Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 25-33
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.20274.2140

Mittag-Leffler-Hyers-Ulam stability of Prabhakar fractional integral equation

A. Moharramnia[a], N. Eghbali*[a], J. M. Rassias[b]

[a] Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran.
[b] Pedagogical Department of Mathematics, The National and Kapodistrian University of Athens, 4, Agamemnonos Str, Aghia Paraskevi, Attikis 15342, Greece.

(Communicated by Saeid Abbasbandy)

Abstract

In this paper, we define and investigate Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassias stability of Prabhakar fractional integral equation.

Keywords: Mittag-Leffler-Hyers-Ulam stability, Mittag-Leffler-Hyers-Ulam-Rassias stability, Prabhakar fractional integral equation
2010 MSC: Primary 26A33; Secondary 34D10, 45N05, 39B72.

1. Introduction

The stability theory for functional equations started with a problem related to the stability of group homomorphism that was considered by Ulam in 1940 ([21]). The first answer to the question of Ulam was given by Hyers in 1941 in the case of Banach spaces in [9]. Thereafter, this type of stability is called the Hyers-Ulam stability. In 1978, Th. M. Rassias [17] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. In fact, he has introduced a new type of stability which is called the Hyers-Ulam-Rassias stability.

Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of a differential equation [2]. With the extension of theory of fractional calculus (the integral and derivative of arbitrary order), the stability analysis of differential system

$$x'(t) = Ax(t), \qquad x(0) = x_0, \qquad A \in \mathbb{R}^{n\times n},$$

*Corresponding author
Email addresses: aramnia@yahoo.com (A. Moharramnia), nasrineghbali@gmail.com;eghbali@uma.ac.ir (N. Eghbali*), jrass@otenet.gr (J. M. Rassias[b])

Received: April 2020 *Accepted:* June 2020

was developed in the last decades for the following fractional differential system

$$D_{\alpha}^{t}x(t) = Ax(t), \qquad x(0) = x_0, \qquad 0 < \alpha \leq 1,$$

where D_{α}^{t} is a fractional differential operator. For the first time in 1996, the stability of the above system with the Caputo fractional derivatives was surveyed by Matignon [13].

Recently some authors ([10, 11, 20, 22, 23]) extended the Ulam stability problem from an integer-order differential equation to a fractional-order differential equation.

Integral equations of various types play an important role in many branches of functional analysis and in their applications, for example in physics, economics and other fields. Also, the fractional differential equations are useful tools in the modelling of many physical phenomena and processes in economics, chemistry, aerodynamics, etc. (for more details see [12, 14, 15]).

There are different types of fractional integral equations. In [5], Eghbali et al. defined the types of Mittag-Leffler-Hyers-Ulam stability of a fractional integral equation to proved that every mapping from this type can be somehow approximated by an exact solution of the considered equation.

In this paper, we investigate the stability analysis of linear differential systems containing the Prabhakar fractional derivatives. This type of fractional derivative was introduced by Garra et al. [7] in terms of the generalized Mittag-Leffler function and was considered as a generalization of the most popular definitions of fractional derivatives. Also see papers [3, 4, 6, 16].

In this paper, we present similar definitions with [5] and prove stability results for the Prabhakar fractional integral equation.

2. Preliminaries

In this section, we introduce notations, definitions and theorems which are used throughout this paper.

Definition 2.1. *The Mittag-Leffler function of one parameter, denoted by $E_{\alpha}(z)$, is defined as*

$$E_{\alpha}(z) = \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

for $z, \alpha \in \mathbb{C}$ with $Re(\alpha) > 0$, where the Euler Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as

$$\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1} \exp(-s) ds.$$

Definition 2.2. *The generalization of $E_{\alpha}(z)$ was studied by Wiman (1905) [24], Agarwal [1] and Humbert and Agarwal [8] defined the function as,*

$$E_{\alpha,\beta}(z) = \sum_{k=o}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} z^k$$

where $z, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0$.

In 1971, The more generalized function was introduced by Prabhakar as

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=o}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)}$$

for $z, \alpha, \beta, \gamma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$, with $\gamma \neq 0$, where $(\gamma)_k = \gamma(\gamma+1)(\gamma+2)...(\gamma+k-1)$ is the Pochhammer symbol [19], and

$$(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$$

or

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} \sum_{k=o}^{\infty} \frac{\Gamma(\gamma+k)}{k!\ \Gamma(\alpha k+\beta)} z^k.$$

In 2007, Shulka and Prajapati [19] introduce the function which is defined as,

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{k=o}^{\infty} \frac{(\gamma)_{qk} z^k}{k!\Gamma(\alpha k+\beta)},$$

where $z, \alpha, \beta, \gamma \in \mathbb{C}, \min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, and $q \in (0,1) \bigcup \mathbb{N}$. In 2012, further generalization of Mittag-Leffler function was defined by Salim [18] as,

$$E^{\gamma,\delta,q}_{\alpha,\beta}(z) = \sum_{k=o}^{\infty} \frac{(\gamma)_{qk} z^k}{(\delta)_{(qk)}\Gamma(\alpha k+\beta)},$$

where $z, \alpha, \beta, \gamma \in \mathbb{C}, \min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, and $q \in (0,1) \bigcup \mathbb{N}$.

Definition 2.3. *The beta function is*

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

that $x, y \in \mathbb{C}$ *and* $Re(x) > 0, Re(y) > 0$.

From the definition of Γ and β functions we have

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

For a nonempty set X, we introduce the definitions of the generalized metric on X.

Definition 2.4. *A function* $d : X \times X \longrightarrow [0, +\infty]$ *is called a generalized metric on* X *if and only if satisfies*

(A_1) $d(x,y) = 0$ *if and only if* $x = y$;
(A_2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
(A_3) $d(x,z) \leq d(x,y) + d(y,z)$ *for all* $x, y, z \in X$.

The above concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such a space a generalized complete metric space.
We now introduce one of the fundamental results of *Banach's* fixed point theorem in a generalized complete metric space.

Theorem 2.5. *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \longrightarrow X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the following are true:*

(a) *The sequence $\Lambda^n x$ convergence to a fixed point x^* of Λ;*
(b) *x^* is the unique fixed point of Λ in*

$$X^* = \{y \in X | d(\Lambda^k x, y) < \infty\};$$

(c) *If $y \in X^*$, then*

$$d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y).$$

Definition 2.6. *For $0 < \gamma < 1$ and $h \in L^1([a, b], \mathbb{R})$, $a < t < b \leq \infty$, the Riemann-Liouville fractional integral of the order γ is defined as*

$$I^{\gamma}_{a^+} h(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} h(s) ds,$$

where $\Gamma(.)$ is the Euler Gamma function.

Definition 2.7. *[15] For $\rho, \mu, \omega, \gamma \in \mathbb{C}, Re(\rho) > 0, m-1 < Re(\mu) < m$ and function $g \in L^1([a, b], \mathbb{R})$, $0 < t < b \leq \infty$, the Prabhakar fractional integral is defined as follows*

$$g(t) = (E^{\gamma}_{\rho,\mu,\omega,0^+} g)(t) = \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}) g(s) ds, \tag{2.1}$$

where $E^{\gamma}_{\rho,\mu}$ is the generalized Mittag-Leffler function.

3. Mittag-Leffler-Hyres-Ulam stability

In this section, we will study Mittag-Leffler-Hyers -Ulam stability of the Prabhakar fractional integral equation.

Definition 3.1. *Equation (2.1) is Mittag-Leffler-Hyers-Ulam stable if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution g of the inequality*

$$|g(t) - \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}) g(s) ds| \leq \varepsilon E_{\beta}(t^{\beta}),$$

there exists a unique solution g_0 of equation (2.1) satisfying the following inequality:

$$|g(t) - g_0(t)| \leq c\varepsilon E_{\beta}(t^{\beta}).$$

Theorem 3.2. *If $\rho, \mu, \omega, \gamma \in \mathbb{C}, Re(\rho) > 0, m-1 < Re(\mu) < m$ and function $g \in L^1([0, b], \mathbb{R}$, $0 < t < b \leq \infty$, let $0 < M_n := \frac{b^{\rho n+\mu}}{\rho n+\mu} < 1$ and $0 < E^{\gamma}_{\rho,\mu}(\omega) < 1$, then Prabhakar fractional integral equation (2.1) is Mittag-Leffler-Hyers-Ulam stable.*

Proof . Let us consider the space of continuous functions

$$X = \{g : [0, b] \longrightarrow \mathbb{R} \mid g \quad is \quad continuous\},$$

endowed with the generalized metric defined by

$$d(g,h) = \inf\{K \in [0,\infty] \mid |g(t)-h(t)| \le K\varepsilon \quad for \quad all \quad t \in [0,b]\},$$

for $\varepsilon > 0$. It is known that (X,d) is a generalized complete metric space.
Define an operator $\Lambda : X \longrightarrow X$ by

$$(\Lambda g)(t) = \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}) g(s) ds, \tag{3.1}$$

for all $g \in X$ and $t \in [0,b]$. Since g is continuous function, Λg is also continuous and this ensures that Λ is a well defined operator. For any $g,h \in X$, let $K \in [0,\infty]$ such that

$$|g(t)-h(t)| \le K\varepsilon \tag{3.2}$$

for any $t \in [0,b]$. This K exists because of definition of (X,d). From the definition of Λ and (3.2) we have

$$|(\Lambda g)(t) - (\Lambda h)(t)| \le \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}) |g(s)-h(s)| ds$$

$$\le K\varepsilon \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}) ds$$

$$\le K\varepsilon \int_0^t (t-s)^{\mu-1} \frac{1}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)} (\omega(t-s)^{\rho})^n ds$$

$$= \frac{K\varepsilon}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)\omega^n}{n!\Gamma(\rho n+\mu)} t^{\rho n+\mu} \int_0^1 (1-x)^{\rho n+\mu-1} dx$$

$$= \frac{K\varepsilon}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)\omega^n}{n!\Gamma(\rho n+\mu)} t^{\rho n+\mu} \frac{1}{\rho n+\mu}$$

$$\le \frac{K\varepsilon}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)\omega^n}{n!\Gamma(\rho n+\mu)} \frac{b^{\rho n+\mu}}{\rho n+\mu} = \frac{K\varepsilon}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)\omega^n}{n!\Gamma(\rho n+\mu)} M_n$$

$$\le \frac{K\varepsilon}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)\omega^n}{n!\Gamma(\rho n+\mu)} = K\varepsilon E^{\gamma}_{\rho,\mu}(\omega),$$

for all $t \in [0,b]$; that is, $d(\Lambda g, \Lambda h) \le K\varepsilon E^{\gamma}_{\rho,\mu}(\omega)$. Hence, we can conclude that $d(\Lambda g,\Lambda h) \le E^{\gamma}_{\rho,\mu}(\omega) d(g,h)$ for any $g,h \in X$, and so the strictly continuous property is verified. Let us take $h_0 \in X$, from the continuous property of h_0 and Λh_0, it follows that there exists a constant $0 < K_1 < \infty$ such that

$$|(\Lambda h_0)(t) - h_0(t)| = |\int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}) h_0(s) ds - h_0(t)| \le K_1\varepsilon,$$

for all $t \in [0,b]$, since h_0 is bounded on $[0,b]$, thus (3.1) implies that $d(\Lambda h_0, h_0) < \infty$. Therefore, according to Theorem 2.5, there exists a continuous function $g_0 : [0,b] \longrightarrow \mathbb{R}$ such that $\Lambda^n h_0 \longrightarrow g_0$ in (X,d) as $n \to \infty$ and $\Lambda g_0 = g_0$; that is, g_0 satisfies the equation (2.1). We will now prove that $\{h \in X | d(h_0, h) < \infty\} = X$ for any $h \in X$, since h and h_0 are bounded in $[0,b]$, there exists a constant $0 < C_h < \infty$ such that

$$|h_0(t) - h(t)| \leq C_h,$$

for any $t \in [0,b]$. Hence, we have $d(h_0, h) < \infty$ for all $h \in X$; that is,

$$\{h \in X | d(h_0, h) < \infty\} = X.$$

Hence, in view of Theorem 2.5, we conclude that g_0 is the unique continuous function which satisfies the equation (2.1). Now we have $d(g, \Lambda g) \leq \varepsilon E_\beta(t^\beta)$. Finally, Theorem 2.5 together with the above inequality implies that

$$d(g, g_0) \leq \frac{1}{1 - E^{\gamma}_{\rho,\mu}(\omega)} d(\Lambda g, g) \leq \frac{1}{1 - E^{\gamma}_{\rho,\mu}(\omega)} \varepsilon E_\beta(t^\beta).$$

This means that the equation (2.1) is Mittag-Leffler-Hyers-Ulam stable. □

4. Mittag-Leffler-Hyres-Ulam-Rassias stability

In this section, we will study Mittag-Leffler-Hyers-Ulam-Rassias stability of the Prabhakar fractional integral equation.

Definition 4.1. *Equation* (2.1) *is Mittag-Leffler-Hyers-Ulam-Rassias stable if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution g of the inequality*

$$|g(t) - \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^\rho) g(s) ds| \leq \varphi(t) \varepsilon E_\beta(t^\beta),$$

there exists a unique solution g_0 of equation (2.1) *satisfying the following inequality:*

$$|g(t) - g_0(t)| \leq c \varphi(t) \varepsilon E_\beta(t^\beta).$$

where $\varphi : [0,b] \longrightarrow \mathbb{R}$ is a continuous function.

Theorem 4.2. *If $\rho, \mu, \omega, \gamma \in \mathbb{C}, Re(\rho) > 0, m-1 < Re(\mu) < m$ and function $g \in L^1([0,b], \mathbb{R})$, $0 < t < b \leq \infty$, and suppose that for each $\varepsilon > 0$ we have*

$$|g(t) - \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^\rho) g(s) ds| \leq \varphi(t) \varepsilon E_\beta(t^\beta),$$

where $\varphi : [0,b] \longrightarrow \mathbb{R}$ is a $L^{\frac{1}{p}}$-integrable function which satisfies

$$\Big(\int_0^t (\varphi(s))^{\frac{1}{p}} ds \Big)^p \leq B \varphi(t),$$

for all $t \in [0,b]$ and $0 < B E^{\gamma}_{\rho,\mu}(\omega) < 1$. Let

$$M_n := b^{\rho n + \mu - p} \Big(\frac{1-p}{\rho n + \mu - p} \Big)^{1-p},$$

if $0 < M_n < 1$, then Prabhakar fractional integral equation (2.1) *is Mittag-Leffler-Hyers-Ulam-Rassias stable.*

Proof . Let us consider the space of continuous functions

$$X = \{g : [0, b] \longrightarrow \mathbb{R} \mid g \quad is \quad continuous\},$$

endowed with the generalized metric defined by

$$d(g, h) = \inf\{K \in [0, \infty] \mid |g(t) - h(t)| \leq K\varepsilon\varphi(t) \quad for \quad all \quad t \in [0, b]\}, \tag{4.1}$$

for $\varepsilon > 0$. It is known that (X, d) is a generalized complete metric space.
For any $g, h \in X$, let $K \in [0, \infty]$ such that

$$|g(t) - h(t)| \leq K\varepsilon\varphi(t), \tag{4.2}$$

for any $t \in [0, b]$. This K exists, because of definition of (X, d).

Define an operator $\Lambda : X \longrightarrow X$ by

$$(\Lambda g)(t) = \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}) g(s) ds,$$

for all $g \in X$ and $t \in [0, b]$.
Now, g is continuous function, so Λg is also continuous and this ensures that Λ is a well defined operator. From the definition of Λ and (4) we have

$$|(\Lambda g)(t) - (\Lambda h)(t)| \leq \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}) |g(s) - h(s)| ds$$

$$\leq K\varepsilon \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}) \varphi(s) ds$$

$$= K\varepsilon \int_0^t (t-s)^{\mu-1} \frac{1}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)} (\omega(t-s)^{\rho})^n \varphi(s) ds$$

$$= \frac{K\varepsilon}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)} .\omega^n \int_0^t (t-s)^{\rho n+\mu-1} .\varphi(s) ds$$

$$\leq \frac{K\varepsilon}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)} .\omega^n \Big(\int_0^t (t-s)^{\frac{\rho n+\mu-1}{1-p}} ds \Big)^{1-p} . \Big(\int_0^t (\varphi(s))^{\frac{1}{p}} ds \Big)^p$$

$$= \frac{K\varepsilon}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)} .\omega^n \Big(\int_0^1 (t-tx)^{\frac{\rho n+\mu-1}{1-p}} t dx \Big)^{1-p} . \Big(\int_0^t (\varphi(s))^{\frac{1}{p}} ds \Big)^p$$

$$\leq \frac{K\varepsilon}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)} .\omega^n t^{\rho n+\mu-p} B\varphi(t) \Big(\int_0^1 (1-x)^{\frac{\rho n+\mu-1}{1-p}} dx \Big)^{1-p}$$

$$= \frac{K\varepsilon B\varphi(t)}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)} .\omega^n t^{\rho n+\mu-p} \Big(\frac{1-p}{\rho n+\mu-p} \Big)^{1-p}$$

$$\leq \frac{K\varepsilon B\varphi(t)}{\Gamma(\gamma)} \sum_{n=o}^{\infty} \frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)} .\omega^n b^{\rho n+\mu-p} \Big(\frac{1-p}{\rho n+\mu-p} \Big)^{1-p}$$

$$\leq \frac{K\varepsilon B\varphi(t)}{\Gamma(\gamma)}\sum_{n=o}^{\infty}\frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)}.\omega^n.M_n \leq \frac{K\varepsilon B\varphi(t)}{\Gamma(\gamma)}\sum_{n=o}^{\infty}\frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)}.\omega^n$$

$$= K\varepsilon B\varphi(t)E^{\gamma}_{\rho,\mu}(\omega),$$

for all $t \in [0,b]$; that is, $d(\Lambda g, \Lambda h) \leq K\varepsilon B\varphi(t)E^{\gamma}_{\rho,\mu}(\omega)$. Hence, we can conclude that $d(\Lambda g, \Lambda h) \leq BE^{\gamma}_{\rho,\mu}(\omega)d(g,h)$ for any $g, h \in X$, and so the strictly continuous property is verified. Let us take $h_0 \in X$, from the continuous property of h_0 and Λh_0, it follows that there exists a constant $0 < K_1 < \infty$ such that

$$|(\Lambda h_0)(t) - h_0(t)| = |\int_0^t (t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho})h_0(s)ds - h_0(t)| \leq K_1\varepsilon\varphi(t),$$

for all $t \in [0,b]$, since h_0 is bounded on $[0,b]$ and $\min_{t\in[0,b]}\varphi(t) > 0$, thus, (4.1) implies that $d(\Lambda h_0, h_0) < \infty$. Therefore, according to Theorem 2.5, there exists a continuous function $g_0 : [0,b] \longrightarrow \mathbb{R}$ such that $\Lambda^n h_0 \longrightarrow g_0$ in (X,d) as $n \to \infty$ and $\Lambda g_0 = g_0$; that is, g_0 satisfies the equation (2.1). We will now prove that $\{h \in X | d(h_0,h) < \infty\} = X$ for any $h \in X$, since h and h_0 are bounded in $[0,b]$ and $\min_{t\in[0,b]}\varphi(t) > 0$, there exists a constant $0 < C_h < \infty$ such that

$$|h_0(t) - h(t)| \leq C_h\varphi(t),$$

for any $t \in [0,b]$. Hence, we have $d(h_0,h) < \infty$ for all $h \in X$; that is,

$$\{h \in X | d(h_0,h) < \infty\} = X.$$

Hence, in view of Theorem 2.5, we conclude that g_0 is the unique continuous function which satisfies the equation (2.1). Now we have $d(g, \Lambda g) \leq \varphi(t)\varepsilon E_{\mu}(t^{\mu})$. Finally, Theorem 2.5 together with the above inequality implies that

$$d(g,g_0) \leq \frac{1}{1-BE^{\gamma}_{\rho,\mu}(\omega)}d(\Lambda g, g) \leq \frac{1}{1-BE^{\gamma}_{\rho,\mu}(\omega)}\varphi(t)\varepsilon E_{\beta}(t^{\beta}).$$

This means that the equation (2.1) is Mittag-Leffler-Hyers-Ulam-Rassias stable. □

References

[1] RP. Agarwal, *A propos d·une note M. Pierre Humbert*, C. R. Acad. Sci. Paris, 236 (1953) 2031–2032.
[2] C. Alsina and R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. 2 (1998) 373–380.
[3] S. Eshaghi and A. Ansari, *Lyapunov inequality for fractional differential equations with Prabhakar derivative*, Math. Inequal. Appl. 19 (2016) 349–358.
[4] S. Eshaghi and A. Ansari, *Finite fractional Sturm-Liouville transforms for generalized fractional derivatives*, Iran. J. Sci. Technol. Trans. A Sci. 41 (2017) 931–937.
[5] N. Eghbali, V. Kalvandi and JM. Rassias, *A fixed point approach to the Mittag-Leffler-Hyers-Ulam stability of a fractional integral equation*, Open Math. 14 (2016) 237–246.
[6] G. Farid, J. Pecaric and Z. Tomovski, *Opial-type inequalities for fractional integral operator involving Mittag-Leffler function*, Fract. Diff. Calc. 5 (2015) 93–106.
[7] R. Garra, R. Gorenflo, F. Polito and Z. Tomovski, *Hilfer-Prabhakar derivatives and some applications*, Appl. Math. Comput. 42(2) (2014) 576–589.
[8] P. Humbert and RP. Agarwal, Sur la function de Mittag-Leffler et quelquesunes deses generalizations, Bull. Sci. Math. (2) (77) (1953) 180–186.
[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA 27 (1941) 222–224.

[10] RW. Ibrahim, *Generalized Ulam-Hyers stability for fractional differential equations*, Int. J. Math. 23 (5) (2012) 9 pp.

[11] RW. Ibrahim, *Ulam stability for fractional differential equation in complex domain*, Abstr. Appl. Anal. 2012 (2012) 1–8.

[12] AA. Killbas, HM. Srivastava and JJ. Trujilo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204 Elsevier, Amsterdam, 2006.

[13] D. Matignon, *Stability results for fractional differential equations with applications to control processing*, Comput. Engrg. Syst. Appl. 2 (1996) 963–968.

[14] KS. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Willey, New York, 1993.

[15] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diago, 1999.

[16] F. Polito and Z. Tomovski, *Some properties of Prabhakar-type fractional calculus operators*, Fract. Differ. Calc. 6 (2016) 73–94.

[17] ThM. Rassias, *On the stability of linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978) 297–300.

[18] TO. Salim and O. Faraj, *A generalization of Mittag-Leffler function and integral operator associated with the fractional calculus*, J. Fract. Calc. Appl. 3(5) (2012) 1–13.

[19] AK. Shukla and JC. Prajapati, On a generalization of Mittag-Leffler function and its properties, J. Math. Anal. Appl. 336 (2007) 79–81.

[20] SE. Takahasi, T. Miura and S. Miyajima, *On the Hyers-Ulam stability of the Banach-space valued differential equation* $y' = \lambda y$, Bull. Korean Math. Soc. 39 (2002) 309–315.

[21] SM. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1968.

[22] JR. Wang, L. Lv and Y. Zhou, *Ulam stability and data dependence for fractional differential equations with Caputo derivative*, Electron. J. Qual. Theory Differ. Equ. 63 (2011) 1–10.

[23] JR. Wang, Y. Zhou and M. Medveď, *Existence and stability of fractional differential equations with Hadamard derivative*, Topol. Meth. Nonl. Anal. 41 (2013) 113–133.

[24] A. Wiman, *Uber de fundamental satz in der theorie der funktionen*, Acta Math. 29 (1905) 191–201.