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Fixed point results in partially ordered partial $b_v(s)$-metric spaces

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Abstract

In this paper, some fixed point results for generalized Geraghty type α-admissible contractive mappings and rational type generalized Geraghty contraction mappings are given in partially ordered partial $b_v(s)$-metric spaces. Also, a modified version of a partial $b_v(s)$-metric space is defined and a fixed point theorem is proved in this space. Finally, some examples are given related to the results.

Keywords: fixed point, α-admissible contraction mappings, Geraghty contraction, partial-$b_v(s)$ metric
2010 MSC: 47H10, 54H25, 47H10, 55M20

1. Introduction and Preliminaries

The Banach contraction principle is one of the most important results in fixed point theory. The importance of this principle stems from the fact that it has large application areas in different disciplines. This principle was generalized by many authors in different ways and some fixed point results were obtained. In 1973, Geraghty [7] introduced a generalization of Banach contraction principle in complete metric space. Later, Amini-Harandi and Emami [2] studied the results of Geraghty in partially ordered complete metric space. Gordji et al. [12] defined the concept of ψ-Geraghty contractive mappings and improved the results of Amini-Harandi and Emami [2]. In 2012 Samet et al. [22] established remarkable fixed point results by defining the concept of α-ψ contraction mapping. Recently, Karapınar and Bae [4] introduced the notion of α- Geraghty type contractive mappings in metric space and proved the existence and uniqueness of such mapping in the concept of

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a complete metric space. In 2017, Erhan [6] proved some fixed point theorem for generalized Geraghty type α-admissible contractive mappings defined in a complete Branciari b-metric spaces.Afterwards, Arshad and Hussaın [3], Prasad and Singh [20], Latif et al. [16] and Roshan et al. [21] published some papers about the existence and uniqueness of fixed points of different kinds of generalizations of Geraghty type contraction mappings.

On the other hand, many different type generalizedmetric spaces were introduced by many authors for many years. In 1994, Matthews [17] introduced the notion of partial metric spaces as a part of the study of denotational semantics of dataflow network. By the time the usual metric is replaced by partial metric. In this type metric spaces, the self-distance of any point may not be zero. In 2013, Shukla [23] introduced the concept of partial b-metric spaces as a generalization of partial metric and b-metric spaces. Afterwards, In 2014, Shukla [24] generalized the rectangular metric spaces and extended the notion of partial metric space by introducing the partial rectangular metric space.

In 2017, Mitrovic and Radenovic introduced the following generalized metric space which is called as $b_v(s)$-metric space.

Definition 1.1. *[18]Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a mapping and $v \in \mathbb{N}$. Then (X, d) is said to be a $b_v(s)$-metric space if there exists a real number $s \geq 1$ such that following conditions hold for all distinct points $x, y, u_1, u_2, ..., u_v \in X \setminus \{x, y\}$:*

1. $d(x, y) = 0$ *iff* $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq s[d(x, u_1) + d(u_1, u_2) + ... + d(u_{v-1}, u_v) + d(u_v, y)]$.

In 2020, Karahan and Isik [13] introduced the concept of partial $b_v(s)$-metric space as follows.

Definition 1.2. *[13]Let X be a nonempty set and $d_p : X \times X \to [0, \infty)$ be a mapping and $v \in \mathbb{N}$. Then (X, d_p) is called a partial $b_v(s)$-metric space if there exists a real number $s \geq 1$ such that following conditions hold for all $x, y, u_1, u_2, ..., u_v \in X$:*

1. $x = y$ *iff* $d_p(x, x) = d_p(x, y) = d_p(y, y)$,
2. $d_p(x, x) \leq d_p(x, y)$,
3. $d_p(x, y) = d_p(y, x)$,
4. $d_p(x, y) \leq s[d_p(x, u_1) + d_p(u_1, u_2) + ... + d_p(u_{v-1}, u_v) + d_p(u_v, y)] - \sum_{i=1}^{v} d_p(u_i, u_i)$.

It is easy to see that every $b_v(s)$-metric space is a partial $b_v(s)$-metric space. However, the converse is not usually true. In the following, we give some properties and definitions related with partial $b_v(s)$-metric space.

Remark 1.3. *[13]Let (X, d_p) be a partial $b_v(s)$-metric space. If $d_p(x, y) = 0$ for $x, y \in X$, then $x = y$.*

Proposition 1.4. *Let E be a nonempty set such that d_1 is a partial metric and d_2 is a $b_v(s)$-metric on E. Then (E, ρ) is a partial $b_v(s)$-metric space where $\rho : E \times E \to [0, \infty)$ is a mapping defined by $\rho(u, w) = d_1(u, w) + d_2(u, w)$ for all $u, w \in E$.*

Definition 1.5. *[13] Let $\{x_n\}$ be a sequence in partial $b_v(s)$-metric space (X, d_p) and $x \in X$. Then:*

1. *The sequence $\{x_n\}$ is called convergent in X and converges to x, if* $\lim_{n\to\infty} d_p(x_n, x) = d_p(x, x)$.

2. *The sequence* $\{x_n\}$ *is called Cauchy sequence in* X *if* $\lim_{n,m\to\infty} d_p(x_n, x_m)$ *exists and is finite.*
3. (X, d_p) *is called complete partial* $b_v(s)$*-metric space if for every Cauchy sequence* $\{x_n\}$ *in* X *there exists* $x \in X$ *such that*

$$\lim_{n,m\to\infty} d_p(x_n, x_m) = \lim_{n\to\infty} d_p(x_n, x) = d_p(x, x).$$

Note that the limit of a convergent sequence has not to be unique in a partial $b_v(s)$-metric space.

In this paper, we consider partially ordered partial $b_v(s)$-metric spaces and α-admissible mappings which are defined by the following ways.

Definition 1.6. *Let* X *be a nonempty set. If* $(X, \preceq)$ *is a partially ordered set and* d_p *is a partial* $b_v(s)$*-metric on* X*, then* $(X, \preceq, d_p)$ *is called as a partially ordered partial* $b_v(s)$*-metric space.*

Definition 1.7. *[22] Let* $T : X \to X$ *be a mapping and* $\alpha : X \times X \to [0, \infty)$ *be a function. Then* T *is said to be* α*-admissible if*

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1 \text{ for } x, y \in X.$$

2. Main Results

In this section, we present our main results about existence and uniqueness of fixed points of special generalized Geraghty type contractive mappings in partially ordered partial $b_v(s)$-metric spaces.

Let $\mathcal{F}_s$ be the class of functions $\beta : [0, \infty) \to \left[0, \frac{1}{s}\right)$ for which

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \text{ whenever } \lim_{n\to\infty} t_n = 0,$$

holds for some $s \geq 1$. If $s = 1$, we obtain the well-known class $\mathcal{F}$ of all Geraghty type contractive mappings introduced in [7]. The following theorem is an extension of [[6], Theorem 2.2] from rectangular b-metric spaces to the case of partially ordered partial $b_v(s)$-metric space.

Theorem 2.1. *Let* $(X, \preceq, d_p)$ *be a complete partially ordered partial* $b_v(s)$*-metric space with a constant* $s > 1$ *and* $\alpha : X \times X \to [0, \infty)$ *be a function. Let* $T : X \to X$ *be a* α*-admissible nondecreasing mapping with respect to "*$\preceq$*" . Assume that*

$$\alpha(x, y) d_p(Tx, Ty) \leq \beta(M_1(x, y)) M_1(x, y) \tag{2.1}$$

for some $\beta \in \mathcal{F}_s$ *and for all* $x, y \in X$ *with* $x \preceq y$ *where*

$$M_1(x, y) = \max\{d_p(x, y), d_p(x, Tx), d_p(y, Ty)\}.$$

Also, suppose that the following assertions hold:

1. $\alpha(x, x) \geq 1$ *for all* $x \in X$ *and there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $x_0 \preceq Tx_0$.
2. T *is continuous mapping.*
3. *For every fixed points* u *and* v *of* T, $\alpha(u, v) \geq 1$.

Then T *has at least one fixed point* u*. Also, if* v *is another fixed point of* T *such that* u *and* v *are comparable, then* $u = v$.

Proof . Let $x_0 \in X$ be an arbitrary initial point such that $\alpha(x_0, Tx_0) \geq 1$ and let define the sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n \text{ for } n \in \mathbb{N}.$$

If there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, then x_n is a fixed point of T and the proof is completed. Otherwise, suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since $x_0 \preceq Tx_0$ and T is a nondecreasing mapping, we have by induction that

$$x_0 \preceq Tx_0 \preceq T^2x_0 \preceq ... \preceq T^n x_0 \preceq T^{n+1}x_0 \preceq ....$$

Since T is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \text{ implies } \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Continuing this process, we get that

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{2.2}$$

Now, we define a sequence $\{e_n\}$ as

$$e_n = d_p(x_{n-1}, x_n).$$

We will prove that the sequence $\{e_n\}$ converges to 0, that is,

$$\lim_{n\to\infty} d_p(x_{n-1}, x_n) = 0.$$

Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (2.1) and (2.2) we have

$$\begin{aligned} d_p(x_n, x_{n+1}) &= d_p(Tx_{n-1}, Tx_n) \\ &\leq \alpha(x_{n-1}, x_n) d_p(Tx_{n-1}, Tx_n) \\ &\leq \beta(M_1(x_{n-1}, x_n)) M_1(x_{n-1}, x_n) \\ &< \frac{1}{s} M_1(x_{n-1}, x_n). \end{aligned} \tag{2.3}$$

for all $n \geq 1$ where,

$$\begin{aligned} M_1(x_{n-1}, x_n) &= \max\{d_p(x_{n-1}, x_n), d_p(x_{n-1}, Tx_{n-1}), d_p(x_n, Tx_n)\} \\ &= \max\{d_p(x_{n-1}, x_n), d_p(x_n, x_{n+1})\}. \end{aligned}$$

Assume that $M_1(x_{n-1}, x_n) = d_p(x_n, x_{n+1})$. Then, we write

$$d_p(x_n, x_{n+1}) < \frac{1}{s} d_p(x_n, x_{n+1})$$

which is not possible. Thus, we have $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ for all $n \geq 1$. Then from the inequality (2.3) we get

$$\begin{aligned} d_p(x_n, x_{n+1}) &< \frac{1}{s} d_p(x_{n-1}, x_n) \\ &< d_p(x_{n-1}, x_n). \end{aligned} \tag{2.4}$$

In other words, the sequence $\{e_n\} = \{d_p(x_{n-1}, x_n)\}$ is positive and decreasing. Thus, it converges to some $e \geq 0$. If we take limit as $n \to \infty$ in (2.3), we obtain

$$\frac{1}{s}e \leq e = \lim_{n\to\infty} e_{n+1} = \lim_{n\to\infty} \beta(e_n) e_n = e \lim_{n\to\infty} \beta(e_n) \leq \frac{1}{s}e.$$

This implies that $\lim_{n\to\infty} \beta(e_n) = \frac{1}{s}$ and therefore

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} d_p(x_{n-1}, x_n) = 0.$$

On the other hand, we observe that repeated application of (2.4) leads to

$$e_{n+1} < \frac{1}{s}e_n < \frac{1}{s^2}e_{n-1} < \ldots < \frac{1}{s^n}e_1. \tag{2.5}$$

for all $n \in \mathbb{N}$. Now, we will prove that $\{x_n\}$ is a Cauchy sequence in the partially ordered partial $b_v(s)$-metric space. In other words, we need to show that $\lim_{m,n\to\infty} d_p(x_n, x_m)$ exists and is finite. Particularly we will show that $\lim_{m,n\to\infty} d_p(x_n, x_m) = 0$. For $m, n \in \mathbb{N}$ with $m > n$ and $n_0 \in \mathbb{N}$, by using inequality (2.5) we obtain

$$\begin{aligned}
d_p(x_n, x_m) &\leq s\left[d_p(x_n, x_{n+1}) + d_p(x_{n+1}, x_{n+2}) + \ldots + d_p(x_{n+v-3}, x_{n+v-2})\right. \\
&\quad \left. + d_p(x_{n+v-2}, x_{n+n_0}) + d_p(x_{n+n_0}, x_{m+n_0}) + d_p(x_{m+n_0}, x_m)\right] \\
&\quad - \sum_{i=1}^{v-2} d_p(x_{n+i}, x_{n+i}) - d_p(x_{n+n_0}, x_{n+n_0}) - d_p(x_{m+n_0}, x_{m+n_0}) \\
&\leq s\left[d_p(x_n, x_{n+1}) + d_p(x_{n+1}, x_{n+2}) + \ldots + d_p(x_{n+v-3}, x_{n+v-2})\right. \\
&\quad \left. + d_p(x_{n+v-2}, x_{n+n_0}) + d_p(x_{n+n_0}, x_{m+n_0}) + d_p(x_{m+n_0}, x_m)\right] \\
&\leq s\left[\left(\frac{1}{s^n} d_p(x_0, x_1) + \frac{1}{s^{n+1}} d_p(x_0, x_1) + \ldots + \frac{1}{s^{n+v-3}} d_p(x_0, x_1)\right)\right. \\
&\quad \left. + \frac{1}{s^n} d_p(x_{v-2}, x_{n_0}) + \frac{1}{s^{n_0}} d_p(x_n, x_m) + \frac{1}{s^m} d_p(x_0, x_{n_0})\right] \\
&= s\left(\frac{1}{s^n} + \frac{1}{s^{n+1}} + \ldots + \frac{1}{s^{n+v-3}}\right) d_p(x_0, x_1) + \frac{1}{s^{n-1}} d_p(x_{v-2}, x_{n_0}) \\
&\quad + \frac{1}{s^{n_0-1}} d_p(x_n, x_m) + \frac{1}{s^{m-1}} d_p(x_0, x_{n_0}).
\end{aligned}$$

So, we get

$$\begin{aligned}
\left(1 - \frac{1}{s^{n_0-1}}\right) d_p(x_n, x_m) &\leq \frac{1}{s^{n-1}} \left(\frac{1 - \frac{1}{s^{v-2}}}{1 - \frac{1}{s}}\right) d_p(x_0, x_1) \\
&\quad + \frac{1}{s^{n-1}} d_p(x_{v-2}, x_{n_0}) + \frac{1}{s^{m-1}} d_p(x_0, x_{n_0}).
\end{aligned}$$

By taking limit from both side, we have

$$\lim_{m,n\to\infty} d_p(x_n, x_m) = 0.$$

Therefore $\{x_n\}$ is a Cauchy sequence in $(X, \preceq, d_p)$. Since X is complete, the sequence $\{x_n\}$ converges to some $u \in X$ such that

$$\lim_{n\to\infty} d_p(x_n, u) = \lim_{m,n\to\infty} d_p(x_n, x_m) = d_p(u, u) = 0. \tag{2.6}$$

Since T is a continuous mapping, then from (2.6) we may write

$$\lim_{n\to\infty} d_p\left(Tx_n, Tu\right) = \lim_{n\to\infty} d_p\left(x_{n+1}, Tu\right) = d_p\left(Tu, Tu\right).$$

Now we show that u is a fixed point of T, i.e., $d_p\left(Tu, u\right) = 0$. We assume on the contrary that $d_p\left(Tu, u\right) \neq 0$. So, we get

$$\begin{aligned} d_p\left(Tu, u\right) &\leq s\left[d_p\left(Tu, x_{n+1}\right) + d_p\left(x_{n+1}, x_{n+2}\right) + ... + d_p\left(x_{n+v-1}, x_{n+v}\right) + d_p\left(x_{n+v}, u\right)\right] \\ &\quad -\sum_{i=1}^{v} d_p\left(x_{n+i}, x_{n+i}\right) \\ &\leq s\left[d_p\left(Tu, x_{n+1}\right) + d_p\left(x_{n+1}, x_{n+2}\right) + ... + d_p\left(x_{n+v-1}, x_{n+v}\right) + d_p\left(x_{n+v}, u\right)\right] \end{aligned}$$

Letting $n \to \infty$ in the above inequality we obtain

$$\begin{aligned} \frac{1}{s} d_p\left(Tu, u\right) &\leq \lim_{n\to\infty} d_p\left(Tu, x_{n+1}\right) + \lim_{n\to\infty} d_p\left(x_{n+1}, x_{n+2}\right) + ... \\ &\quad + \lim_{n\to\infty} d_p\left(x_{n+v-1}, x_{n+v}\right) + \lim_{n\to\infty} d_p\left(x_{n+v}, u\right) \\ &= d_p\left(Tu, Tu\right). \end{aligned}$$

So, we get $\frac{1}{s} d_p\left(Tu, u\right) \leq d_p\left(Tu, Tu\right)$. It follows from $\alpha\left(u, u\right) \geq 1$ that

$$\begin{aligned} \frac{1}{s} d_p\left(Tu, u\right) &\leq d_p\left(Tu, Tu\right) \leq \alpha\left(u, u\right) d_p\left(Tu, Tu\right) \\ &\leq \beta\left(M_1\left(u, u\right)\right) M_1\left(u, u\right), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} M_1\left(u, u\right) &= \max\left\{d_p\left(u, u\right), d_p\left(u, Tu\right), d_p\left(u, Tu\right)\right\} \\ &= d_p\left(u, Tu\right). \end{aligned}$$

Hence, from inequality (2.7) we have

$$\frac{1}{s} d_p\left(Tu, u\right) \leq \beta\left(d_p\left(u, Tu\right)\right) d_p\left(u, Tu\right).$$

As a result, $\frac{1}{s} \leq \limsup_{n\to\infty} \beta\left(d_p\left(u, Tu\right)\right) \leq \frac{1}{s}$. We concluded $\limsup_{n\to\infty} d_p\left(u, Tu\right) = 0$ and so $d_p\left(u, Tu\right) = 0$ which is a contradiction. Therefore, we obtain $Tu = u$, that is, u is a fixed point of T. Since the existence of a fixed point is proved, we need to prove only the uniqueness under the given conditions. Assume that T has two distinct comparable fixed points u and v.We put these points in the contractive condition (2.1) and use the fact that $\alpha\left(u, v\right) \geq 1$, we obtain,

$$d_p\left(u, v\right) \leq \alpha\left(u, v\right) d_p\left(Tu, Tv\right) \leq \beta\left(M_1\left(u, v\right)\right) M_1\left(u, v\right) < \frac{1}{s} M_1\left(u, v\right),$$

where

$$M_1\left(u, v\right) = \max\left\{d_p\left(u, v\right), d_p\left(u, Tu\right), d_p\left(v, Tv\right)\right\} = d_p\left(u, v\right).$$

This implies

$$d_p\left(u, v\right) < \frac{1}{s} d_p\left(u, v\right),$$

which is a contradiction and hence we get $d_p\left(u, v\right) = 0$, that is $u = v$. □

In the next theorem we replace the continuity of the mapping T by a property of the space. Thus, the following theorem is an enlargement of [[6], Theorem 2.4] from Branciari b metric spaces to the case of partially ordered partial $b_v\left(s\right)$ metric space.

Theorem 2.2. *Let $(X, \preceq, d_p)$ be a complete partially ordered partial $b_v(s)$-metric space with a constant $s > 1$ and $\alpha : X \times X \to [0, \infty)$ be a function. Let $T : X \to X$ be an α-admissible nondecreasing mapping with respect to "$\preceq$". Suppose that*

$$\alpha(x,y)\, d_p(Tx,Ty) \leq \beta(M_1(x,y))\, M_1(x,y) \quad x,y \in X \tag{2.8}$$

for some $\beta \in \mathcal{F}_s$ and for all $x, y \in X$ with $x \preceq y$ where

$$M_1(x,y) = \max\{d_p(x,y), d_p(x,Tx), d_p(y,Ty)\}.$$

Suppose also that

1. *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $x_0 \preceq Tx_0$.*
2. *For any nondecreasing sequence $\{x_n\} \in X$ which converges to x, we have $x_n \preceq x$ and for $\{x_n\}$ which satisfies $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*
3. *For every pair z and w of fixed points of T, we have $\alpha(z, w) \geq 1$.*

Then T has at least one fixed point z. Also, if v is another fixed point of T such that z and v are comparable, then $z = v$.

Proof . Taking $x_0 \in X$ as the element satisfying the condition (1), we construct sequence $\{x_n\}$ as usual, that is $x_n = Tx_{n-1}$, for $n \in \mathbb{N}$. The convergence of this sequence can be shown exactly as in the proof of Theorem 2.1. Let z be the limit of $\{x_n\}$, that is,

$$\lim_{n\to\infty} d_p(x_n, z) = \lim_{m,n\to\infty} d_p(x_n, x_m) = d_p(z,z) = 0.$$

We will show that z is a fixed point of T. For any nondecreasing sequence $\{x_n\} \in X$ which converges to z we know from the assumption that $x_n \preceq z$. Also, we can show in the same way as in the proof of Theorem 2.1 that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $d_p(x_n, x_{n+1}) \to 0$ for $n \to \infty$. So, the condition (2) in the statement of the theorem implies that

$$\alpha(x_n, z) \geq 1, \text{ for all } n \in \mathbb{N}.$$

Then, for $Tz \neq z$ we have

$$\begin{aligned} d_p(Tz,z) &\leq s\left[d_p(Tz,x_{n+1}) + d_p(x_{n+1},x_{n+2}) + ... + d_p(x_{n+v-1},x_{n+v}) + d_p(x_{n+v},z)\right] \\ &\quad -\sum_{i=1}^{v} d_p(x_{n+i},x_{n+i}) \\ &\leq s\left[d_p(Tz,x_{n+1}) + d_p(x_{n+1},x_{n+2}) + ... + d_p(x_{n+v-1},x_{n+v}) + d_p(x_{n+v},z)\right] \end{aligned}$$

Letting $n \to \infty$ in the above inequality we obtain

$$\begin{aligned} d_p(Tz,z) &\leq s\lim_{n\to\infty} d_p(Tz,x_{n+1}) + s\lim_{n\to\infty} d_p(x_{n+1},x_{n+2}) + ... \\ &\quad + s\lim_{n\to\infty} d_p(x_{n+v-1},x_{n+v}) + s\lim_{n\to\infty} d_p(x_{n+v},z) \\ &= s\lim_{n\to\infty} d_p(Tz,x_{n+1}). \end{aligned}$$

Also, we have

$$\begin{aligned} sd_p(x_{n+1},Tz) &= sd_p(Tx_n,Tz) \leq s\alpha(x_n,z)\, d_p(Tx_n,Tz) \\ &\leq s\beta(M_1(x_n,z))\, M_1(x_n,z) \\ &< M_1(x_n,z), \end{aligned} \tag{2.9}$$

where

$$M_1(x_n, z) = \max\{d_p(x_n, z), d_p(x_n, Tx_n), d_p(z, Tz)\}. \tag{2.10}$$

So, we get

$$d_p(Tz, z) < \max\{d_p(x_n, z), d_p(x_n, Tx_n), d_p(z, Tz)\}.$$

By taking limit we conclude that

$$\begin{aligned} d_p(Tz, z) &< \lim_{n\to\infty}[\max\{d_p(x_n, z), d_p(x_n, Tx_n), d_p(z, Tz)\}] \\ &= d_p(z, Tz) \end{aligned}$$

which is a contradiction. Then $d_p(z, Tz) = 0$, thus z is a fixed point of T. The rest of the proof can be shown similarly to the proof of Theorem 2.1. □

From Theorem 2.1, we can easily conclude that the following result.

Theorem 2.3. *Let $(X, \preceq, d_p)$ be a complete partially ordered partial $b_v(s)$-metric spaces with a parameter $s > 1$ and $T : X \to X$ be an α-admissible nondecreasing mapping with respect to $\preceq$. Suppose that*

$$\alpha(x, y)\, d_p(Tx, Ty) \le \beta(d_p(x, y))\, d_p(x, y) \tag{2.11}$$

for some $\beta \in \mathcal{F}_s$ and for all $x, y \in X$ with $x \preceq y$. Then T has at least one fixed point z. If v is another fixed point of T such that z and v are comparable, then $z = v$. Moreover, if we suppose that the following conditions are satisfied, then the Picard sequence $\{x_n\}$ converges strongly to the fixed point of T.

1. There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ and $x_0 \preceq Tx_0$.

2. For any nondecreasing sequence $\{x_n\} \in X$ which converges to x, we have $x_n \preceq x$ and for $\{x_n\}$ which satisfies $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$.

3. For every pair z and w of fixed points of T, we have $\alpha(z, w) \ge 1$.

Now, we will give an example which satisfies the conditions of Theorem 2.3.

Example 2.4. *Let $X = \{1, 2, 3, 4, 5\}$ and the function $d_p : X \times X \to [0, \infty)$ be defined by*

$$d_p(x, y) = \begin{cases} 0, & \text{if } x = y \text{ and } x, y \in \{3, 4\}, \\ \frac{9}{10}, & \text{if } x \text{ or } y \in \{1, 2\},\ x \ne y, \\ \frac{1}{10}, & \text{otherwise,} \end{cases}$$

for all $x, y \in X$. It is easy to see that (X, d_p) is a complete partially ordered partial $b_v(s)$-metric space where $v = 3$ and $s = \frac{5}{4}$. Also, if we define a mapping $\alpha : X \times X \to \mathbb{R}$ by

$$\alpha(x, y) = \begin{cases} 2 & x, y \in \{3, 4, 5\}, \\ 1 & \text{otherwise,} \end{cases},$$

a mapping $T : X \times X \to [0, \infty)$ by $T2 = T3 = T4 = T5 = 4$, $T1 = 3$, and a mapping $\beta : [0, \infty) \to \left[0, \frac{4}{5}\right)$ by $\beta(t) = \frac{4}{5}e^{-t}$, then it is clear that T is a nondecreasing α-admissible mapping which satisfies the inequality (2.11). Then T has a unique fixed point $z = 4$ and the Picard sequence converges to unique fixed point.

The following theorem is an enlargement of [[5],Theorem 2.1] from the framework of $b_v(s)$-metric spaces to the case of partially ordered partial $b_v(s)$-metric spaces.

Theorem 2.5. *Let $(X, \preceq, d_p)$ be a complete partially ordered partial $b_v(s)$-metric spaces with parameter $s > 1$ and $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$ such that there exists $x_0 \in X$ with $x_0 \preceq Tx_0$. Suppose that*

$$d_p(Tx, Ty) \leq \beta(M_2(x, y)) M_2(x, y) \tag{2.12}$$

for some $\beta \in \mathcal{F}_s$ and all comparable elements $x, y \in X$, where

$$M_2(x, y) = \max\left\{d_p(x, y), \frac{d_p(x, Tx) d_p(y, Ty)}{1 + d_p(Tx, Ty)}, \frac{d_p(x, Tx) d_p(y, Ty)}{1 + d_p(x, y)}, \frac{d_p(x, Tx) d_p(x, Ty)}{1 + d_p(x, Ty) + d_p(y, Tx)}\right\}.$$

If $(X, \preceq, d_p)$ has a sequential limit comparison property, i.e., for any nondecreasing sequence $\{x_n\} \in X$ which converges to x, we have $x_n \preceq x$, then T has a fixed point. Furthermore, the set of fixed points of T is well ordered if and only if T has a unique fixed point.

Proof . *Starting with a given x_0 such that $x_0 \preceq Tx_0$, let $\{x_n\}$ be a sequence defined by $x_n = T^n x_0$. If $x_n = x_{n+1}$ for all $n \in \mathbb{N}$, then x_n is a fixed point of T. Therefore, we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since $x_0 \preceq Tx_0$ and T is an nondecreasing mapping, we obtain by induction*

$$x_0 \preceq x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq \ldots .$$

First, we will show that $\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0$. Since x_n and x_{n+1} are comparable for each $n \in \mathbb{N}$, then by (2.12) we have

$$d_p(x_n, x_{n+1}) = d_p(Tx_{n-1}, Tx_n) \leq \beta(M_2(x_{n-1}, x_n)) M_2(x_{n-1}, x_n), \tag{2.13}$$

where

$$\begin{aligned} M_2(x_{n-1}, x_n) &= \max\left\{d_p(x_{n-1}, x_n), \frac{d_p(x_{n-1}, Tx_{n-1}) d_p(x_n, Tx_n)}{1 + d_p(Tx_{n-1}, Tx_n)},\right. \\ &\quad \left.\frac{d_p(x_{n-1}, Tx_{n-1}) d_p(x_n, Tx_n)}{1 + d_p(x_{n-1}, x_n)}, \frac{d_p(x_{n-1}, Tx_{n-1}) d_p(x_{n-1}, Tx_n)}{1 + d_p(x_{n-1}, Tx_n) + d_p(x_n, Tx_{n-1})}\right\} \\ &= \max\left\{d_p(x_{n-1}, x_n), \frac{d_p(x_{n-1}, x_n) d_p(x_n, x_{n+1})}{1 + d_p(x_n, x_{n+1})},\right. \\ &\quad \left.\frac{d_p(x_{n-1}, x_n) d_p(x_n, x_{n+1})}{1 + d_p(x_{n-1}, x_n)}, \frac{d_p(x_{n-1}, x_n) d_p(x_{n-1}, x_{n+1})}{1 + d_p(x_{n-1}, x_{n+1}) + d_p(x_n, x_n)}\right\} \\ &\leq \max\{d_p(x_{n-1}, x_n), d_p(x_n, x_{n+1})\}. \end{aligned}$$

If $\max\{d_p(x_{n-1}, x_n), d_p(x_n, x_{n+1})\} = d_p(x_n, x_{n+1})$, *then from (2.13) we have*

$$\begin{aligned} d_p(x_n, x_{n+1}) &\leq \beta(M_2(x_{n-1}, x_n)) M_2(x_{n-1}, x_n) \\ &\leq \frac{1}{s} d_p(x_n, x_{n+1}). \end{aligned}$$

This is a contradiction. Therefore, we get $\max\{d_p(x_{n-1}, x_n), d_p(x_n, x_{n+1})\} = d_p(x_{n-1}, x_n)$. *So, from (2.13) we have*

$$d_p(x_n, x_{n+1}) \leq \frac{1}{s} d_p(x_{n-1}, x_n).$$

If we repeat this process, we obtain

$$d_p(x_n, x_{n+1}) \leq \frac{1}{s^n} d_p(x_0, x_1) \tag{2.14}$$

for all $n \in \mathbb{N}$*. Then we have* $\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0$*. Now we will prove that* $\{x_n\}$ *is a Cauchy sequence in partially ordered partial* $b_v(s)$*-metric space. In this case,we need to show that* $\lim_{m,n\to\infty} d_p(x_n, x_m)$ *exists and is finite. Particularly, we will show that* $\lim_{m,n\to\infty} d_p(x_n, x_m) = 0$*. For* $m, n \in \mathbb{N}$ *with* $m > n$ *and* $n_0 \in \mathbb{N}$*, by using (2.14) and the triangular inequality we obtain*

$$\begin{aligned}
d_p(x_n, x_m) &\leq s\left[d_p(x_n, x_{n+1}) + d_p(x_{n+1,} x_{n+2}) + ... + d_p(x_{n+v-3,} x_{n+v-2})\right.\\
&\quad \left. + d_p(x_{n+v-2,} x_{n+n_0}) + d_p(x_{n+n_0}, x_{m+n_0}) + d_p(x_{m+n_0}, x_m)\right]\\
&\quad - \sum_{i=1}^{v-2} d_p(x_{n+i}, x_{n+i}) - d_p(x_{n+n_0}, x_{n+n_0}) - d_p(x_{m+n_0}, x_{m+n_0})\\
&\leq s\left[d_p(x_n, x_{n+1}) + d_p(x_{n+1,} x_{n+2}) + ... + d_p(x_{n+v-3,} x_{n+v-2})\right.\\
&\quad \left. + d_p(x_{n+v-2,} x_{n+n_0}) + d_p(x_{n+n_0}, x_{m+n_0}) + d_p(x_{m+n_0}, x_m)\right]\\
&\leq s\left[\frac{1}{s^n} d_p(x_0, x_1) + \frac{1}{s^{n+1}} d_p(x_0, x_1) + ... + \frac{1}{s^{n+v-3}} d_p(x_0, x_1)\right.\\
&\quad \left. + \frac{1}{s^n} d_p(x_{v-2}, x_{n_0}) + \frac{1}{s^{n_0}} d_p(x_n, x_m) + \frac{1}{s^m} d_p(x_0, x_{n_0})\right]\\
&= \left(\frac{1}{s^{n-1}} + \frac{1}{s^n} + ... + \frac{1}{s^{n+v-4}}\right) d_p(x_0, x_1) + \frac{1}{s^{n-1}} d_p(x_{v-2}, x_{n_0})\\
&\quad + \frac{1}{s^{n_0-1}} d_p(x_n, x_m) + \frac{1}{s^{m-1}} d_p(x_0, x_{n_0}),
\end{aligned}$$

and so, we have

$$\begin{aligned}
\left(1 - \frac{1}{s^{n_0-1}}\right) d_p(x_n, x_m) &\leq \frac{1}{s^{n-1}} \left(\frac{1 - \frac{1}{s^{v-2}}}{1 - \frac{1}{s}}\right) d_p(x_0, x_1)\\
&\quad + \frac{1}{s^{n-1}} d_p(x_{v-2}, x_{n_0}) + \frac{1}{s^{m-1}} d_p(x_0, x_{n_0}).
\end{aligned}$$

By taking limit from both side for $m, n \to \infty$*, we get*

$$\lim_{m,n\to\infty} d_p(x_n, x_m) = 0.$$

Therefore $\{x_n\}$ *is a Cauchy sequence in* X*. Since* X *is complete, there exists* $u \in X$ *such that*

$$\lim_{n\to\infty} d_p(x_n, u) = \lim_{m,n\to\infty} d_p(x_n, x_m) = d_p(u, u) = 0.$$

Now, we will show that u *is a fixed point of* T*. We know from the sequential limit comparision property of the space that* x_n *and* u *are comparable for all* $n \in \mathbb{N}$*. Further, there exists* $n \in \mathbb{N}$ *such that* $u, Tu \notin \{x_{n+1}, x_{n+2}, ...\}$*. Then we have*

$$\begin{aligned}
d_p(u, Tu) &\leq s\left[d_p(Tu, x_{n+1}) + d_p(x_{n+1}, x_{n+2}) + ... + d_p(x_{n+v-1}, x_{n+v}) + d_p(x_{n+v}, u)\right]\\
&\quad - \sum_{i=1}^{v} d_p(x_{n+i}, x_{n+i})\\
&\leq s\left[d_p(Tu, x_{n+1}) + d_p(x_{n+1}, x_{n+2}) + ... + d_p(x_{n+v-1}, x_{n+v}) + d_p(x_{n+v}, u)\right].
\end{aligned}$$

By taking limitsup in the above inequality for $n \to \infty$*, we obtain*

$$\begin{aligned}
\frac{1}{s} d_p(u, Tu) &\leq \limsup_{n\to\infty} d_p(Tu, x_{n+1})\\
&\leq \limsup_{n\to\infty} \beta(M_2(u, x_n)) M_2(u, x_n)\\
&\leq 0.
\end{aligned}$$

Indeed, the inequality $\limsup_{n\to\infty} \beta\left(M_2\left(u,x_n\right)\right)=0$ *can be showed by the following:*

$$\begin{aligned}
\limsup_{n\to\infty} M_2(x_n,u) &= \limsup_{n\to\infty} \max\left\{d_p\left(x_n,u\right), \frac{d_p\left(x_n,Tx_n\right)d_p\left(u,Tu\right)}{1+d_p\left(Tx_n,Tu\right)}\right. \\
&\quad \left., \frac{d_p\left(x_n,Tx_n\right)d_p\left(u,Tu\right)}{1+d_p\left(x_n,u\right)}, \frac{d_p\left(x_n,Tx_n\right)d_p\left(x_n,Tu\right)}{1+d_p\left(x_n,Tu\right)+d_p\left(u,Tx_n\right)}\right\} \\
&= \limsup_{n\to\infty} \max\left\{d_p\left(x_n,u\right), \frac{d_p\left(x_n,x_{n+1}\right)d_p\left(u,Tu\right)}{1+d_p\left(x_{n+1},Tu\right)}\right. \\
&\quad \left., \frac{d_p\left(x_n,x_{n+1}\right)d_p\left(u,Tu\right)}{1+d_p\left(x_n,u\right)}, \frac{d_p\left(x_n,x_{n+1}\right)d_p\left(x_n,Tu\right)}{1+d_p\left(x_n,Tu\right)+d_p\left(u,Tx_n\right)}\right\} \\
&= 0.
\end{aligned}$$

Therefore,we have $Tu=u$ *. Hence* u *is a fixed point of* T*. Finally, assume that the set of fixed point of* T *is well ordered. We need to show that* T *has a unique fixed point. Assume on the contrary,* u *and* v *are two fixed points of the* T *such that* $u\neq v$*. Then, by (2.12) we have*

$$\begin{aligned}
d_p\left(u,v\right) &= d_p\left(Tu,Tv\right) \leq \beta\left(M_2\left(u,v\right)\right)M_2\left(u,v\right) \\
&< \frac{1}{s}M_2\left(u,v\right)
\end{aligned}$$

where

$$\begin{aligned}
M_2(u,v) &= \max\left\{d_p\left(u,v\right), \frac{d_p\left(u,Tu\right)d_p\left(v,Tv\right)}{1+d_p\left(Tu,Tv\right)}, \frac{d_p\left(u,Tu\right)d_p\left(v,Tv\right)}{1+d_p\left(u,v\right)}, \frac{d_p\left(u,Tu\right)d_p\left(u,Tv\right)}{1+d_p\left(u,Tv\right)+d_p\left(v,Tv\right)}\right\} \\
&= \max\left\{d_p\left(u,v\right),0,0,0\right\} \\
&= d_p\left(u,v\right).
\end{aligned}$$

Hence, we get $d_p\left(u,v\right)<\frac{1}{s}d_p\left(u,v\right)$ *which is a contradiction. Hence* $u=v$ *and* T *has a unique fixed point. Conversely, if* T *has a unique fixed point, it is clear that the set of fixed points of* T *is well ordered. Morever, for any fixed point* u*, let assume that* $d_p\left(u,u\right)>0$*. Then, we get*

$$\begin{aligned}
d_p\left(u,u\right) &= d_p\left(Tu,Tu\right) \leq \beta\left(M_2\left(u,u\right)\right)M_2\left(u,u\right) \\
&< \frac{1}{s}M_2\left(u,u\right)
\end{aligned}$$

where

$$\begin{aligned}
M_2(u,u) &= \max\left\{d_p\left(u,u\right), \frac{d_p\left(u,Tu\right)d_p\left(u,Tu\right)}{1+d_p\left(Tu,Tu\right)}, \frac{d_p\left(u,Tu\right)d_p\left(u,Tu\right)}{1+d_p\left(u,u\right)}, \frac{d_p\left(u,Tu\right)d_p\left(u,Tu\right)}{1+d_p\left(u,Tu\right)+d_p\left(u,Tu\right)}\right\} \\
&= \max\left\{d_p\left(u,u\right),0,0,0\right\} \\
&= d_p\left(u,u\right).
\end{aligned}$$

So, we obtain $d_p\left(u,u\right)<\frac{1}{s}d_p\left(u,u\right)$ *which is a contradiction. So, assumption is wrong, that is* $d_p\left(u,u\right)=0$*. This completes the proof.* □

3. A Modified Partial $b_v\left(s\right)$-Metric Space

It is known from the Proposition 1.4 that the sum of a $b_v\left(s\right)$-metric and a partial metric is a partial $b_v\left(s\right)$-metric. This means that defining a partial $b_v\left(s\right)$-metric by using a $b_v\left(s\right)$-metric is possible. But, on the contrary that, defining a $b_v\left(s\right)$-metric by using a partial $b_v\left(s\right)$-metric is not possible in this concept. In this part, we introduce a modified partial $b_v\left(s\right)$-metric space such that each partial $b_v\left(s\right)$-metric generates a $b_v\left(s\right)$-metric. While defining this space, we inspired by the following partial b-metric space introduced by Mustafa et al.[19].

Definition 3.1. *[19]Let X be a nonempty set and $s \geq 1$ be a given real numbers. A mapping $p_b : X \times X \to [0, \infty)$ is said to be a modified partial b-metric spaces if for all $x, y, z \in X$ the following condition hold:*

1. $x = y$ *iff* $p_b(x, x) = p_b(x, y) = p_b(y, y)$,
2. $p_b(x, x) \leq p_b(x, y)$,
3. $p_b(x, y) = p_b(y, x)$,
4. $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y) - p_b(z, z)] + \left(\frac{1-s}{2}\right)[p_b(x, x) + p_b(y, y)]$.

We are now in a position to define our modified space.

Definition 3.2. *Let X be a nonempty set and $p_{b_v} : X \times X \to [0, \infty)$ be a mapping and $v \in \mathbb{N}$. Then, (X, p_{b_v}) is said to be a modified partial $b_v(s)$-metric space if there exists a real number $s \geq 1$ such that following conditions hold for all $x, y, u_1, u_2, ..., u_{v-1}, u_v \in X$:*

1. $x = y \Leftrightarrow p_{b_v}(x, x) = p_{b_v}(x, y) = p_{b_v}(y, y)$,
2. $p_{b_v}(x, x) \leq p_{b_v}(x, y)$,
3. $p_{b_v}(x, y) = p_{b_v}(y, x)$,
4. $p_{b_v}(x, y) \leq s\left[p_{b_v}(x, u_1) + p_{b_v}(u_1, u_2) + ... + p_{b_v}(u_{v-1}, u_v) + p_{b_v}(u_v, y) - \sum_{i=1}^{v} p_{b_v}(u_i, u_i)\right]$
$+\frac{(1-s)}{2}[p_{b_v}(x, x) + p_{b_v}(y, y)]$.

Remark 3.3. *In definition (3.2), if we take $v = 1$, then we derive the modified partial b-metric spaces.*

From the triangular inequality of definition (3.2), we have

$$\begin{aligned} p_{b_v}(x, y) &\leq s\left[p_{b_v}(x, u_1) + p_{b_v}(u_1, u_2) + ... + p_{b_v}(u_{v-1}, u_v) + p_{b_v}(u_v, y) - \sum_{i=1}^{v} p_{b_v}(u_i, u_i)\right] \\ &\quad + \frac{(1-s)}{2}[p_{b_v}(x, x) + p_{b_v}(y, y)] \\ &\leq s[p_{b_v}(x, u_1) + p_{b_v}(u_1, u_2) + ... + p_{b_v}(u_{v-1}, u_v) + p_{b_v}(u_v, y)] - \sum_{i=1}^{v} p_{b_v}(u_i, u_i). \end{aligned}$$

Hence, a modified partial $b_v(s)$-metric space is also a partial $b_v(s)$-metric space. On the other hand, since a modified partial $b_v(s)$-metric is a partial metric with $s = v = 1$, it should be noted that the class of a modified partial $b_v(s)$-metric spaces is larger than the class of partial metric spaces.

Proposition 3.4. *Let (X, d) be a $b_v(s)$-metric space and $p > 1$ and $k \geq 0$ be real numbers. If $p_{b_v} : X \times X \to [0, \infty)$ is a mapping defined by $p_{b_v}(x, y) = d(x, y)^p + k$, then (X, p_{b_v}) is a modified partial $b_v(s)$-metric space with $s = v^{p-1}$.*

Proof . We will show that p_{b_v} is a modified partial $b_v(s)$-metric with $s = v^{p-1}$. Obviously,by using the convexity of the function $f(x) = x^p$ for $x \geq 0$ and Jensen inequality, we may write

$(x_1+x_2+...+x_{v-1}+x_v)^p \leq v^{p-1}\left(x_1^p+x_2^p+...+x_{v-1}^p+x_v^p\right)$ for $x_1, x_2, ..., x_{v-1}, x_v \geq 0$. Thus, for each $x, y, x_1, x_2, ..., x_{v-1}, x_v \in X$, we obtain

$$\begin{aligned}
p_{b_v}(x,y) &= d(x,y)^p + k \\
&\leq [d(x,x_1)+d(x_1,x_2)+...+d(x_{v-1},x_v)+d(x_v,y)]^p + k \\
&\leq v^{p-1}[d(x,x_1)^p+d(x_1,x_2)^p+...+d(x_{v-1},x_v)^p+d(x_v,y)^p] + k \\
&= v^{p-1}[d(x,x_1)^p+k+d(x_1,x_2)^p+k+...+d(x_{v-1},x_v)^p+k+d(x_v,y)^p+k-vk-k] - v \\
&= v^{p-1}\left[p_{b_v}(x,x_1)+p_{b_v}(x_1,x_2)+...+p_{b_v}(x_{v-1},x_v)+p_{b_v}(x_v,y)-\sum_{i=1}^{v} p_{b_v}(x_i,x_i)\right] \\
&\quad + \left(\frac{1-v^{p-1}}{2}\right)[p_{b_v}(x,x)+p_{b_v}(y,y)].
\end{aligned}$$

Hence, p_{b_v} is a modified partial $b_v(s)$ metric on X. □

In the following proposition, we show that a modified partial $b_v(s)$-metric space generates a $b_v(s)$-metric space.

Proposition 3.5. *Let (X, p_{b_v}) be a modified partial $b_v(s)$-metric space. Then, modified partial $b_v(s)$-metric p_{b_v} defines a $b_v(s)$-metric d by the following way:*

$$d(x,y) = 2p_{b_v}(x,y) - p_{b_v}(x,x) - p_{b_v}(y,y),$$

Proof . Let $x, y, u_1, u_2, ..., u_{v-1}, u_v \in X$

$$\begin{aligned}
d(x,y) &= 2p_{b_v}(x,y) - p_{b_v}(x,x) - p_{b_v}(y,y) \\
&\leq 2s[p_{b_v}(x,u_1)+p_{b_v}(u_1,u_2)+...+p_{b_v}(u_{v-1},u_v)+p_{b_v}(u_v,y)-p_{b_v}(u_1,u_1)-p_{b_v}(u_2,u_2)-... \\
&\quad -p_{b_v}(u_{v-1},u_{v-1})-p_{b_v}(u_v,u_v)] + 2\frac{(1-s)}{2}[p_{b_v}(x,x)+p_{b_v}(y,y)] - p_{b_v}(x,x) - p_{b_v}(y,y) \\
&= s[2p_{b_v}(x,u_1)-p_{b_v}(u_1,u_1)-p_{b_v}(x,x)+2p_{b_v}(u_1,u_2)-p_{b_v}(u_1,u_1)-p_{b_v}(u_2,u_2)+... \\
&\quad +2p_{b_v}(u_{v-1},u_v)-p_{b_v}(u_{v-1},u_{v-1})-p_{b_v}(u_v,u_v)+2p_{b_v}(u_v,y)-p_{b_v}(u_v,u_v)-p_{b_v}(y,y)] \\
&= s[d(x,u_1)+d(u_1,u_2)+...+d(u_{v-1},u_v)+d(u_v,y)].
\end{aligned}$$

□

Let $(X, \preceq)$ be an ordered set. The mappings $T, S : X \to X$ are said to be weakly increasing mappings, if $Tx \preceq STx$ and $Sx \preceq TSx$ for all $x \in X$. The following theorem is a generalization of [[1], Theorem 2.2.] from the concept of partially ordered partial b-metric spaces.

Theorem 3.6. *Let $(X, \preceq, p_{b_v})$ be a complete partially ordered modified partial $b_v(s)$-metric space with constant $s > 1$. Suppose that $T, S : X \to X$ are two weakly increasing mappings which satisfy the following condition for all $x, y \in X$ with $x \preceq y$ and $\beta \in \mathcal{F}_s$:*

$$sp_{b_v}(Tx, Sy) \leq \beta(M(x,y))M(x,y) + LN(x,y), \tag{3.1}$$

where $L \geq 0$,

$$M(x,y) = \max\left\{p_{b_v}(x,y), \frac{p_{b_v}(x,Tx)\,p_{b_v}(y,Sy)}{1+p_{b_v}(Tx,Sy)}\right\}$$

and

$$N(x,y) = \min\{d(x,y), d(x,Tx), d(y,Sy), d(y,Tx), d(x,Sy)\}$$

where d is a $b_v(s)$-metric. Suppose that there exist a $x_0 \in X$ such that $Tx_0 \preceq STx_0$ and X has a sequential limit comparison property. Then T and S have a common fixed point. Moreover, if the set of common fixed points of T and S is comparable, then they have a unique common fixed point.

Proof . By using assumption there exists $x_0 \in X$ such that $Tx_0 \preceq STx_0$. Let $\{x_n\}$ be a sequence in X defined by the following way:

$$Tx_{2n} = x_{2n+1} \text{ and } Sx_{2n+1} = x_{2n+2}, \text{ for } \forall n \geq 0$$

Note that

$$x_1 = Tx_0 \preceq STx_0 = Sx_1 = x_2 \preceq TSx_1 = x_3.$$

So, we obtain

$$x_1 \preceq x_2 \preceq x_3 \preceq x_4 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ....$$

Since x_{2n} and x_{2n+1} are comparable and T and S satisfy (3.1), we have

$$\begin{aligned} sp_{b_v}(x_{2n+1}, x_{2n+2}) &= sp_{b_v}(Tx_{2n}, Sx_{2n+1}) \\ &\leq \beta(M(x_{2n}, x_{2n+1}))\, M(x_{2n}, x_{2n+1}) + LN(x_{2n}, x_{2n+1}), \end{aligned}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\left\{p_{b_v}(x_{2n}, x_{2n+1}), \frac{p_{b_v}(x_{2n}, Tx_{2n})\, p_{b_v}(x_{2n+1}, Sx_{2n+1})}{p_{b_v}(Tx_{2n}, Sx_{2n+1})}\right\} \\ &= \max\left\{p_{b_v}(x_{2n}, x_{2n+1}), \frac{p_{b_v}(x_{2n}, x_{2n+1})\, p_{b_v}(x_{2n+1}, x_{2n+2})}{p_{b_v}(x_{2n+1}, x_{2n+2})}\right\} \\ &\leq \max\{p_{b_v}(x_{2n}, x_{2n+1}), p_{b_v}(x_{2n}, x_{2n+1})\} = p_{b_v}(x_{2n}, x_{2n+1}) \end{aligned}$$

and

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= \min\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n}, Sx_{2n+1}), d(x_{2n+1}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1})\} \\ &= \min\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ &= \min\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), 0, d(x_{2n+1}, x_{2n+2})\} = 0. \end{aligned}$$

Thus, we can write

$$\begin{aligned} sp_{b_v}(x_{2n+1}, x_{2n+2}) &\leq \beta(p_{b_v}(x_{2n}, x_{2n+1}))\, p_{b_v}(x_{2n}, x_{2n+1}) \\ &\leq \frac{1}{s} p_{b_v}(x_{2n}, x_{2n+1}) \leq p_{b_v}(x_{2n}, x_{2n+1}). \end{aligned}$$

Similarly we can show that

$$sp_{b_v}(x_{2n+2}, x_{2n+3}) \leq p_{b_v}(x_{2n+1}, x_{2n+2}).$$

Hence, we conclude that

$$p_{b_v}(x_n, x_{n+1}) \leq \frac{1}{s} p_{b_v}(x_{n-1}, x_n) \text{ for each } n \in \mathbb{N}. \tag{3.2}$$

So, by repeating this process, we obtain

$$p_{b_v}(x_n, x_{n+1}) \leq \frac{1}{s^n} p_{b_v}(x_0, x_1)$$

for all $n \in \mathbb{N}$. Then we obtain $\lim_{n\to\infty} p_{b_v}(x_n, x_{n+1}) = 0$. Now we prove that the sequence $\{x_n\}$ is a Cauchy sequence in partially ordered modified partial $b_v(s)$-metric spaces. Namely, we need to show that $\lim_{n\to\infty} p(x_n, x_m)$ exists and finite. Particularly,we will show that $\lim_{m,n\to\infty} p_{b_v}(x_n, x_m) = 0$. For $m, n \in \mathbb{N}$ with $m > n$ and $n_0 \in \mathbb{N}$, by using (3.2) and the triangular inequality we obtain

$$\begin{aligned}
p_{b_v}(x_n, x_m) &\leq s\left[p_{b_v}(x_n, x_{n+1}) + p_{b_v}(x_{n+1,}x_{n+2}) + ... + p_{b_v}(x_{n+v-3,}x_{n+v-2}) + p_{b_v}(x_{n+v-2,}x_{n+n_0})\right.\\
&\quad + p_{b_v}(x_{n+n_0}, x_{m+n_0}) + p_{b_v}(x_{m+n_0}, x_m) - \sum_{i=1}^{v-2} p_{b_v}(x_{n+i}, x_{n+i})\\
&\quad \left.- p_{b_v}(x_{n+n_0}, x_{n+n_0}) - p_{b_v}\left(x_{m+n_0}, x_{m+n_0}\right)\right]\\
&\quad + \frac{(1-s)}{2}\left[p_{b_v}(x_n, x_n) + p_{b_v}(x_m, x_m)\right]\\
&\leq s\left[p_{b_v}(x_n, x_{n+1}) + p_{b_v}(x_{n+1,}x_{n+2}) + ... + p_{b_v}(x_{n+v-3,}x_{n+v-2}) + \right.\\
&\quad \left. p_{b_v}(x_{n+v-2,}x_{n+n_0}) + p_{b_v}(x_{n+n_0}, x_{m+n_0}) + p_{b_v}(x_{m+n_0}, x_m)\right]\\
&= s\left[\left(\frac{1}{s^n} p_{b_v}(x_0, x_1) + \frac{1}{s^{n+1}} p_{b_v}(x_0, x_1) + ... + \frac{1}{s^{n+v-3}} p_{b_v}(x_0, x_1)\right)\right.\\
&\quad \left. + \frac{1}{s^n} p_{b_v}(x_{v-2}, x_{n_0}) + \frac{1}{s^{n_0}} p_{b_v}(x_n, x_m) + \frac{1}{s^m} d_p(x_0, x_{n_0})\right]\\
&= \left(\frac{1}{s^{n-1}} + \frac{1}{s^n} + ... + \frac{1}{s^{n+v-4}}\right) p_{b_v}(x_0, x_1) + \frac{1}{s^{n-1}} p_{b_v}(x_{v-2}, x_{n_0})\\
&\quad + \frac{1}{s^{n_0-1}} d_p(x_n, x_m) + \frac{1}{s^{m-1}} d_p(x_0, x_{n_0}).
\end{aligned}$$

So, we get

$$\begin{aligned}
\left(1 - \frac{1}{s^{n_0-1}}\right) p_{b_v}(x_n, x_m) &\leq \frac{1}{s^{n-1}} \left(\frac{1 - \frac{1}{s^{v-2}}}{1 - \frac{1}{s}}\right) p_{b_v}(x_0, x_1)\\
&\quad + \frac{1}{s^{n-1}} p_{b_v}(x_{v-2}, x_{n_0}) + \frac{1}{s^{m-1}} p_{b_v}(x_0, x_{n_0}).
\end{aligned}$$

Taking limit as $n, m \to \infty$ in the above inequality, we have

$$\lim_{m,n\to\infty} p_{b_v}(x_n, x_m) = 0.$$

Since the space is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} p_{b_v}(x_n, u) = \lim_{m,n\to\infty} p_{b_v}(x_n, x_m) = p_{b_v}(u, u) = 0.$$

Now, we show that u is a fixed point of T. Since X has a sequential limit comparison property, we know that $x_n \preceq u$. Then, we have

$$\begin{aligned}
p_{b_v}(Tu, u) &\leq s\left[p_{b_v}(Tu, x_{2n+2}) + p_{b_v}(x_{2n+2}, x_{2n+3}) + ... + p_{b_v}(x_{2n+v}, x_{2n+v+1}) + p_{b_v}(x_{2n+v}, u)\right.\\
&\quad \left. - \sum_{i=1}^{v} p_{b_v}(u_{2n+i}, u_{2n+i})\right] + \frac{(1-s)}{2}\left[p_{b_v}(Tu, Tu) + p_{b_v}(u, u)\right]\\
&\leq s\left[p_{b_v}(Tu, x_{2n+2}) + p_{b_v}(x_{2n+2}, x_{2n+3}) + ... + p_{b_v}(x_{2n+v}, x_{2n+v+1}) + p_{b_v}(x_{2n+v}, u)\right].
\end{aligned}$$

By taking limit as $n \to \infty$ from both side, we have

$$\begin{aligned}
p_{b_v}(Tu, u) &\leq s \lim_{n\to\infty} p_{b_v}(Tu, x_{2n+2}) = s \lim_{n\to\infty} p_{b_v}(Tu, Sx_{2n+1})\\
&\leq \lim_{n\to\infty} \beta(M(u, x_{2n+1}))\, M(u, x_{2n+1}) + L \lim_{n\to\infty} N(u, x_{2n+1}), \qquad (3.3)
\end{aligned}$$

where

$$\begin{aligned}\lim_{n\to\infty} M\left(u,x_{2n+1}\right) &= \lim_{n\to\infty}\max\left\{p_{b_v}\left(u,x_{2n+1}\right),\frac{p_{b_v}\left(u,Tu\right)p_{b_v}\left(x_{2n+1},Sx_{2n+1}\right)}{1+p\left(Tu,Sx_{2n+1}\right)}\right\}\\ &= \lim_{n\to\infty}\max\left\{p_{b_v}\left(u,x_{2n+1}\right),\frac{p_{b_v}\left(u,Tu\right)p_{b_v}\left(x_{2n+1},x_{2n+2}\right)}{1+p_{b_v}\left(Tu,x_{2n+2}\right)}\right\}\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} N\left(u,x_{2n+1}\right) &= \lim_{n\to\infty}\min\left\{d\left(u,x_{2n+1}\right),d\left(u,Tu\right),d\left(x_{2n+1},Sx_{2n+1}\right),d\left(x_{2n+1},Tu\right),d\left(u,Sx_{2n+1}\right)\right\}\\ &= \lim_{n\to\infty}\min\left\{d\left(u,x_{2n+1}\right),d\left(u,Tu\right),d\left(x_{2n+1},x_{2n+2}\right),d\left(x_{2n+1},Tu\right),d\left(u,x_{2n+2}\right)\right\}\\ &= 0.\end{aligned}$$

So, we get $p_{b_v}\left(Tu,u\right)\leq 0$ which shows that $Tu=u$. From (3.1), we obtain

$$sp_{b_v}\left(u,Su\right)=sp_{b_v}\left(Tu,Su\right)\leq\beta\left(M\left(u,u\right)\right)M\left(u,u\right)+LN\left(u,u\right),$$

where

$$\begin{aligned}M\left(u,u\right) &= \max\left\{p_{b_v}\left(u,u\right),\frac{p_{b_v}\left(u,Tu\right)p_{b_v}\left(u,Su\right)}{1+p_{b_v}\left(Tu,Su\right)}\right\}\\ &= 0\end{aligned}$$

and

$$\begin{aligned}N\left(u,u\right) &= \min\left\{d\left(u,u\right),d\left(u,Tu\right),d\left(u,Su\right),d\left(u,Tu\right),d\left(u,Su\right)\right\}\\ &= \min\left\{0,0,d\left(u,Su\right),0,d\left(u,Su\right)\right\}\\ &= 0.\end{aligned}$$

Hence, we have

$$sp_{b_v}\left(u,Su\right)\leq 0.$$

Therefore, we obtain $Su=u$. Consequently, u is a common fixed point of T and S . Finally, we will prove the uniqueness of the common fixed point. Let $v=Tv=Sv$ be another comparable common fixed point for T and S. Then, it follows from (3.1) that

$$sp_{b_v}\left(u,v\right)=sp_{b_v}\left(Tu,Su\right)\leq\beta\left(M\left(u,v\right)\right)M\left(u,v\right)+LN\left(u,v\right)$$

where

$$\begin{aligned}M\left(u,v\right) &= \max\left\{p_{b_v}\left(u,v\right),\frac{p_{b_v}\left(u,Tu\right)p_{b_v}\left(v,Sv\right)}{1+p_{b_v}\left(Tu,Sv\right)}\right\}\\ &= p_{b_v}\left(u,v\right)\end{aligned}$$

and

$$\begin{aligned}N\left(u,v\right) &= \min\left\{d\left(u,v\right),d\left(u,Tu\right),d\left(v,Sv\right),d\left(u,Tu\right),d\left(u,Su\right)\right\}\\ &= \min\left\{d\left(u,v\right),0,0,0,0\right\}\\ &= 0.\end{aligned}$$

So, we get

$$\begin{aligned} sp_{b_v}(u,v) &\leq \beta\left(p_{b_v}(u,v)\right)p_{b_v}(u,v) \\ &< \frac{1}{s}p_{b_v}(u,v) \end{aligned}$$

which is a contradiction. Therefore, $u = v$, that is T and S have a unique common fixed point. □

Open Problem: In the main results section, we proved some theorems in partial $b_v(s)$-metric spaces equipped with partial order relations. On the other hand, nowadays, some papers have been published about orthogonal metric spaces and R-metric spaces, for detail please see [8, 9, 10, 11, 14, 15]. So, it is an open problem whether it is possible to prove our main results in orthogonal and R-metric spaces.

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