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# About solving stochastic Itô-Volterra integral equations using the spectral collocation method

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**Abstract**

The purpose of this paper is to propose the spectral collocation method to solve linear and nonlinear stochastic Itô-Volterra integral equations (SVIEs). The proposed approach is different from other numerical techniques as we consider the Legendre Gauss type quadrature for estimating Itô integrals. The main characteristic of the presented method is that it reduces SVIEs into a system of algebraic equations. Thus, we can solve the problem by Newton's method. Furthermore, the convergence analysis of the approach is established. The method is computationally attractive, and to reveal the accuracy, validity, and efficiency of the proposed method, some numerical examples and convergence analysis are included.

*Keywords:* stochastic; Itô-Volterra integral equations; collocation; shifted Legendre polynomials; Gauss type quadrature.
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## 1. Introduction

Stochastic Itô-Volterra integral equations SVIEs have been used recently to solve a rang of problems in economics, sociology, biology, medical models, anthropology, as well as in engineering and finance [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. These equations arise when a noise source, which is a Gaussian white noise, is introduced into Volterra integral equations. Some papers have focused on the existence of solution for these equations [12, 13, 14]. On many occasions, it is not possible to find the exact solutions of SVIEs. Because of this fact, together with the excellent evolution of the numerical method, it is important to find their approximate solutions by using some numerical techniques

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[6, 9, 14, 15, 16, 17, 18, 19, 20]. Also, numerical methods have been proposed to solve stochastic differential equations and stochastic integral equations [6, 8, 19, 20, 21, 22, 23, 24]. However, there are a few papers considering nonlinear stochastic differential equations such as [25, 26, 27]. Solving these problems are still difficult either numerically or theoretically.
In this paper, the spectral collocation method with numerical integration will be developed to solve the following nonlinear SVIE

$$y(t) = y_0 + \int_0^t k_1(t,\tau)b(\tau, y(\tau))d\tau + \int_0^t k_2(t,\tau)\sigma(\tau, y(\tau))dB(\tau), \quad t \in D = [0,l], \tag{1.1}$$

where $k_1(t,s)$ and $k_2(t,s)$ are known functions on $D \times D$. Also, $b(t, y(t))$ and $\sigma(t, y(t))$, for $t \in D$ are stochastic processes defined on the same probability space $(\Omega, \mathbf{F}, P)$ with a filtration $\{\mathbf{F}_t | t \geq 0\}$ which is increasing and right continuous with $\mathbf{F}_0$ containing all $P$-null sets. Furthermore, $y(t)$ is the unknown random function to be found. In addition, $y_0 = y(0)$, and $B(t)$ is a standard Brownian motion defined on the probability space.
It is worth noting that, the main advantage of the proposed method is that it reduces the problem under consideration into solving a system of algebraic equations by expanding the solution in the shifted Legendre function basis and using the spectral collocation method. In addition, the Gauss-Legendre type quadrature for estimating Itô-integral is used. Finally, we can solve the system of algebraic equations by Newton's method to obtain the unknown coefficients.
This paper is organized as follows: In Section 2, some basic definitions and preliminaries for the Legendre polynomials, stochastic processes and Itô integral are presented. The approximation of the solution of the equation (1.1) using a shifted Legendre collocation approach is introduced in Section 3. Section 4 is devoted to convergence analysis of the method. The accuracy of the proposed method is demonstrated by considering some numerical examples in section 5. Finally, a brief conclusion is given in the last section.

## 2. Preliminaries and notations

### *2.1. Function approximation*

Given an open interval $I = (a,b)$ and a generic positive weight function $w$ on $I$. Let $\mathbf{P}_m$, $m \geq 1$, be the space of polynomials of degree less than or equal to $m$, and the sequence $\{\phi_j\}_{j=0}^{\infty}$ be a complete orthonormal set of functions in $L^2_w(I)$ with $\phi_j \in \mathbf{P}_j,\ j = 0, 1, \cdots$.
The inner product and its corresponding norm in $L^2_w(I)$ are defined respectively as

$$(u,v)_w = \int_a^b u(t)v(t)w(t)dt, \text{ for all } u, v \in L^2_w(I),$$
$$\|u\|_{L^2_w} = \sqrt{(u,u)_w}.$$

*Theorem 1 [28]..* For any $u \in L^2_w(I)$ and $m \in \mathbf{N}$, there exists a unique $q^*_m \in \mathbf{P}_m$ such that,

$$\|u - q^*_m\|_{L^2_w} = \inf_{q_m \in \mathbf{P}_m} \|u - q_m\|_{L^2_w},$$

where

$$q^*_m(x) = \sum_{k=0}^{m} \hat{u}_k \phi_k(x) \quad \text{with} \quad \hat{u}_k = \frac{(u,\phi_k)_w}{\|p_k\|^2_w},$$

and $\{\phi_k\}_{k=0}^m$ forms an $L^2_w$-orthogonal basis for $\mathbf{P}_m$.
In particular, we denote the best approximation polynomial $q^*_m$ by $\pi_m u$, which is the $L^2_w$-orthogonal projection of $u$.

*2.2. The shifted Legendre polynomials*

The well-known Legendre polynomials are defined on the interval $[-1, 1]$. We define the so-called shifted Legendre polynomials of degree $i$ on the interval $[0, l]$ as follows

$$\Psi_i(t) = L_i(\frac{2}{l}t - 1), \quad i = 0, 1, \ldots,$$

where $L_i(t)$ is the Legendre polynomial of degree $i$ on $[-1, 1]$.
The shifted Legendre polynomials are orthogonal with respect to the weight function $w = 1$ over $[0, l]$, namely

$$\int_0^l \Psi_i(x)\Psi_j(x)dx = \begin{cases} \frac{l}{2i+1}, & i = j, \\ 0, & i \neq j. \end{cases}$$

Therefore, if $y$ is an arbitrary element in $L^2(0, l)$, by theorem 1, $y$ has the unique best approximation $\pi_N y$, such that

$$\pi_N y(t) = \sum_{i=0}^{N} c_i \Psi_i(t) = C^T \Psi(t),$$

where vectors $C$ and $\Psi(t)$ are given by

$$C = [c_0, c_1, \ldots, c_N]^T, \quad \Psi(t) = [\Psi_0(t), \Psi_1(t), \ldots, \Psi_N(t)]^T.$$

Moreover, the coefficients $c_i$ are given by

$$c_i = \frac{2i+1}{l} \int_0^l y(\tau)\Psi_i(\tau)d\tau, \quad i = 0, 1, 2, \ldots N.$$

*2.3. Some properties of the Itô integral*

*Definition 1. [29].* (Brownian motion process)
A real-valued stochastic process $B(t)$, $t \in [0, T]$ is called Brownian motion, if it satisfies the following properties:
($i$) (Independence of increments), $B(t) - B(s)$, for $t > s$, is independent of the past, that is, of $B(u), 0 \leq u \leq s$, or of $\mathbf{F}_s$, the $\sigma$-field generated by $B(u), u \leq s$.
($ii$) (Normal increments) $B(t) - B(s)$ has Normal distribution with mean 0 and variance $t - s$.
($iii$) (Continuity of paths) $B(t), t \geq 0$ is continuous function of $t$.

*Definition 2..* Let $\{\aleph_t\}_{t\geq 0}$ be an increasing family of $\sigma-$algebras of subsets of $\Omega$. A process $g(t, \omega)$ : $[0, \infty) \times \Omega \to \mathbf{R}^n$ is called $\aleph_t$-adapted if for each $t \geq 0$ the function $\omega \to g(t, \omega)$ is $\aleph_t$-measurable.

*Definition 3..* Let $\nu = \nu(T, S)$ be the class of functions $f(t, \omega) : [0, \infty) \to \Omega \times \mathbf{R}$ such that
($i$) The function $(t, \omega) \to f(t, \omega)$ is $\mathtt{B} \times \mathbf{F}$-measurable, where $\mathtt{B}$ denotes the Borel algebra on $[0, \infty)$ and $\mathbf{F}$ is the $\sigma$-algebra on $\Omega$.
($ii$) $f$ is adapted to $\mathbf{F}_t$, where $\mathbf{F}_t$ is the $\sigma$- algebra generated by the random variables $B(s)$, $s \leq t$.
($iii$) $E[\int_S^T f^2(t, \omega)dt] < \infty$.

*Definition 4..* A function $\phi \in \nu$ is called *elementary* if it has the form

$$\phi(t, \omega) = \sum_j e_j(\omega)\chi_{[t_j, t_{j+1})}(t),$$

where $\chi$ denotes the characteristic (indicator) function. Note that since $\phi \in \nu$ each function $e_j$ must be $\mathbf{F}_t$-measurable.

*Definition 5. (The Itô integral).* Let $f \in \nu(S,T)$, then the Itô integral of $f$ is defined by

$$\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty}\int_S^T \varphi_n(t,\omega)dB_t(\omega), \quad (\lim \quad in \quad L^2(P))$$

where, $\varphi_n$ is a sequence of elementary functions such that

$$E[\int_S^T (f(t,\omega)-\varphi_n(t,\omega))^2 dt] \to 0, \ as\ n \to \infty.$$

*Theorem 2 ([29], Integration by parts).* Suppose $f(s,\omega) = f(s)$ only depends on $s$ and $f$ is continuous and of bounded variation in $[0,t]$. Then

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df_s.$$

*Theorem 3. (The Itô isometry).* Let $f \in \nu(S,T)$, then

$$E\left[\left(\int_S^T f(t,\omega)dB_t(\omega)\right)^2\right] = E\left[\int_S^T f^2(t,\omega)dt\right].$$

**Proof.** see ([29], p. 29)

## 3. Numerical solution

This section is devoted to approximation of the solution of equation (1.1) using a shifted Legendre collocation approach. To solve this equation, we first approximate function $y(t)$ as

$$y(t) \simeq y_N(t) = C^T\Psi(t), \tag{3.1}$$

with $C, \Psi(t)$ defined in the previous section. Substituting (3.1) into (1.1) yields

$$y_N(t) \simeq y_0 + \int_0^t k_1(t,\tau)b(\tau, y_N(\tau))d\tau + \int_0^t k_2(t,\tau)\sigma(\tau, y_N(\tau))dB(\tau). \tag{3.2}$$

For simplicity, we set

$$\begin{aligned} k_1(t,\tau)b(\tau,y(\tau)) = F_t(\tau,y(\tau)), \quad k_1(t_i,\tau)b(\tau,y(\tau)) = F_i(\tau,y(\tau)) \\ k_2(t,\tau)\sigma(\tau,y(\tau)) = G_t(\tau,y(\tau)), \quad k_2(t_i,\tau)\sigma(\tau,y(\tau)) = G_i(\tau,y(\tau)) \end{aligned}$$

hence, (3.2) becomes

$$y_N(t) \simeq y_0 + \int_0^t F_t(\tau, y_N(\tau))d\tau + \int_0^t G_t(\tau, y_N(\tau))dB(\tau). \tag{3.3}$$

On the other hand, using Theorem 2 yields

$$\int_0^t G_t(\tau, y_N(\tau))dB(\tau) = G_t(t, y_N(t))B(t) - \int_0^t \bar{G}_t(\tau, y_N(\tau))B(\tau)d\tau, \tag{3.4}$$

where

$$\bar{G}_t(\tau, y_N(\tau)) = \frac{\partial}{\partial \tau} G_t(\tau, y_N(\tau)).$$

Inserting (3.4) into (3.3) leads to

$$\begin{aligned} y_N(t) &\simeq y_0 + G_t(t, y_N(t))B(t) + \int_0^t F_t(\tau, y_N(\tau))d\tau \\ &- \int_0^t \bar{G}_t(\tau, y_N(\tau))B(\tau)d\tau. \end{aligned} \tag{3.5}$$

However, the second integral term in (3.5) can not be evaluated exactly. So, we transform $[0, t]$ to $[-1, 1]$ and use a Gaussian type quadrature rule to approximate the integrals. More precisely, under the linear transformation

$$\tau := \frac{t}{2}(\theta + 1), \quad \theta \in [-1, 1], \quad \tau \in [0, t],$$

(3.5) becomes

$$\begin{aligned} y_N(t) &\simeq y_0 + G_t(t, y_N(t))B(t) + \frac{t}{2} \int_{-1}^1 F_t(\tau(t, \theta), y_N(\tau(t, \theta)))d\theta \qquad (3.6) \\ &- \frac{t}{2} \int_{-1}^1 \bar{G}_t(\tau(t, \theta), y_N(\tau(t, \theta)))B(\tau(t, \theta))d\theta. \end{aligned}$$

Therefore, the integral terms are approximated by Legendre-Gauss-Radau quadrature formula with the nodes and weights denoted by $\{\theta_j, w_j\}_{j=0}^M$ as

$$\begin{aligned} y_N(t) &\simeq y_0 + G_t(t, y_N(t))B(t) \\ &+ \frac{t}{2} \sum_{j=0}^M w_j \left[ F_t(\tau(t, \theta_j), y_N(\tau(t, \theta_j))) - \bar{G}_t(\tau(t, \theta_j), y_N(\tau(t, \theta_j)))B(\tau(t, \theta_j)) \right]. \end{aligned} \tag{3.7}$$

To find the solution $y_N(t)$, we collocate (3.7) at $N + 1$ points. For suitable collocation points we use the shifted Chebyshev-Gauss collocation points

$$t_i = \frac{1}{2} \left( 1 - \cos(\frac{(2i + 1)\pi}{2N + 2}) \right), \quad i = 0, 1, \ldots, N.$$

Now, inserting $t_i$ into (3.7) leads to

$$\begin{aligned} y_N(t_i) &\simeq y_0 + G_i(t_i, y_N(t_i))B(t_i) \qquad (3.8) \\ &+ \frac{t_i}{2} \sum_{j=0}^M w_j \left[ F_i(\tau(t_i, \theta_j), y_N(\tau(t_i, \theta_j))) - \bar{G}_i(\tau(t_i, \theta_j), y_N(\tau(t_i, \theta_j)))B(\tau(t_i, \theta_j)) \right], \end{aligned}$$

$$\text{for} \quad i = 0, 1, ..., N.$$

It is worthwhile to point out the collocation points $\{t_i\}_{i=0}^N$ and $\{\theta_j\}_{j=0}^M$ are chosen differently in types and numbers.

For simplicity, we define the sequence $\{z_l\}_{l=0}^{(N+1)(M+1)+N}$ as

$$z_l = \begin{cases} t_l, & \text{for} \quad l = 0, \cdots, N, \\ \tau(t_i, \theta_j), & l = N + 1 + i(M + 1) + j, \ i = 0, 1, \cdots N, \ \text{and} \ j = 0, 1, \cdots, M. \end{cases}$$

Hence, (3.8) becomes

$$\begin{aligned} y_N(z_i) &\simeq y_0 + G_i(z_i, y_N(z_i))B(z_i) \\ &+ \frac{z_i}{2} \sum_{l=N+i(M+1)+1}^{N+(i+1)(M+1)} w_j \left[ F_i(z_l, y_N(z_l)) - \bar{G}_i(z_l, y_N(z_l))B(z_l) \right], \end{aligned} \tag{3.9}$$

for $i = 0, 1, \ldots, N$.
To obtain the values of $B$ at points $\{z_l\}_{l=0}^{(N+1)(M+1)+N}$, we apply the Definition 1 in which the Brawnian motion has Normal distribution as follows

$$B(t) - B(s) \sim \sqrt{t-s}N(0,1),$$

where $t > s$. So, the points $\{z_l\}_{l=0}^{(N+1)(M+1)+N}$ must be sorted in ascending order. Let $\{(s_i, q_i)\}_{i=0}^{(N+1)(M+1)+N}$ be a sequence such that $s_i \in \{0, 1, 2, \ldots, (N+1)(M+2)\}$, and $\{q_i\}_{i=0}^{(N+1)(M+1)+N}$ is an increasing sequence where $q_i = z_{s_i}$ for $i = 0, \cdots, (N+1)(M+1)+N$. By building the Brownian motion on nodes $\{q_i\}_{i=0}^{(N+1)(M+1)+N}$, we get

$$B(z_{s_i}) = B(q_i), \quad i = 0, 1, \ldots, (N+1)(M+1)+N.$$

Hence, we have the values of $B$ at points $\{z_j\}_{j=0}^{(N+1)(M+1)+N}$. The resulting non-linear system (3.9) is solved by Newton's method with respect to the unknown vector $C$. Finally, we obtain an approximate solution for the problem (1.1) by substituting $C$ in (3.1).

## 4. Convergence analysis

In the following theorems, suppose that the functions $b(t, y(t)), \sigma(t, y(t))$ satisfy the Lipschitz and the linear growth conditions as follows
$(i)$ For every $T$ and $R$, there is a constant $\alpha$ depending only on $T$ and $R$ such that for all $|y_1|, |y_2| \leq R$ and all $0 \leq t \leq T$,

$$|b(t, y_1(t)) - b(t, y_2(t))| + |\sigma(t, y_1(t)) - \sigma(t, y_2(t))| \leq \alpha|y_1 - y_2|, \tag{4.1}$$

and
$(ii)$

$$|b(t, y(t))| + |\sigma(t, y(t))| \leq \alpha(1 + |y|) \tag{4.2}$$

*Theorem 4.* Let $y(t)$ and $y_N(t)$ be the exact solution and approximate solution of (1.1), respectively; Furthermore, assume that conditions (4.1), (4.2), and
$(i)$ $||y(t)|| < \infty, \quad t \in D$,
$(ii)$ $||k_i(t, s)|| \leq K_i, \quad (t, s) \in D \times D, \quad i = 1, 2$,
hold; then,

$$||y(t) - y_N(t)|| \to 0,$$

where

$$||y(t)||^2 = E[|y(t)|^2].$$

*Proof.*. Let us $e_N(t) = y(t) - y_N(t)$ be an error function of approximate solution $y_N(t)$ to the exact solution $y(t)$, so we can write

$$y(t) - y_N(t) = \int_0^t k_1(t,\tau)(b(\tau, y(\tau)) - b(\tau, y_N(\tau)))d\tau$$
$$+ \int_0^t k_2(t,\tau)(\sigma(\tau, y(\tau)) - \sigma(\tau, y_N(\tau)))dB(\tau) + J(t), \tag{4.3}$$

where, $J(t)$ is the the error estimate between the the Gauss-Legendre quadrature formula and the exact integral

$$J(t) = \frac{t}{2}\int_{-1}^{1} \left(F_t(\tau(t,\theta), y_N(\tau(t,\theta)))d\theta - \bar{G}_t(\tau(t,\theta), y_N(\tau(t,\theta)))B(\tau(t,\theta))\right) d\theta$$
$$- \frac{t}{2}\sum_{j=0}^{M} w_j \left[F_t(\tau(t,\theta_j), y_N(\tau(t,\theta_j))) - \bar{G}_t(\tau(t,\theta_j), y_N(\tau(t,\theta_j)))B(\tau(t,\theta_j))\right], \tag{4.4}$$

which if we choose $M$ sufficiently large, then the integration error tend to 0.
Using (4.3) leads to

$$||e_N(t)|| \leq ||I_1(t)|| + ||I_2(t)|| + ||J(t)||, \tag{4.5}$$

where,

$$I_1(t) = \int_0^t k_1(t,\tau)(b(\tau, y(\tau)) - b(\tau, y_N(\tau)))d\tau, \tag{4.6}$$

and,

$$I_2(t) = \int_0^t k_2(t,\tau)(\sigma(\tau, y(\tau)) - \sigma(\tau, y_N(\tau)))dB(\tau). \tag{4.7}$$

For $I_1$, using condition $(ii)$ and Lipschitz condition (4.1) lead to

$$||I_1(t)|| \leq \int_0^t ||k_1(t,\tau)\left(b(\tau, y(\tau)) - b(\tau, y_N(\tau))\right)||d\tau, \leq K_1\alpha \int_0^t ||e_N(\tau)||d\tau \tag{4.8}$$

and for $I_2$, from the Itô isometry, and condition $(ii)$, we get

$$||I_2(t)|| = ||\int_0^t k_2(t,\tau)(\sigma(\tau, y(\tau)) - \sigma(\tau, y_N(\tau)))dB(\tau)||$$
$$\leq \int_0^t ||k_2(t,\tau)(\sigma(\tau, y(\tau)) - \sigma(\tau, y_N(\tau)))||d\tau \tag{4.9}$$
$$\leq K_2\alpha \int_0^t ||e_N(\tau)||d\tau.$$

Hence, inserting (4.8) and (4.9) into (4.5) yields

$$||e_N(t)|| \leq (K_1 + K_2)\alpha \int_0^t ||e_N(\tau)||d\tau + ||J(t)||. \tag{4.10}$$

Thus, by using Gronwall's Lemma and $||J(t)|| \to 0$, we obtain

$$||e_N|| \to 0,$$

so, $y_N \to y$ in $L^2$.

## 5. Numerical Examples

In this section, some examples are given to illustrate the applicability and accuracy of the proposed method. The algorithm associated with the numerical method was performed using Maple 18. Let $y(t)$ be the exact solution of (1.1) and $y_N(t)$ be the approximate solution, then we define the absolute error for some collocation points in the interval $[0, l]$ as

$$e^{(n)}(t_j) = \frac{\sum_{i=0}^{n} |(y^i - y_N^i)(t_j)|}{n}, \quad 0 \leq t_j \leq l, \quad j = 0, 1, \ldots,$$

in which $e^{(n)}(t_j)$ is the error mean at $t_j$ with $n$ iteration, and $y^i(t_j)$, and $y_N^i(t_j)$ are the exact solution and approximate solution at $t_j$ in $i$th iteration, respectively.

*Example 1..* Let us consider the following linear stochastic Itô-Volterra integral equation [9]

$$y(t) = 1 + \int_0^t \tau^2 y(\tau) d\tau + \int_0^t \tau y(\tau) dB(\tau), \quad t \in [0, 0.5], \tag{5.1}$$

where $y(t)$ is the unknown stochastic process defined on the probability space $(\Omega, \mathbf{F}, P)$, and $B(t)$ is a Brownian motion process. The exact solution is

$$y(t) = exp\left(\frac{t^3}{6} + \int_0^t \tau dB(\tau)\right).$$

The collocation spectral method presented in Section 3 is employed for deriving numerical solution of this (SVIE). Here, we collocate (5.1) in $N + 1$ points $t_i, i = 0, ..., N$. The error mean of the numerical results is shown in Table 1 for $N = M = 10$ at collocation points $t_i,\ i = 0, 1, \ldots, 5$ where $t_5 = 0.5$. Also, the error means for $M = N = 20$ are shown in Table 2, where $t_{10} = 0.5$. The errors for $M = 10, N = 20$ are shown in Table 3. Note that from now on, $N$ is the number of collocation points, and $M$ is the number of nodes of the Gauss-Legendre quadrature. The results emphasize that when we increase the value of $M, N$, the errors decrease. Moreover, Figure 1 shows the exact solution and numerical solution computed by the presented method for $N = M = 20$. The numerical results reveal the accuracy of the proposed method for solving this SVIE.

Table 1: The errors for Example 1 at collocation points for $M = N = 10$.

| $t_i$ | $t_0$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ |
|---|---|---|---|---|---|---|
| $e^{(50)}(t_i)$ | 8.125E-7 | 5.264E-5 | 5.146E-4 | 2.861E-3 | 1.058E-2 | 2.603E-2 |
| $e^{(100)}(t_i)$ | 7.11E-7 | 4.881E-5 | 4.602E-4 | 2.577E-3 | 9.656E-3 | 2.433E-2 |

Table 2: The errors for Example 1 at collocation points for $M = N = 20$.

| $t_i$ | $t_0$ | $t_2$ | $t_4$ | $t_6$ | $t_8$ | $t_{10}$ |
|---|---|---|---|---|---|---|
| $e^{(50)}(t_i)$ | 5.649E-8 | 2.092E-5 | 3.119E-4 | 2.141E-3 | 9.169E-3 | 2.377E-2 |
| $e^{(100)}(t_i)$ | 7.299E-8 | 2.265E-5 | 3.077E-4 | 2.095E-3 | 8.601E-3 | 2.571E-2 |

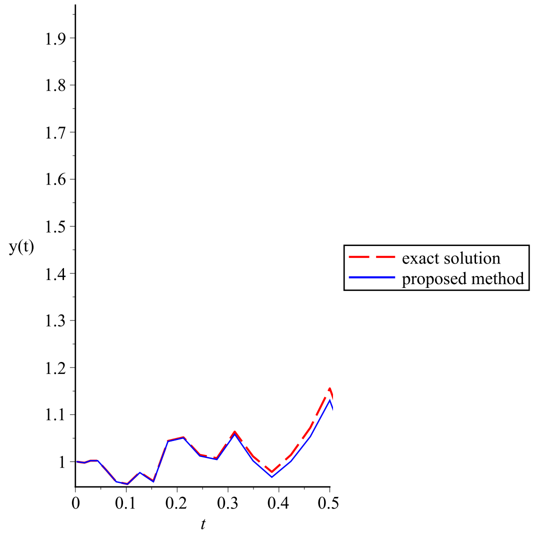


Figure 1: The Graph of the exact solution and the approximate solution for $M = N = 20$ for Example 1.

*Example 2.*. Consider the following stochastic Volterra integral equation [9]

$$y(t) = \frac{1}{12} + \int_0^t \cos(\tau)y(\tau)d\tau + \int_0^t \sin(\tau)y(\tau)dB(\tau), \quad t \in [0,1],$$

where $y(t)$ is the unknown stochastic process defined on the probability space $(\Omega, \mathbf{F}, P)$, and $B(t)$ is a Brownian motion process. The exact solution of this stochastic Itô-Volterra integral equation is

$$y(t) = \frac{1}{12} \exp\left(\frac{-t}{4} + \sin(t) + \frac{\sin(2t)}{8} + \int_0^t \sin(\tau)dB(\tau)\right).$$

The absolute errors of the numerical results for $M = N = 10$, $M = 10, N = 20$, and $M = N = 20$ at some collocation points $0 < t_i < 1$ are represented in Table 4, Table 5, and Table 6, respectively. The tables shows that by increasing $M, N$, the approximate solution can quickly converge to the exact solution. Also, the numerical findings demonstrate the accuracy of the proposed method. As can be seen, by less computation, we get good accuracy. Also, Figure 2 illustrates the exact solution and approximate solution with present method, which are found to be in excellent agreement.

Table 3: The errors for Example 2 at collocation points for $M = N = 10$.

| $t_i$ | $t_0$ | $t_2$ | $t_4$ | $t_6$ | $t_8$ | $t_{10}$ |
|---|---|---|---|---|---|---|
| $e^{(50)}(t_i)$ | 4.887E-6 | 1.295E-4 | 1.392E-3 | 9.059E-3 | 2.361E-2 | 3.452E-2 |
| $e^{(100)}(t_i)$ | 4.546E-6 | 8.908E-5 | 1.329E-3 | 7.555E-3 | 2.058E-2 | 3.012E-2 |

Table 4: The errors for Example 2 at collocation points for $M = 10, N = 20$.

| $t_i$ | $t_0$ | $t_4$ | $t_8$ | $t_{12}$ | $t_{16}$ | $t_{20}$ |
|---|---|---|---|---|---|---|
| $e^{(50)}(t_i)$ | 7.203E-7 | 4.388E-5 | 9.689E-4 | 8.572E-3 | 2.256E-2 | 3.773E-2 |
| $e^{(100)}(t_i)$ | 8.234E-7 | 3.966E-5 | 8.691E-4 | 7.500E-3 | 2.128E-2 | 3.761E-2 |

Table 5: The errors for Example 2 at collocation points for $M = N = 20$.

| $t_i$ | $t_0$ | $t_4$ | $t_8$ | $t_{12}$ | $t_{16}$ | $t_{20}$ |
|---|---|---|---|---|---|---|
| $e^{(50)}(t_i)$ | 7.940E-7 | 4.442E-5 | 9.738E-4 | 7.003E-3 | 1.804E-2 | 2.859E-2 |
| $e^{(100)}(t_i)$ | 6.974E-7 | 4.029E-5 | 1.049.918E-4 | 7.056E-3 | 1.948E-2 | 2.867E-2 |

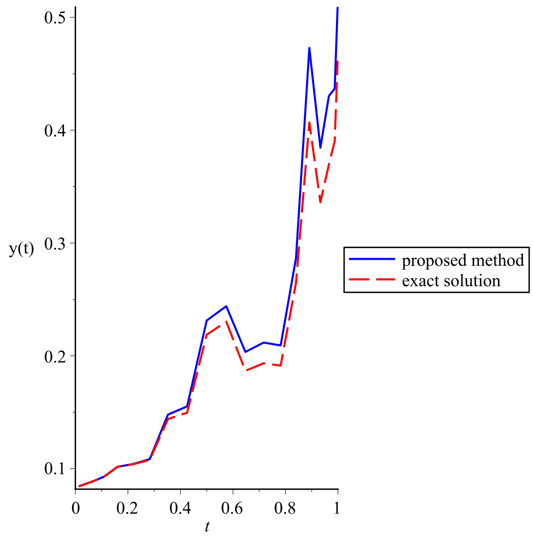


Figure 2: The Graph of the exact solution and the approximate solution for $M = N = 20$ for Example 2.

*Example 3.*. Consider the following nonlinear stochastic Itô-Volterra integral equation [8]

$$y(t) = 0.5 + \int_0^t y(\tau)(1 - y(\tau))d(\tau) + \int_0^t 0.25y(\tau)dB(\tau), \quad t \in [0, 1], \tag{5.2}$$

with the exact solution

$$y(t) = \frac{0.5\exp(0.96875t + 0.25B(t))}{1 + 0.5\int_0^t \exp(0.96875\tau + 0.25B(\tau))d\tau},$$

where $y(t)$ is the unknown stochastic process defined on the probability space $(\Omega, \mathbf{F}, P)$ and $B(t)$ is a Brownian motion process. The numerical results are shown in Table 7, and Table 8 for $M = N = 10$, and $M = N = 20$, respectively. As the numerical results show, the proposed method is accurate and efficient for solving nonlinear SVIEs. In order to compare the present method with the analytic solution, the resulting graph of (5.2) is shown in Figure 3.

Table 6: The errors for Example 3 at collocation points for $M = N = 10$.

| $t_i$ | $t_0$ | $t_2$ | $t_4$ | $t_6$ | $t_8$ | $t_{10}$ |
|---|---|---|---|---|---|---|
| $e^{(50)}(t_i)$ | 1.343E-4 | 2.235E-3 | 6.384E-3 | 1.143E-2 | 1.576E-2 | 1.844E-2 |
| $e^{(100)}(t_i)$ | 1.727E-4 | 2.377E-3 | 6.893E-3 | 1.174E-3 | 1.532E-2 | 1.971E-2 |

*Example 4.*. Consider the following Itô-Volterra integral equation [5]

$$y(t) = \frac{1}{3} + \int_0^t \ln(\tau + 1)y(\tau)d(\tau) + \int_0^t \sqrt{\ln(\tau + 1)}y(\tau)dB(\tau), \quad t \in [0, 1],$$

Table 7: The errors for Example 3 at collocation points for $M = N = 20$.

| $t_i$ | $t_0$ | $t_4$ | $t_8$ | $t_{12}$ | $t_{16}$ | $t_{20}$ |
|---|---|---|---|---|---|---|
| $e^{(50)}(t_i)$ | 3.568E-5 | 1.517E-3 | 4.209E-3 | 9.711E-3 | 1.545E-2 | 1.622E-2 |
| $e^{(100)}(t_i)$ | 3.388E-5 | 1.313.E-3 | 3.650E-3 | 6.224E-3 | 1.0429E-2 | 1.488E-2 |

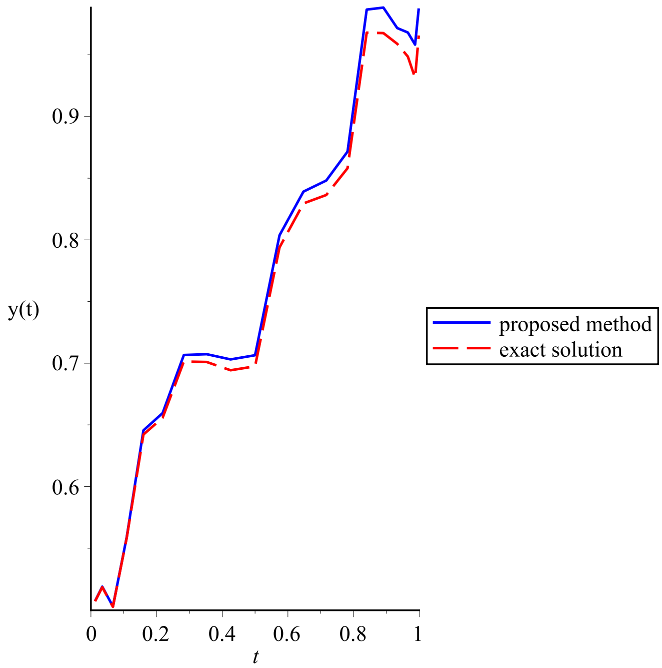


Figure 3: The Graph of the exact solution and the approximate solution for $M = N = 20$ for Example 3.

where $y(t)$ is the unknown stochastic process defined on the probability space $(\Omega, \mathbf{F}, P)$ and $B(t)$ is a Brownian motion process. The exact solution of this SVIE is

$$y(t) = \frac{1}{3} \exp\left( \frac{-t}{2} + \frac{1}{2} t \ln(t+1) + \frac{1}{2} \ln(t+1) + \int_0^t \sqrt{\ln(s+1)} dB(s) \right).$$

The proposed method in section 3 is employed for solving this SIVE. Table 9 lists the errors for $N = 10$ and $M = 10, 15$ at some collocation points $0 < t_i < 1$. Also, Table 10 lists the errors at $N = 20$, and $M = 10, 15, 20$. It is observed that by increasing $N$ and $M$, the errors decrease. The approximate solution for $M = N = 20$ and the exact solution are shown in Figure 4.

Table 8: The errors $e^{(50)}(t_i)$ for Example 4 at collocation points for $M = 10, 15$ and $N = 10$.

| $M/t_i$ | $t_0$ | $t_2$ | $t_4$ | $t_6$ | $t_8$ | $t_{10}$ |
|---|---|---|---|---|---|---|
| $M = 10$ | 9.778E-6 | 1.799E-3 | 1.181E-2 | 4.021E-2 | 7.160E-2 | 1.046E-1 |
| $M = 15$ | 1.067E-5 | 1.557E-3 | 1.303E-2 | 4.052E-2 | 8.069E-2 | 9.650E-2 |

## 6. Conclusion

For some SDEs that can be written as Volterra integral equations (1.1), it is impossible to find the exact solutions. So, it would be convenient to determine their numerical solutions based on stochastic

Table 9: The errors $e^{(50)}(t_i)$ for Example 4 at collocation points for $M = 10, 15, 20$ and $N = 20$.

| $t_i$ | $M = 10, N = 20$ | $M = 15, N = 20$ | $M = 20, N = 20$ |
|---|---|---|---|
| $t_0$ | 1.536E-5 | 1.469E-6 | 1.694E-6 |
| $t_2$ | 1.285E-4 | 1.538E-4 | 1.596E-4 |
| $t_4$ | 9.483E-4 | 1.203E-3 | 1.077E-3 |
| $t_6$ | 4.279E-3 | 4.330E-3 | 4.008E-3 |
| $t_8$ | 1.114E-2 | 1.158E-2 | 9.987E-3 |
| $t_{10}$ | 2.155E-2 | 2.071E-2 | 2.252E-2 |
| $t_{12}$ | 3.966E-2 | 4.161E-2 | 3.877E-2 |
| $t_{14}$ | 5.779E-2 | 6.528E-2 | 5.735E-2 |
| $t_{16}$ | 9.728E-2 | 9.021E-2 | 7.071E-2 |
| $t_{18}$ | 1.241E-1 | 1.0681E-1 | 8.758E-2 |
| $t_{20}$ | 1.507E-1 | 1.136E-1 | 1.072E-1 |

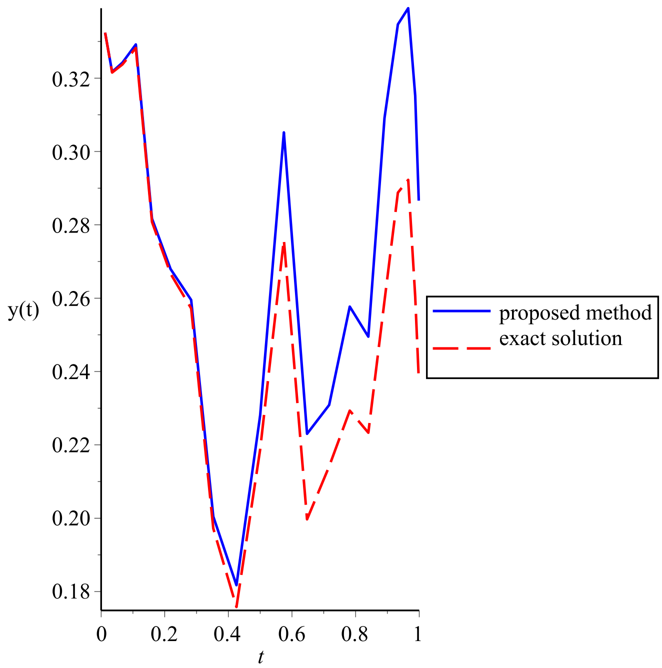


Figure 4: The Graph of the exact solution and the approximate solution for $M = N = 20$ for Example 4.

numerical analysis. In this paper, a numerical method for obtaining a numerical spectral solution for SVIEs was discussed. The derivation of this method is essentially based on shifted Legendre functions and Gauss quadrature formula. The main characteristic of the presented method is that it reduces SVIE into a system of algebraic equations that can be solved by Newton's method. Moreover, this method is used to solve non-linear SIVEs. Furthermore, the error analysis of the approach is established. In this paper, the applicability and accuracy of this method were shown by some linear and non-linear examples. The results show that the present method is easy to implement and it is a powerful mathematical tool for finding the numerical solution of SVIEs.

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