ are continuous. A function $f : K \longrightarrow \mathbb{R}$, if*

$$f\big(mx + t\eta(y,x,m)\big) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{1.8}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, with $m \in (0,1]$, then we say that $f(x)$ is a generalized (m, h_1, h_2)-preinvex function with respect to η. If the inequality (1.8) reverses, then f is said to be (m, h_1, h_2)-preincave on K.

We are in position to introduced the notion of generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex functions.

Definition 1.17. *Let $K \subseteq \mathbb{R}$ be an open m–invex set with respect to the mapping $\xi : K \times K \times (0,1] \longrightarrow \mathbb{R}$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\vartheta : I \longrightarrow K$ are continuous functions. A function $f : K \longrightarrow (0,+\infty)$ is said to be generalized relative semi–$(r; m, p, q, h_1, h_2)$–preinvex, if*

$$f\big(m\vartheta(x) + t\xi(\vartheta(y), \vartheta(x), m)\big) \leq M_r(h_1(t), h_2(t); mf(x), f(y), p, q) \tag{1.9}$$

holds for all $x, y \in I$ and $t \in [0,1]$, for $p, q > -1$ and some fixed $m \in (0,1]$, where

$$M_r(h_1(t), h_2(t); mf(x), f(y), p, q) = \begin{cases} \Big[mh_1^p(t)f^r(x) + h_2^q(t)f^r(y)\Big]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \big[mf(x)\big]^{h_1^p(t)}\big[f(y)\big]^{h_2^q(t)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(x)$ and $f(y)$.

Remark 1.18. *In definition 1.17, if we choose $r = p = q = 1$ and $\vartheta(x) = x$, $\forall x \in I$, then we get Definition 1.16. If we choose $r = p = q = 1$, $\vartheta(x) = x$, $\forall x \in I$ and $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{m\sqrt{t}}{2\sqrt{1-t}}$, then we get MT_m-preinvex function, see [16].*

Remark 1.19. *For $r = p = q = 1$, let us discuss some special cases in definition 1.17 as follows:*

(I) Taking $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-$(m,s)-$Breckner-preinvex functions.
(II) Choosing $h_1(t) = h_2(t) = 1$, then we get generalized relative semi-(m,P)-preinvex functions.

(III) Taking $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex functions.
(IV) Choosing $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get generalized relative semi-(m,h)-preinvex functions.
(V) Taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-(m,tgs)-preinvex functions.
(VI) Choosing $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex functions.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

Motivated by the above literatures, the main objective of this article is to establish in Section 2 some new estimates on generalizations to Ostrowski type conformable fractional integral inequalities associated with differentiable generalized relative semi–$(r;m,p,q,h_1,h_2)$–preinvex mappings on m–invex set. It is pointed out that some new special cases will be deduced from main results. In Section 3, some conclusions and future research will be given.

2. Main results

In this section, in order to prove our main results regarding some generalizations of Ostrowski type inequalities for differentiable generalized relative semi–$(r;m,p,q,h_1,h_2)$–preinvex functions via conformable fractional integrals, we need the following integral identity.

Lemma 2.1. *Let $\vartheta : I \longrightarrow K$ be a continuous function. Suppose $K = [m\vartheta(a), m\vartheta(a)+\xi(\vartheta(b),\vartheta(a),m)] \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\xi : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and let $\xi(\vartheta(b),\vartheta(a),m) > 0$. Assume that $f : K \longrightarrow \mathbb{R}$ be a differentiable function on K° and $f' \in L(K)$. Then for $r \geq 0$ with $\alpha \in (n, n+1]$, $n = 0,1,2,...$, the following integral identity holds:*

$$\frac{\beta(n+1,\alpha-n)}{(r+1)^{n+1}\xi(\vartheta(b),\vartheta(a),m)}$$
$$\times\Bigg[\xi^\alpha(\vartheta(x),\vartheta(a),m) f\left(m\vartheta(a) + \frac{\xi(\vartheta(x),\vartheta(a),m)}{r+1}\right)$$
$$-\xi^\alpha(\vartheta(x),\vartheta(b),m) f\left(m\vartheta(b) + \frac{\xi(\vartheta(x),\vartheta(b),m)}{r+1}\right)\Bigg]$$
$$-\frac{1}{\xi(\vartheta(b),\vartheta(a),m)} \times \Bigg\{ \int_{m\vartheta(a)}^{m\vartheta(a)+\frac{\xi(\vartheta(x),\vartheta(a),m)}{r+1}} (t-m\vartheta(a))^n$$
$$\times\big[\xi(\vartheta(x),\vartheta(a),m) - (r+1)(t-m\vartheta(a))\big]^{\alpha-n-1} f(t)dt$$
$$-\int_{m\vartheta(b)}^{m\vartheta(b)+\frac{\xi(\vartheta(x),\vartheta(b),m)}{r+1}} (t-m\vartheta(b))^n$$
$$\times\big[\xi(\vartheta(x),\vartheta(b),m) - (r+1)(t-m\vartheta(b))\big]^{\alpha-n-1} f(t)dt\Bigg\}$$

$$= \frac{\xi^{\alpha+1}(\vartheta(x),\vartheta(a),m)}{(r+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.1}$$

$$\times \int_0^1 \beta_t(n+1,\alpha-n) f'\left(m\vartheta(a) + \left(\frac{t}{r+1}\right)\xi(\vartheta(x),\vartheta(a),m)\right) dt$$

$$- \frac{\xi^{\alpha+1}(\vartheta(x),\vartheta(b),m)}{(r+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}$$

$$\times \int_0^1 \beta_t(n+1,\alpha-n) f'\left(m\vartheta(b) + \left(\frac{t}{r+1}\right)\xi(\vartheta(x),\vartheta(b),m)\right) dt.$$

We denote

$$I_{f,\xi,\vartheta}(x;\alpha,n,m,r,a,b) = \frac{\xi^{\alpha+1}(\vartheta(x),\vartheta(a),m)}{(r+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.2}$$

$$\times \int_0^1 \beta_t(n+1,\alpha-n) f'\left(m\vartheta(a) + \left(\frac{t}{r+1}\right)\xi(\vartheta(x),\vartheta(a),m)\right) dt$$

$$- \frac{\xi^{\alpha+1}(\vartheta(x),\vartheta(b),m)}{(r+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}$$

$$\times \int_0^1 \beta_t(n+1,\alpha-n) f'\left(m\vartheta(b) + \left(\frac{t}{r+1}\right)\xi(\vartheta(x),\vartheta(b),m)\right) dt.$$

Proof . Integrating by parts equation (2.2), we get

$$I_{f,\xi,\vartheta}(x;\alpha,n,m,r,a,b) = \frac{\xi^{\alpha+1}(\vartheta(x),\vartheta(a),m)}{(r+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}$$

$$\times \left[\frac{(r+1)\beta_t(n+1,\alpha-n)}{\xi(\vartheta(x),\vartheta(a),m)} f\left(m\vartheta(a) + \left(\frac{t}{r+1}\right)\xi(\vartheta(x),\vartheta(a),m)\right)\Bigg|_0^1\right.$$

$$\left. - \frac{r+1}{\xi(\vartheta(x),\vartheta(a),m)} \int_0^1 t^n(1-t)^{\alpha-n-1} f\left(m\vartheta(a) + \left(\frac{t}{r+1}\right)\xi(\vartheta(x),\vartheta(a),m)\right) dt\right]$$

$$- \frac{\xi^{\alpha+1}(\vartheta(x),\vartheta(b),m)}{(r+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}$$

$$\times \left[\frac{(r+1)\beta_t(n+1,\alpha-n)}{\xi(\vartheta(x),\vartheta(b),m)} f\left(m\vartheta(b) + \left(\frac{t}{r+1}\right)\xi(\vartheta(x),\vartheta(b),m)\right)\Bigg|_0^1\right.$$

$$\left. - \frac{r+1}{\xi(\vartheta(x),\vartheta(b),m)} \int_0^1 t^n(1-t)^{\alpha-n-1} f\left(m\vartheta(b) + \left(\frac{t}{r+1}\right)\xi(\vartheta(x),\vartheta(b),m)\right) dt\right]$$

$$= \frac{\beta(n+1,\alpha-n)}{(r+1)^{n+1}\xi(\vartheta(b),\vartheta(a),m)}$$

$$\times \left[\xi^{\alpha}(\vartheta(x),\vartheta(a),m) f\left(m\vartheta(a) + \frac{\xi(\vartheta(x),\vartheta(a),m)}{r+1}\right)\right.$$

$$
\begin{aligned}
&-\xi^{\alpha}(\vartheta(x),\vartheta(b),m)f\left(m\vartheta(b)+\frac{\xi(\vartheta(x),\vartheta(b),m)}{r+1}\right)\Bigg]\\
&-\frac{1}{\xi(\vartheta(b),\vartheta(a),m)}\times\Bigg\{\int_{m\vartheta(a)}^{m\vartheta(a)+\frac{\xi(\vartheta(x),\vartheta(a),m)}{r+1}}(t-m\vartheta(a))^{n}\\
&\times\left[\xi(\vartheta(x),\vartheta(a),m)-(r+1)(t-m\vartheta(a))\right]^{\alpha-n-1}f(t)dt\\
&-\int_{m\vartheta(b)}^{m\vartheta(b)+\frac{\xi(\vartheta(x),\vartheta(b),m)}{r+1}}(t-m\vartheta(b))^{n}\\
&\times\left[\xi(\vartheta(x),\vartheta(b),m)-(r+1)(t-m\vartheta(b))\right]^{\alpha-n-1}f(t)dt\Bigg\}.
\end{aligned}
$$

The proof of Lemma 2.1 is completed. □

Remark 2.2. *In Lemma 2.1, if we choose $m=1$, $r=0$, $\alpha=n+1$, where $n=0,1,2,...$, $\xi(\vartheta(y),\vartheta(x),m)=\vartheta(y)-m\vartheta(x)$ and $\vartheta(x)=x$, $\forall\, x\in I$, we get ([22], Lemma 1).*

Remark 2.3. *In Lemma 2.1, if we choose $r=0$, we get the following equality for conformable fractional integrals:*

$$
\begin{aligned}
&\frac{\beta(n+1,\alpha-n)}{\xi(\vartheta(b),\vartheta(a),m)}\\
&\times\Big[\xi^{\alpha}(\vartheta(x),\vartheta(a),m)f\left(m\vartheta(a)+\xi(\vartheta(x),\vartheta(a),m)\right)\\
&-\xi^{\alpha}(\vartheta(x),\vartheta(b),m)f\left(m\vartheta(b)+\xi(\vartheta(x),\vartheta(b),m)\right)\Big]\\
&-\frac{n!}{\xi(\vartheta(b),\vartheta(a),m)}\\
&\times\left[\left(I_{\alpha}^{(m\vartheta(a)+\xi(\vartheta(x),\vartheta(a),m))}f\right)(m\vartheta(a))-\left(I_{\alpha}^{(m\vartheta(b)+\xi(\vartheta(x),\vartheta(b),m))}f\right)(m\vartheta(b))\right]\\
&=\frac{\xi^{\alpha+1}(\vartheta(x),\vartheta(a),m)}{\xi(\vartheta(b),\vartheta(a),m)}\int_0^1\beta_t(n+1,\alpha-n)f'\left(m\vartheta(a)+t\xi(\vartheta(x),\vartheta(a),m)\right)dt \qquad (2.3)\\
&-\frac{\xi^{\alpha+1}(\vartheta(x),\vartheta(b),m)}{\xi(\vartheta(b),\vartheta(a),m)}\int_0^1\beta_t(n+1,\alpha-n)f'\left(m\vartheta(b)+t\xi(\vartheta(x),\vartheta(b),m)\right)dt.
\end{aligned}
$$

Using relation (2.2), the following results can be obtained for the corresponding version for power of the first derivative.

Theorem 2.4. *Let $\alpha\in(n,n+1]$ where $n=0,1,2,...$, $r_1\in[0,1]$, $0<r\leq 1$ and $p_1,p_2>-1$. Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\xi:K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$. Suppose $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\vartheta:I\longrightarrow K$ are continuous functions. Assume that $f:K\longrightarrow(0,+\infty)$ be a differentiable function on K° where $\xi(\vartheta(b),\vartheta(a),m)>0$. If $f'(x)^q$ is* *positive generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvex functions, $q>1$, $p^{-1}+q^{-1}=1$, then the following inequality holds:*

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,r_1,a,b)\right| \le \frac{\delta^{\frac{1}{p}}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.4}$$

$$\times\left\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\Big[mf'(a)^{rq}I^r(h_1(t))+f'(x)^{rq}I^r(h_2(t))\Big]^{\frac{1}{rq}}\right.$$

$$\left.+|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\Big[mf'(b)^{rq}I^r(h_1(t))+f'(x)^{rq}I^r(h_2(t))\Big]^{\frac{1}{rq}}\right\},$$

where

$$\delta=\int_0^1\left[\beta_t(n+1,\alpha-n)\right]^p dt$$

and

$$I(h_i(t))=\int_0^1 h_i^{\frac{p_i}{r}}\left(\frac{t}{r_1+1}\right)dt,\ \forall\, i=1,2.$$

Proof . From relation (2.2), positive generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvexity of $f'(x)^q$, Hölder's inequality, Minkowski's inequality and properties of the modulus, we have

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,r_1,a,b)\right| \le \frac{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}}{(r_1+1)^{n+2}|\xi(\vartheta(b),\vartheta(a),m)|}$$

$$\times\int_0^1\beta_t(n+1,\alpha-n)\left|f'\left(m\vartheta(a)+\left(\frac{t}{r_1+1}\right)\xi(\vartheta(x),\vartheta(a),m)\right)\right|dt$$

$$+\frac{|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}}{(r_1+1)^{n+2}|\xi(\vartheta(b),\vartheta(a),m)|}$$

$$\times\int_0^1\beta_t(n+1,\alpha-n)\left|f'\left(m\vartheta(b)+\left(\frac{t}{r_1+1}\right)\xi(\vartheta(x),\vartheta(b),m)\right)\right|dt$$

$$\le\frac{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\left(\int_0^1\left[\beta_t(n+1,\alpha-n)\right]^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left(f'\left(m\vartheta(a)+\left(\frac{t}{r_1+1}\right)\xi(\vartheta(x),\vartheta(a),m)\right)\right)^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\left(\int_0^1\left[\beta_t(n+1,\alpha-n)\right]^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left(f'\left(m\vartheta(b)+\left(\frac{t}{r_1+1}\right)\xi(\vartheta(x),\vartheta(b),m)\right)\right)^q dt\right)^{\frac{1}{q}}$$

$$\le\frac{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\delta^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left[mh_1^{p_1}\left(\frac{t}{r_1+1}\right)(f'(a))^{rq}+h_2^{p_2}\left(\frac{t}{r_1+1}\right)(f'(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$+\frac{|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\delta^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left[mh_1^{p_1}\left(\frac{t}{r_1+1}\right)(f'(b))^{rq}+h_2^{p_2}\left(\frac{t}{r_1+1}\right)(f'(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$\leq\frac{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\delta^{\frac{1}{p}}$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}(f'(a))^q h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r+\left(\int_0^1(f'(x))^q h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\right\}^{\frac{1}{rq}}$$

$$+\frac{|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\delta^{\frac{1}{p}}$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}(f'(b))^q h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r+\left(\int_0^1(f'(x))^q h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\right\}^{\frac{1}{rq}}$$

$$=\frac{\delta^{\frac{1}{p}}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}$$

$$\times\Bigg\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\Big[mf'(a)^{rq}I^r(h_1(t))+f'(x)^{rq}I^r(h_2(t))\Big]^{\frac{1}{rq}}$$

$$+|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\Big[mf'(b)^{rq}I^r(h_1(t))+f'(x)^{rq}I^r(h_2(t))\Big]^{\frac{1}{rq}}\Bigg\}.$$

The proof of Theorem 2.4 is completed. □
We point out some special cases of Theorem 2.4.

Corollary 2.5. *In Theorem 2.4 for $r_1 = 0$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, h_1, h_2)$-preinvex functions via conformable fractional integrals:*

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,0,a,b)\right|\leq\frac{\delta^{\frac{1}{p}}}{\xi(\vartheta(b),\vartheta(a),m)}\tag{2.5}$$

$$\times\Bigg\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\Big[mf'(a)^{rq}I^r(h_1(t))+f'(x)^{rq}I^r(h_2(t))\Big]^{\frac{1}{rq}}$$

$$+|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\Big[mf'(b)^{rq}I^r(h_1(t))+f'(x)^{rq}I^r(h_2(t))\Big]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.6. *In Corollary 2.5 if we choose* $\alpha = n+1$ *where* $n = 0, 1, 2, \ldots,$ *and* $0 < f'(x) \leq K, \forall\, x \in I$*, we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2, h_1, h_2)$*-preinvex functions via fractional integrals:*

$$\Bigg|\frac{1}{\xi(\vartheta(b),\vartheta(a),m)} \times\Big[\xi^{n+1}(\vartheta(x),\vartheta(a),m)f\left(m\vartheta(a)+\xi(\vartheta(x),\vartheta(a),m)\right) -\xi^{n+1}(\vartheta(x),\vartheta(b),m)f\left(m\vartheta(b)+\xi(\vartheta(x),\vartheta(b),m)\right)\Big] -\frac{(n+1)!}{\xi(\vartheta(b),\vartheta(a),m)} \times\Big[J^{n+1}_{(m\vartheta(a)+\xi(\vartheta(x),\vartheta(a),m))^-}f(m\vartheta(a)) - J^{n+1}_{(m\vartheta(b)+\xi(\vartheta(x),\vartheta(b),m))^-}f(m\vartheta(b))\Big]\Bigg|$$

$$\leq \frac{K\Big[mI^r(h_1(t))+I^r(h_2(t))\Big]^{\frac{1}{rq}}}{(n+1)(p(n+1)+1)^{\frac{1}{p}}} \times\left[\frac{|\xi(\vartheta(x),\vartheta(a),m)|^{n+2}+|\xi(\vartheta(x),\vartheta(b),m)|^{n+2}}{\xi(\vartheta(b),\vartheta(a),m)}\right]. \tag{2.6}$$

Corollary 2.7. *In Theorem 2.4 for* $h_1(t) = h(1-t)$ *and* $h_2(t) = h(t)$*, we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2, h)$*-preinvex functions:*

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,r_1,a,b)\right| \leq \frac{\delta^{\frac{1}{p}}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.7}$$
$$\times\Bigg\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\Big[mf'(a)^{rq}I^r(h(1-t))+f'(x)^{rq}I^r(h(t))\Big]^{\frac{1}{rq}} +|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\Big[mf'(b)^{rq}I^r(h(1-t))+f'(x)^{rq}I^r(h(t))\Big]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.8. *In Theorem 2.4 for* $h_1(t) = (1-t)^s$*,* $h_2(t) = t^s$*, we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2, s)$*-Breckner-preinvex functions:*

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,r_1,a,b)\right| \leq \frac{r^{\frac{1}{q}}\delta^{\frac{1}{p}}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.8}$$
$$\times\Bigg\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\Bigg[m\left(\frac{(r_1+1)^{\frac{sp_1}{r}+1}-r_1^{\frac{sp_1}{r}+1}}{(r+sp_1)(r_1+1)^{\frac{sp_1}{r}}}\right)^r f'(a)^{rq} +\frac{f'(x)^{rq}}{(r+sp_2)^r(r_1+1)^{sp_2}}\Bigg]^{\frac{1}{rq}}$$

$$+|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\left[m\left(\frac{(r_1+1)^{\frac{sp_1}{r}+1}-r_1^{\frac{sp_1}{r}+1}}{(r+sp_1)(r_1+1)^{\frac{sp_1}{r}}}\right)^r f'(b)^{rq}\right.$$
$$\left.+\frac{f'(x)^{rq}}{(r+sp_2)^r(r_1+1)^{sp_2}}\right]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.9. *In Theorem 2.4 for $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ and $r>s\cdot\max\{p_1,p_2\}$, we have the following inequality for generalized relative semi-$(r;m,p_1,p_2,s)$-Godunova-Levin-Dragomir-preinvex functions:*

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,r_1,a,b)\right|\leq\frac{r^{\frac{1}{q}}\delta^{\frac{1}{p}}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.9}$$
$$\times\Bigg\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\left[m\left(\frac{(r_1+1)^{1-\frac{sp_1}{r}}-r_1^{1-\frac{sp_1}{r}}}{(r-sp_1)(r_1+1)^{-\frac{sp_1}{r}}}\right)^r f'(a)^{rq}\right.$$
$$\left.+\frac{f'(x)^{rq}}{(r-sp_2)^r(r_1+1)^{-sp_2}}\right]^{\frac{1}{rq}}$$
$$+|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\left[m\left(\frac{(r_1+1)^{1-\frac{sp_1}{r}}-r_1^{1-\frac{sp_1}{r}}}{(r-sp_1)(r_1+1)^{-\frac{sp_1}{r}}}\right)^r f'(b)^{rq}\right.$$
$$\left.+\frac{f'(x)^{rq}}{(r-sp_2)^r(r_1+1)^{-sp_2}}\right]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.10. *In Theorem 2.4 for $r_1=0$ and $h_1(t)=h_2(t)=t(1-t)$, we have the following inequality for generalized relative semi-$(r;m,p_1,p_2,tgs)$-preinvex functions:*

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,0,a,b)\right|\leq\frac{\delta^{\frac{1}{p}}}{\xi(\vartheta(b),\vartheta(a),m)} \tag{2.10}$$
$$\times\Bigg\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\left[mf'(a)^{rq}\beta^r\left(1+\frac{p_1}{r},1+\frac{p_1}{r}\right)\right.$$
$$\left.+f'(x)^{rq}\beta^r\left(1+\frac{p_2}{r},1+\frac{p_2}{r}\right)\right]^{\frac{1}{rq}}$$
$$+|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\left[mf'(b)^{rq}\beta^r\left(1+\frac{p_1}{r},1+\frac{p_1}{r}\right)\right.$$
$$\left.+f'(x)^{rq}\beta^r\left(1+\frac{p_2}{r},1+\frac{p_2}{r}\right)\right]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.11. *In Theorem 2.4 for* $r_1 = 0$ *and* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $r > \frac{1}{2} \cdot \max\{p_1, p_2\}$*, we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2)$*-*MT*-preinvex functions:*

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,0,a,b)\right| \le \frac{\delta^{\frac{1}{p}}}{\xi(\vartheta(b),\vartheta(a),m)} \tag{2.11}$$
$$\times\left\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\left[mf'(a)^{rq}\left(\frac{1}{2}\right)^{rp_1}\beta^r\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)\right.\right.$$
$$\left.+f'(x)^{rq}\left(\frac{1}{2}\right)^{rp_2}\beta^r\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\right]^{\frac{1}{rq}}$$
$$+|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\left[mf'(b)^{rq}\left(\frac{1}{2}\right)^{rp_1}\beta^r\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)\right.$$
$$\left.\left.+f'(x)^{rq}\left(\frac{1}{2}\right)^{rp_2}\beta^r\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\right]^{\frac{1}{rq}}\right\}.$$

Theorem 2.12. *Let* $\alpha \in (n, n+1]$ *where* $n = 0, 1, 2, ...$, $r_1 \in [0,1]$, $0 < r \le 1$ *and* $p_1, p_2 > -1$. *Let* $K \subseteq \mathbb{R}$ *be an open* m*-invex subset with respect to* $\xi : K \times K \times (0,1] \longrightarrow \mathbb{R}$ *for some fixed* $m \in (0,1]$. *Suppose* $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ *and* $\vartheta : I \longrightarrow K$ *are continuous functions. Assume that* $f : K \longrightarrow (0,+\infty)$ *be a differentiable function on* K° *where* $\xi(\vartheta(b),\vartheta(a),m) > 0$. *If* $f'(x)^q$ *is positive generalized relative semi-*$(r; m, p_1, p_2, h_1, h_2)$*-preinvex functions,* $q \ge 1$*, then the following inequality holds:*

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,r_1,a,b)\right| \le \frac{\left[\beta(n+1,\alpha-n)-\beta(n+2,\alpha-n)\right]^{1-\frac{1}{q}}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.12}$$
$$\times\left\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\left[mf'(a)^{rq}J^r(h_1(t))+f'(x)^{rq}J^r(h_2(t))\right]^{\frac{1}{rq}}\right.$$
$$\left.+|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\left[mf'(b)^{rq}J^r(h_1(t))+f'(x)^{rq}J^r(h_2(t))\right]^{\frac{1}{rq}}\right\},$$

where

$$J(h_i(t)) = \int_0^1 \beta_t(n+1,\alpha-n)h_i^{\frac{p_i}{r}}\left(\frac{t}{r_1+1}\right)dt,\ \forall\, i = 1,2.$$

Proof . From relation (2.2), positive generalized relative semi-$(r; m, p_1, p_2, h_1, h_2)$-preinvexity of $f'(x)^q$, the well–known power mean inequality, Minkowski's inequality and properties of the modulus, we have

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,r_1,a,b)\right| \le \frac{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}}{(r_1+1)^{n+2}|\xi(\vartheta(b),\vartheta(a),m)|}$$
$$\times\int_0^1 \beta_t(n+1,\alpha-n)\left|f'\left(m\vartheta(a)+\left(\frac{t}{r_1+1}\right)\xi(\vartheta(x),\vartheta(a),m)\right)\right|dt$$

$$+\frac{|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}}{(r_1+1)^{n+2}|\xi(\vartheta(b),\vartheta(a),m)|}$$

$$\times\int_0^1 \beta_t(n+1,\alpha-n)\left|f'\left(m\vartheta(b)+\left(\frac{t}{r_1+1}\right)\xi(\vartheta(x),\vartheta(b),m)\right)\right|dt$$

$$\leq\frac{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\left(\int_0^1 \beta_t(n+1,\alpha-n)dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \beta_t(n+1,\alpha-n)\left(f'\left(m\vartheta(a)+\left(\frac{t}{r_1+1}\right)\xi(\vartheta(x),\vartheta(a),m)\right)\right)^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\left(\int_0^1 \beta_t(n+1,\alpha-n)dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \beta_t(n+1,\alpha-n)\left(f'\left(m\vartheta(b)+\left(\frac{t}{r_1+1}\right)\xi(\vartheta(x),\vartheta(b),m)\right)\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\left[\beta(n+1,\alpha-n)-\beta(n+2,\alpha-n)\right]^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \beta_t(n+1,\alpha-n)\left[mh_1^{p_1}\left(\frac{t}{r_1+1}\right)(f'(a))^{rq}+h_2^{p_2}\left(\frac{t}{r_1+1}\right)(f'(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$+\frac{|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\left[\beta(n+1,\alpha-n)-\beta(n+2,\alpha-n)\right]^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \beta_t(n+1,\alpha-n)\left[mh_1^{p_1}\left(\frac{t}{r_1+1}\right)(f'(b))^{rq}+h_2^{p_2}\left(\frac{t}{r_1+1}\right)(f'(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$\leq\frac{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\left[\beta(n+1,\alpha-n)-\beta(n+2,\alpha-n)\right]^{1-\frac{1}{q}}$$

$$\times\Bigg\{\left(\int_0^1 m^{\frac{1}{r}}(f'(a))^q\beta_t(n+1,\alpha-n)h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r$$

$$+\left(\int_0^1 (f'(x))^q\beta_t(n+1,\alpha-n)h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\Bigg\}^{\frac{1}{rq}}$$

$$+\frac{|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}\left[\beta(n+1,\alpha-n)-\beta(n+2,\alpha-n)\right]^{1-\frac{1}{q}}$$

$$\times\Bigg\{\left(\int_0^1 m^{\frac{1}{r}}(f'(b))^q\beta_t(n+1,\alpha-n)h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r$$

$$+\left(\int_0^1 (f'(x))^q\beta_t(n+1,\alpha-n)h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\Bigg\}^{\frac{1}{rq}}$$

$$= \frac{\left[\beta(n+1,\alpha-n)-\beta(n+2,\alpha-n)\right]^{1-\frac{1}{q}}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)}$$

$$\times\left\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\left[mf'(a)^{rq}J^r(h_1(t))+f'(x)^{rq}J^r(h_2(t))\right]^{\frac{1}{rq}}\right.$$

$$\left.+|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\left[mf'(b)^{rq}J^r(h_1(t))+f'(x)^{rq}J^r(h_2(t))\right]^{\frac{1}{rq}}\right\}.$$

The proof of Theorem 2.12 is completed. □
We point out some special cases of Theorem 2.12.

Corollary 2.13. *In Theorem 2.12 for $r_1 = 0$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, h_1, h_2)$-preinvex functions via conformable fractional integrals:*

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,0,a,b)\right| \leq \frac{\left[\beta(n+1,\alpha-n)-\beta(n+2,\alpha-n)\right]^{1-\frac{1}{q}}}{\xi(\vartheta(b),\vartheta(a),m)} \tag{2.13}$$

$$\times\left\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\left[mf'(a)^{rq}J^r(h_1(t))+f'(x)^{rq}J^r(h_2(t))\right]^{\frac{1}{rq}}\right.$$

$$\left.+|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\left[mf'(b)^{rq}J^r(h_1(t))+f'(x)^{rq}J^r(h_2(t))\right]^{\frac{1}{rq}}\right\}.$$

Corollary 2.14. *In Corollary 2.13 if we choose $\alpha = n+1$, where $n = 0, 1, 2, ...,$ and $0 < f'(x) \leq K$, $\forall\, x \in I$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, h_1, h_2)$-preinvex functions via fractional integrals:*

$$\left|\frac{1}{\xi(\vartheta(b),\vartheta(a),m)}\right.$$

$$\times\Big[\xi^{n+1}(\vartheta(x),\vartheta(a),m)f\left(m\vartheta(a)+\xi(\vartheta(x),\vartheta(a),m)\right)$$

$$-\xi^{n+1}(\vartheta(x),\vartheta(b),m)f\left(m\vartheta(b)+\xi(\vartheta(x),\vartheta(b),m)\right)\Big]$$

$$-\frac{(n+1)!}{\xi(\vartheta(b),\vartheta(a),m)}$$

$$\left.\times\left[J^{n+1}_{(m\vartheta(a)+\xi(\vartheta(x),\vartheta(a),m))^-}f(m\vartheta(a))-J^{n+1}_{(m\vartheta(b)+\xi(\vartheta(x),\vartheta(b),m))^-}f(m\vartheta(b))\right]\right|$$

$$\leq \frac{K\left[mJ^r(h_1(t))+J^r(h_2(t))\right]^{\frac{1}{rq}}}{((n+1)(n+2))^{1-\frac{1}{q}}} \tag{2.14}$$

$$\times\left[\frac{|\xi(\vartheta(x),\vartheta(a),m)|^{n+2}+|\xi(\vartheta(x),\vartheta(b),m)|^{n+2}}{\xi(\vartheta(b),\vartheta(a),m)}\right].$$

Corollary 2.15. *In Theorem 2.12 for $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, h)$-preinvex functions:*

$$\left|I_{f,\xi,\vartheta}(x;\alpha,n,m,r_1,a,b)\right| \leq \frac{\left[\beta(n+1,\alpha-n)-\beta(n+2,\alpha-n)\right]^{1-\frac{1}{q}}}{(r_1+1)^{n+2}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.15}$$

$$\times\Bigg\{|\xi(\vartheta(x),\vartheta(a),m)|^{\alpha+1}\Big[mf'(a)^{rq}J^r(h(1-t))+f'(x)^{rq}J^r(h(t))\Big]^{\frac{1}{rq}}$$

$$+|\xi(\vartheta(x),\vartheta(b),m)|^{\alpha+1}\Big[mf'(b)^{rq}J^r(h(1-t))+f'(x)^{rq}J^r(h(t))\Big]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.16. *In Theorem 2.12 for $r_1 = 0$, $\alpha = n+1$, where $n = 0,1,2,\ldots$, and $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, s)$-Breckner-preinvex functions:*

$$\left|I_{f,\xi,\vartheta}(x;n+1,n,m,0,a,b)\right| \leq \frac{1}{(n+1)(n+2)^{1-\frac{1}{q}}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.16}$$

$$\times\Bigg\{|\xi(\vartheta(x),\vartheta(a),m)|^{n+2}\left[mf'(a)^{rq}\beta^r\left(n+2,1+\frac{sp_1}{r}\right)+\frac{f'(x)^{rq}}{\left(\frac{sp_2}{r}+n+2\right)^r}\right]^{\frac{1}{rq}}$$

$$+|\xi(\vartheta(x),\vartheta(b),m)|^{n+2}\left[mf'(b)^{rq}\beta^r\left(n+2,1+\frac{sp_1}{r}\right)+\frac{f'(x)^{rq}}{\left(\frac{sp_2}{r}+n+2\right)^r}\right]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.17. *In Theorem 2.12 for $r_1 = 0$, $\alpha = n+1$ where $n = 0,1,2,\ldots$, and $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $r > s\cdot\max\{p_1,p_2\}$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, s)$-Godunova-Levin-Dragomir-preinvex functions:*

$$\left|I_{f,\xi,\vartheta}(x;n+1,n,m,0,a,b)\right| \leq \frac{1}{(n+1)(n+2)^{1-\frac{1}{q}}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.17}$$

$$\times\Bigg\{|\xi(\vartheta(x),\vartheta(a),m)|^{n+2}\left[mf'(a)^{rq}\beta^r\left(n+2,1-\frac{sp_1}{r}\right)+\frac{f'(x)^{rq}}{\left(n+2-\frac{sp_2}{r}\right)^r}\right]^{\frac{1}{rq}}$$

$$+|\xi(\vartheta(x),\vartheta(b),m)|^{n+2}\left[mf'(b)^{rq}\beta^r\left(n+2,1-\frac{sp_1}{r}\right)+\frac{f'(x)^{rq}}{\left(n+2-\frac{sp_2}{r}\right)^r}\right]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.18. *In Theorem 2.12 for $r_1 = 0$, $\alpha = n+1$ where $n = 0,1,2,\ldots$, and $h_1(t) = h_2(t) = t(1-t)$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, tgs)$-preinvex functions:*

$$\left|I_{f,\xi,\vartheta}(x;n+1,n,m,0,a,b)\right| \leq \frac{1}{(n+1)(n+2)^{1-\frac{1}{q}}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.18}$$

$$\times\Bigg\{|\xi(\vartheta(x),\vartheta(a),m)|^{n+2}\Bigg[mf'(a)^{rq}\beta^r\left(n+2+\frac{p_1}{r},1+\frac{p_1}{r}\right)$$

$$+f'(x)^{rq}\beta^r\left(n+2+\frac{p_2}{r},1+\frac{p_2}{r}\right)\Bigg]^{\frac{1}{rq}}$$

$$+|\xi(\vartheta(x),\vartheta(b),m)|^{n+2}\Bigg[mf'(b)^{rq}\beta^r\left(n+2+\frac{p_1}{r},1+\frac{p_1}{r}\right)$$

$$+f'(x)^{rq}\beta^r\left(n+2+\frac{p_2}{r},1+\frac{p_2}{r}\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.19. *In Theorem 2.12 for $r_1 = 0$, $\alpha = n+1$ where $n = 0,1,2,...$, and $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ and $r > \frac{1}{2}\cdot\max\{p_1,p_2\}$, we have the following inequality for generalized relative semi-$(r;m,p_1,p_2)$-MT-preinvex functions:*

$$\left|I_{f,\xi,\vartheta}(x;n+1,n,m,0,a,b)\right| \leq \frac{1}{(n+1)(n+2)^{1-\frac{1}{q}}\xi(\vartheta(b),\vartheta(a),m)} \tag{2.19}$$

$$\times\Bigg\{|\xi(\vartheta(x),\vartheta(a),m)|^{n+2}\Bigg[mf'(a)^{rq}\left(\frac{1}{2}\right)^{rp_1}\beta^r\left(n+2-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)$$

$$+f'(x)^{rq}\left(\frac{1}{2}\right)^{rp_2}\beta^r\left(n+2+\frac{p_2}{2r},1-\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}$$

$$+|\xi(\vartheta(x),\vartheta(b),m)|^{n+2}\Bigg[mf'(b)^{rq}\left(\frac{1}{2}\right)^{rp_1}\beta^r\left(n+2-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)$$

$$+f'(x)^{rq}\left(\frac{1}{2}\right)^{rp_2}\beta^r\left(n+2+\frac{p_2}{2r},1-\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Remark 2.20. *Applying our Theorems 2.4 and 2.12, we can deduce some new inequalities using special means associated with positive generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvex functions. We omit their proofs and the details are left to the interested readers.*

3. Conclusions

Motivated by this new interesting class of generalized relative semi–$(r;m,p,q,h_1,h_2)$–preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain. Since this kind of functions has large applications in many mathematical areas, our results can be applied to obtain several new fascinating inequalities in convex analysis, special functions, quantum mechanics, related optimization theory, mathematical inequalities and also may stimulate further research in different areas of pure and applied sciences. We conclude that our methods considered here may be a stimulant for further investigations concerning Ostrowski, Hermite–Hadamard and Simpson type integral inequalities for various kinds of preinvex functions involving local fractional integrals, fractional integral operators, Caputo k–fractional derivatives, q–calculus, (p,q)–calculus, time scale calculus and conformable fractional integrals.

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