Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 115-133
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.20860.2207

A new method for solving three-dimensional nonlinear Fredholm integral equations by Haar wavelet

Manochehr Kazemi[a,*], Vali Torkashvand[b], Reza Ezzati[c]

[a]*Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran*
[b]*Department of Mathematics, Farhangian University, Tehran, Iran. Member of Young Researchers and Elite club Shahr-e-Qods, Branch Islamic Azad University, Tehran, Iran*
[c]*Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran*

(Communicated by Javad Damirchi)

Abstract

In this paper, a new iterative method of successive approximations based on Haar wavelets is proposed for solving three-dimensional nonlinear Fredholm integral equations. The convergence of the method is verified. The error estimation and numerical stability of the proposed method are provided in terms of Lipschitz condition. Conducting numerical experiments confirm the theoretical results of the proposed method and endorse the accuracy of the method.

Keywords: Three-dimensional integral equations; Three-dimensional Haar wavelet ; Lipschitz condition; Successive approximations.
2010 MSC: 47H09, 47H10.

1. Introduction

In this research, the three-dimensional (3-D) Haar wavelets constructed on $I = [a, b] \times [c, d] \times [e, h]$ are applied to solve the 3-D nonlinear Fredholm integral equations of the second kind (3D-NFIEs):

$$U(s,t,r) - \lambda \int_e^h \int_c^d \int_a^b H(s,t,r,x,y,z)\varphi(U(x,y,z))dxdydz = f(s,t,r), \quad (s,t,r) \in I, \tag{1.1}$$

*Corresponding author
Email addresses: univer_ka@yahoo.com, m.kazemi@aiau.ac.ir (Manochehr Kazemi) (Manochehr Kazemi), vali_torkashvand@yahoo.com (Vali Torkashvand), reza_ezati@yahoo.com (Reza Ezzati)

Received: July 2020 *Accepted:* September 2020

where $U(s,t,r)$ is an unknown function on I and , $f(s,t,r)$, $H(s,t,r,x,y,z)$ are known functions. The multidimensional integral equations provide important tools for modeling a wide range of phenomena and processes as well as solving boundary problems for differential equations [40, 38, 35, 17, 18, 36, 24]. Analytical solutions of integral equations with two or more variables are not available or difficult to acquire, especially in the nonlinear cases. Therefore, in many cases, it is necessary to use numerical methods to approximate the solution. The Galerkin and collocation methods are the two commonly numerical methods applied for solving integral equations [23, 41, 16, 8]. Numerical solutions of three-dimensional linear and nonlinear integral equations have been introduced, including differential tranform methods (DTM)[11, 42], block-pulse functions (BPFs)[35], degenerate kernel method [12], triangular functions (TFs)[37], Jacobi polynomials[39], Chebyshev polynomials [34, 21], Bernstein's polynomials [29, 33], Legendre Gauss-Lobatto collocation (L- GL- C) [1], radial basis functions (RBFs) [20],wavelet method [9], operational matrices [6], neural network method [5], Adomian decomposition method [4] and successive approximations methods [13, 15, 26]. The theorems on the existence and uniqueness of the solution for the multidimentional integral equations can be found in [2, 38, 3, 27, 28, 32].
The Haar wavelet is one of the simple and popular wavelets which its function was introduced by Alfred Haar in 1910 [22] and then developed by other researchers. The wavelet methods are efficient in providing tools for solving mathematical problems including differential and integral equations [30].
In this research, an iterative method based on successive approximations employing the Haar wavelet methods is presented for generating a numerical solution for solving Eq. 1.1. This approach is proposed to solve the integral equations, in contrast with the current available numerical methods which are generally ended up with linear systems and might have the singularity problem. Some reseachers, [25, 10], applied uniform Haar wavelets for integration of triple real integrals. Here, we aim to extend the proposed numerical method to solve (3D-NFIEs). The presented research, in this paper is new and can be more efficient than current suggested methods proposed by the authors of [11, 42, 35, 12, 37, 39, 1, 33, 20, 19].
The organization of the paper is as follows: Approximating of any three variable function $f(x,y,z)$ by 3-D Haar wavelet and quadrature formula for triple integral by Haar wavelet are described in Section 2. In Section 3, a sequence of successive approximations is introduced by using the explained quadrature formula. Also convergence of suggested iterative method is analyzed and numerical stability of method is studied by considering the small change in the first iteration. In Section 4, the convergence and stability of the proposed method is numerically confirmed. Finally, conclusions are provided in Section 5.

2. Haar wavelet method

Definition 2.1. *[14] The Haar scaling function, so-called as the father wavelet, is defined on the interval* $[a,b)$ *as follows*

$$\phi(x) = \begin{cases} 1 & , a \leqslant x < b, \\ 0 & , otherwise. \end{cases}$$

Definition 2.2. *[14] The mother wavelet for the Haar wavelets family is also defined on the interval* $[a,b)$ *as follows*

$$\psi(x) = \begin{cases} 1 & , a \leqslant x < \frac{a+b}{2}, \\ -1 & , \frac{a+b}{2} \leqslant x < b, \\ 0 & , otherwise. \end{cases}$$

All the other functions in the Haar wavelets family are defined on subintervals of $[a, b)$ and are generated from $\psi(x)$ by the operations of dilation and translation. Each function in the Haar wavelets family defined for $x \in [a, b)$ except the scaling function can be expressed as

$$h_i(x) = \Psi(2^j - k) = \begin{cases} 1 & , \alpha \leqslant x < \beta, \\ -1 & , \beta \leqslant x < \gamma, \\ 0 & , otherwise, \end{cases}$$

where

$$\alpha = a + (b-a)\frac{k}{n}, \quad \beta = a + (b-a)\frac{k+0.5}{n}, \quad \gamma = a + (b-a)\frac{k+1}{n}, \quad , i = 2, 3, ..., 2N$$

In the above definition the integer $n = 2^j, \quad j = 0, 1, ..., J$ shows the level of the wavelet and $k = 0, 1, ..., n-1$ is the translation parameter. The maximal level of resolution is the integer J. The wavelet number i is calculated according the formula $i = n + k + 1$. In the case of minimal values $n = 1, k = 0$, we have $i = 2$. The maximum of i is $i = 2N = 2^{J+1}$.
For $i = 1, 2$, the function $h_1(x)$ is called scaling function whereas $h_2(x)$ is the mother wavelet for the Haar wavelet family.

2.1. Three-dimensional Haar wavelet

Consider $(x, y, z) \in I$. We will define $N_1 = 2^{J_1}$, $N_2 = 2^{J_2}$ and $N_3 = 2^{J_3}$ where J_1, J_2 and J_3 are the maximal levels of resolution. Now, divide the interval $[a, b]$, $[c, d]$ and $[e, h]$ respectively into $2N_1, 2N_2$ and $2N_3$ subintervals, each of length $\delta_x = \frac{b-a}{2N_1}$, $\delta_y = \frac{d-c}{2N_2}$ and $\delta_z = \frac{h-e}{2N_3}$ respectively. Similar to the 1D case, a set of 3D Haar wavelets functions $\{h_{i_1,i_2,i_3}(x, y, z) |\ i_1 = 1, 2, .., 2N_1,\ i_2 = 1, 2, .., 2N_2,\ i_3 = 1, 2, .., 2N_3\}$ are defined on the region $x \in [a, b), y \in [c, d)$ and $z \in [e, h)$ as:

$$h_{i_1,i_2,i_3}(x, y, z) = h_{i_1}(x)h_{i_2}(y)h_{i_3}(z) = \Psi(2^{j_1} - k_1)\Psi(2^{j_2} - k_2)\Psi(2^{j_3} - k_3)$$

where

$$h_{i_1}(x) = \begin{cases} 1 & , \xi_1 \leqslant x < \xi_2, \\ -1 & , \xi_2 \leqslant x < \xi_3, \\ 0 & , otherwise, \end{cases}$$

$$h_{i_2}(y) = \begin{cases} 1 & , \zeta_1 \leqslant y < \zeta_2, \\ -1 & , \zeta_2 \leqslant y < \zeta_3, \\ 0 & , otherwise, \end{cases}$$

$$h_{i_3}(z) = \begin{cases} 1 & , \eta_1 \leqslant z < \eta_2, \\ -1 & , \eta_2 \leqslant z < \eta_3, \\ 0 & , otherwise, \end{cases}$$

with

$$\xi_1 = a + 2k_1\frac{N_1}{n_1}\delta_x, \qquad \xi_2 = a + (2k_1 + 1)\frac{N_1}{n_1}\delta_x, \qquad \xi_3 = a + 2(k_1 + 1)\frac{N_1}{n_1}\delta_x$$

$$\zeta_1 = c + 2k_2\frac{N_2}{n_2}\delta_y, \qquad \zeta_2 = c + (2k_2 + 1)\frac{N_2}{n_2}\delta_y, \qquad \zeta_3 = c + 2(k_2 + 1)\frac{N_2}{n_2}\delta_y$$

$$\eta_1 = e + 2k_3\frac{N_3}{n_3}\delta_z, \qquad \eta_2 = e + (2k_3 + 1)\frac{N_3}{n_3}\delta_z, \qquad \eta_3 = e + 2(k_3 + 1)\frac{N_3}{n_3}\delta_z$$

The integers $j_1 = 0, 1, ..., J_1, j_2 = 0, 1, ..., J_2$ and $j_3 = 0, 1, ..., J_3$ show the levels of the wavelet. Therefore, $k_1 = 0, 1, ..., n_1 - 1, k_2 = 0, 1, ..., n_2 - 1$ and $k_3 = 0, 1, ..., n_3 - 1$ are the translation

parameters, where $n_1 = 2^{j_1}, n_2 = 2^{j_2}$ and $n_3 = 2^{j_3}$. The indexes i_1, i_2 and i_3 are determined by $i_1 = n_1 + k_1 + 1, i_2 = n_2 + k_2 + 1$ and $i_3 = n_3 + k_3 + 1$ respectively.
Any function $f(x, y, z)$ defined on $[a, b] \times [c, d] \times [e, h]$ can be expressed in terms of 3-D Haar wavelets as follows

$$f(x,y,z) = \sum_{i_3=1}^{2N_3} \sum_{i_2=1}^{2N_2} \sum_{i_1=1}^{2N_1} a_{i_1 i_2 i_3} h_{i_1}(x) h_{i_2}(y) h_{i_3}(z),$$

where the wavelet coefficients $a_{i_1 i_2 i_3}, i_1 = 1, 2, ..., 2N_1, i_2 = 1, 2, ..., 2N_2, i_3 = 1, 2, ..., 2N_3$ are to be determined.
In this paper, it is assumed $N_1 = N_2 = N_3 = N$, for the Haar wavelets approximations in which the collocation points are as follow

$$(x_i, y_j, z_k), \qquad i, j, k = 1, 2, ...2N, \tag{2.1}$$

where

$$x_i = a + (b-a)\frac{2i-1}{4N}, \qquad y_j = c + (d-c)\frac{2j-1}{4N}, \qquad z_k = e + (h-e)\frac{2k-1}{4N}. \tag{2.2}$$

Definition 2.3. *For $L_1, L_2, L_3 \geq 0$, the function $f : I \to \mathbb{R}$ is L_1, L_2, L_3-Lipschitz if*

$$|f(x_1, y_1, z_1) - f(x_2, y_2, z_2)| \leq L_1|x_1 - x_2| + L_2|y_1 - y_2| + L_3|z_1 - z_2|,$$

$\forall x_1, x_2 \in [a, b], y_1, y_2 \in [c, d]$ and $z_1, z_2 \in [e, h]$.

Theorem 2.4. *Consider the triple integral*

$$\int_e^h \int_c^d \int_a^b f(x,y,z)dxdydz, \tag{2.3}$$

where $f : I \to \mathbb{R}$ is continuous integrable function of L_1, L_2, L_3-Lipschitz type. Using the quadrature formula with respect to Haar wavelets the above triple integral can be approximated as follows:

$$S_N(f) = \frac{(b-a)(d-c)(h-e)}{8N^3} \sum_{k=1}^{2N} \sum_{j=1}^{2N} \sum_{i=1}^{2N} f\left(a + \frac{\delta_x}{2}(2i-1), c + \frac{\delta_y}{2}(2j-1), e + \frac{\delta_z}{2}(2k-1)\right), \tag{2.4}$$

where $N = 2^J$ is the maximal level of resolution of Haar wavelets [25]. Also, for 3D continuous integrable functions of L_1, L_2, L_3-Lipschitz type, the following error estimate is true:

$$|\int_e^h \int_c^d \int_a^b f(x,y,z)dxdydz - S_N(f)| \leq L(b-a)(d-c)(h-e)\big(\delta_x + \delta_y + \delta_z\big), \tag{2.5}$$

where

$$L = \max\{L_1, L_2, L_3\}.$$

Proof .

$$
\begin{aligned}
&\Big|\int_e^h\int_c^d\int_a^b f(x,y,z)dxdydz - S_N(f)\Big|\\
&=\Big|\int_e^h\int_c^d\int_a^b \Big(f(x,y,z)-\frac{1}{8N^3}\sum_{i,j,k=1}^{2N} f(a+\frac{\delta_x}{2}(2i-1),c+\frac{\delta_y}{2}(2j-1),e+\frac{\delta_z}{2}(2k-1))\Big)dxdydz\Big|\\
&\le \frac{1}{8N^3}\int_e^h\int_c^d\int_a^b\sum_{k=1}^{2N}\sum_{j=1}^{2N}\sum_{i=1}^{2N}\Big|f(x,y,z)-f\big(a+\frac{\delta_x}{2}(2i-1),c+\frac{\delta_y}{2}(2j-1),e+\frac{\delta_z}{2}(2k-1)\big)\Big|dxdydz\\
&\le \frac{1}{8N^3}\int_e^h\int_c^d\int_a^b\sum_{k=1}^{2N}\sum_{j=1}^{2N}\sum_{i=1}^{2N}\big(L_1|x-(a+\frac{\delta_x}{2}(2i-1))|+L_2|y-(c+\frac{\delta_y}{2}(2j-1))|\\
&+L_3|z-(e+\frac{\delta_z}{2}(2k-1))|\big)dxdydz
\end{aligned}
$$

According to the given relation

$$(x,y,z)\in[a+\frac{\delta_x}{2}(2i-1),a+\frac{\delta_x}{2}(2i+1))\times[c+\frac{\delta_y}{2}(2j-1),c+\frac{\delta_y}{2}(2j+1))\times[e+\frac{\delta_z}{2}(2k-1),e+\frac{\delta_z}{2}(2k+1))$$

we get

$$\Big|\int_e^h\int_c^d\int_a^b f(x,y,z)dxdydz - S_N(f)\Big|\le L(b-a)(d-c)(h-e)\big(\delta_x+\delta_y+\delta_z\big).$$

Thus, the proof is complete. □

3. Main results

3.1. The sequence of successive approximations

Here, we consider the three-dimensional nonlinear Eq. (1.1), where $\lambda>0$, $H(s,t,r,x,y,z)$ is kernel function on $I\times I$ and U,f are continuous functions and also $\varphi:\mathbb{R}\to\mathbb{R}$ is a continuous function. We assume that H is continuous and therefore it is uniformly continuous with respect to (s,t,r). This attribute mentions implies that there exists $M_H>0$ such that

$$M_H=\max\{\big|H(s,t,r,x,y,z)\big|;s,x\in[a,b],t,y\in[c,d],r,z\in[e,h]\}.$$

Let $\mathbf{X}=\{f:[a,b]\times[c,d]\times[e,h]\to\mathbb{R};f\ \ is\ \ continuous\}$ be the space of three-dimensional continuous functions with the metric

$$d(f,g)=\big\|f-g\big\|=\sup\{\big|f(s,t,r)-g(s,t,r)\big|;s\in[a,b],t\in[c,d],r\in[e,h]\},\tag{3.1}$$

Theorem 3.1. *Let $H(s,t,r,x,y,z)$ be continuous for $s,x\in[a,b]$, $t,y\in[c,d]$, $r,z\in[e,h]$ and $f\in\boldsymbol{X}$. Furthermore, suppose that there is $\rho>0$, such that*

$$|\varphi(\Phi_1(\xi,\eta,\zeta))-\varphi(\Phi_2(\xi,\eta,\zeta))|\le\rho|\Phi_1(\xi,\eta,\zeta)-\Phi_2(\xi,\eta,\zeta)|,\quad\forall(\xi,\eta,\zeta)\in I,\quad\forall\Phi_1,\Phi_2\in\boldsymbol{X}.\tag{3.2}$$

If $\sigma = \rho\lambda M_H(b-a)(d-c)(h-e) < 1$, then Eq. (1.1) has a unique solution $U^* \in \boldsymbol{X}$, which can be accessed by the following successive approximations method

$$U_0(s,t,r) = f(s,t,r),$$
$$U_m(s,t,r) = f(s,t,r) + \lambda \int_e^h \int_c^d \int_a^b H(s,t,r,x,y,z)\varphi(U_{m-1}(x,y,z))dxdydz, \qquad m \geq 1 \tag{3.3}$$

Also, $(U_m)_{m\geq 1}$ converges to U^*. Moreover, the following error estimates hold

$$d(U^*, U_m) \leq \frac{\sigma^m}{1-\sigma} d(U_0, U_1), \tag{3.4}$$

$$d(U^*, U_m) \leq \frac{\sigma}{1-\sigma} d(U_{m-1}, U_m) \tag{3.5}$$

and choosing $U_0 \in \boldsymbol{X}, U_0 = f$, the inequality (3.4) becomes

$$d(U^*, U_m) \leq \frac{\sigma^{m+1}}{\rho(1-\sigma)} M_0 \tag{3.6}$$

where

$$M_0 = \sup\{|\varphi(f(s,t,r))|; s \in [a,b], t \in [c,d], r \in [e,h]\}.$$

Proof . First of all, we define the operators $\Gamma : \mathbf{X} \to \mathbf{X}$ by

$$\Gamma(U)(s,t,r) = f(s,t,r) + \lambda \int_e^h \int_c^d \int_a^b H(s,t,r,x,y,z)\varphi\big(U(x,y,z)\big)dxdydz, \ \forall (s,t,r) \in I, \ \forall U \in \mathbf{X}.$$

We prove that Γ maps $\mathbf{X}$ into $\mathbf{X}$. To this purpose, we see that for all $\varepsilon > 0$ there are $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_1 + M_U\lambda(b-a)(d-c)(h-e)\varepsilon_2 < \varepsilon$. Since f is continuous on compact set of I, we infer that it is uniformly continuous and therefore for $\varepsilon_1 > 0$ exists $\delta_1 > 0$ such that

$$\mid f(s',t',r') - f(s'',t'',r'') \mid < \varepsilon_1 \quad \forall (s',t',r'), (s'',t'',r'') \in I,$$

with $sqrt\big((s'-s'')^2 + (t'-t'')^2 + (r'-r'')^2\big) < \delta_1$.
As mentioned above, H also is uniformly continuous thus, for $\varepsilon_2 > 0$ exists $\delta_2 > 0$ such that

$$|H(s',t',r',x,y,z) - H(s'',t'',r'',x,y,z)| < \varepsilon_2 \quad \forall (s',t',r'), (s'',t'',r'') \in I,$$

with $sqrt\big((s'-s'')^2 + (t'-t'')^2 + (r'-r'')^2\big) < \delta_2$.
Let $\delta = \min\{\delta', \delta''\}$ and $(s',t',r'), (s'',t'',r'') \in I$, with $sqrt\big((s'-s'')^2 + (t'-t'')^2 + (r'-r'')^2\big) < \delta$.

We obtain

$$
\begin{aligned}
& \mid \Gamma(U)(s',t',r') - \Gamma(U)(s'',t'',r'') \mid \leq \mid f(s',t',r') - f(s'',t'',r'') \mid \\
+ & \mid \lambda \int_e^h \int_c^d \int_a^b H(s',t',r',x,y,z)\varphi\big(U(x,y,z)\big)dxdydz \mid \\
- & \lambda \int_e^h \int_c^d \int_a^b H(s'',t'',r'',x,y,z)\varphi\big(U(x,y,z)\big)dxdydz \mid \\
< & \varepsilon_1 + \lambda \int_e^h \int_c^d \int_a^b \mid H(s',t',r',x,y,z)\varphi\big(U(x,y,z)\big) \\
- & H(s'',t'',r'',x,y,z)\varphi\big(U(x,y,z)\big)|dxdydz \\
= & \varepsilon_1 + \lambda \int_e^h \int_c^d \int_a^b |H(s',t',r',x,y,z) - H(s'',t'',r'',x,y,z)|. \mid \varphi\big(U(x,y,z)\big) \mid dxdydz \\
< & \varepsilon_1 + \lambda\varepsilon_2 \int_e^h \int_c^d \int_a^b \sup_{a\leq x\leq b,\ c\leq y\leq d,\ e\leq z\leq h} \mid \varphi\big(U(x,y,z)\big) \mid dxdydz \\
\leq & \varepsilon_1 + M_U\lambda(b-a)(d-c)(h-e)\varepsilon_2 < \varepsilon,
\end{aligned}
$$

where

$$
M_U = \sup_{a\leq x\leq b,\ c\leq y\leq d,\ e\leq z\leq h} \mid \varphi\big(U(x,y,z)\big) \mid,
$$

we derive

$$
\mid \Gamma(U)(s',t',r') - \Gamma(U)(s'',t'',r'') \mid < \varepsilon.
$$

This shows that $\Gamma(U)$ is uniformly continuous for any $U \in \mathbf{X}$, and so continuous on I, and hence Γ maps $\mathbf{X}$ into $\mathbf{X}$. Let set $U_0 \in \mathbf{X}$, and define Picard iterative sequence $U_m = \Gamma(U_{m-1}), m \in \mathbb{N}$. We show that the operator Γ is, for any $f \in \mathbf{X}$, a contraction with respect to the norm (3.1). So, for $U, G \in \mathbf{X}$ and $(s,t,r) \in I$, we have

$$
\begin{aligned}
& |\Gamma(U)(s,t,r) - \Gamma(G)(s,t,r)| \\
\leq & \Big|\lambda \int_e^h \int_c^d \int_a^b H(s,t,r,x,y,z)\varphi\big(U(x,y,z)\big)dxdydz \\
- & \lambda \int_e^h \int_c^d \int_a^b H(s,t,r,x,y,z)\varphi\big(G(x,y,z)\big)dxdydz\Big| \\
\leq & \lambda \int_e^h \int_c^d \int_a^b \mid H(s,t,r,x,y,z)\varphi\big(U(x,y,z)\big) \\
- & H(s,t,r,x,y,z)\varphi\big(G(x,y,z)\big) \mid dxdydz \\
= & \lambda \int_e^h \int_c^d \int_a^b \mid H(s,t,r,x,y,z) \mid\mid \varphi\big(U(x,y,z)\big) - \varphi\big(G(x,y,z)\big) \mid dxdy \\
\leq & \lambda M_H \int_e^h \int_c^d \int_a^b \mid \varphi\big(U(x,y,z)\big) - \varphi\big(G(x,y,z)\big) \mid dxdydz \\
\leq & \rho\lambda M_H(b-a)(d-c)(h-e) \parallel U - G \parallel .
\end{aligned}
$$

Consequently

$$
\parallel \Gamma(U) - \Gamma(G) \parallel \leq \sigma \parallel U - G \parallel .
$$

In view of the Banach fixed point theorem and crucial condition $\sigma < 1$, we infer that integral equation Eq. (1.1) has an unique solution U^* in $\mathbf{X}$. Also, the same Banach's fixed point principle leads to the estimates (3.4) and (3.5). Choosing $U_0 = f$, we have

$$\begin{aligned}
& \parallel U_0 - U_1 \parallel \\
= & \sup_{\substack{a\le s\le b\\ c\le t\le d\\ e\le r\le h}} \mid f(s,t,r) - f(s,t,r) - \lambda \int_e^h \int_c^d \int_a^b H(s,t,r,x,y,z)\varphi\big(U_0(x,y,z)dxdydz \mid \\
\le & \sup_{\substack{a\le s\le b\\ c\le t\le d\\ e\le r\le h}} \lambda \int_e^h \int_c^d \int_a^b \mid H(s,t,r,x,y,z)\varphi\big(U_0(x,y,z)\big) \mid dxdydz \\
\le & M_H \lambda \int_e^h \int_c^d \int_a^b \sup_{a\le x\le b, c\le y\le d, e\le z\le h} \mid \varphi\big(f(x,y,z)\big) \mid dxdydz \\
= & \lambda M_H (b-a)(d-c)(h-e)M_0.
\end{aligned}$$

In this way we obtain the inequality (3.6), which completes the proof.
□ Now, we propose the numerical method to solve (1.1). We consider a uniform partition $D = (D_x, D_y, D_z)$ of $[a,b]\times[c,d]\times[e,h]$ with

$$\begin{aligned}
& D_x : a = s_0 < s_1 < s_2 < ... < s_{2n-1} < s_{2N} = b, \\
& D_y : c = t_0 < t_1 < t_2 < ... < t_{2n-1} < t_{2N} = d, \\
& D_z : e = r_0 < r_1 < r_2 < ... < r_{2n-1} < r_{2N} = h,
\end{aligned}$$

and also $s_i = a + i\delta_x$, $t_j = c + j\delta_y$, $r_k = e + k\delta_z$,where $\delta_x = \frac{b-a}{2N}$, $\delta_y = \frac{d-c}{2N}$, $\delta_z = \frac{h-e}{2N}$, $i, j, k = \overline{0, 2N}$.

Applying the quadrature rule (2.4) and (2.5) to approximate of the integrals in (3.3) we obtain,

$$\begin{aligned}
& \overline{U}_0(s,t,r) = f(s,t,r) \\
& \overline{U}_m(s,t,r) = f(s,t,r) + \delta_x\delta_y\delta_z \sum_{k=1}^{2N}\sum_{j=1}^{2N}\sum_{i=1}^{2N} H\bigg(s,t,r,a+\frac{\delta_x}{2}(2i-1), c+\frac{\delta_y}{2}(2j-1), e+\frac{\delta_z}{2}(2k-1)\bigg) \\
& \qquad \varphi(\overline{U}_{m-1}(a+\frac{\delta_x}{2}(2i-1), c+\frac{\delta_y}{2}(2j-1), e+\frac{\delta_z}{2}(2k-1))). \qquad (3.7)
\end{aligned}$$

3.2. Convergence Analysis

Here, we examine the convergence of the suggested iterative method to obtain the numerical solution of equation (1.1) under the following conditions:

(i) there is $\beta > 0$ such that
$\mid f(s,t,r) - f(s',t',r') \mid \leqslant \beta\big(|(s-s'| + |t-t'| + |r-r'|\big), \quad \forall (s,t,r), (s',t',r') \in I,$

(ii) there exist $\mu, \nu > 0$ such that
$\mid H(s,t,r,x,y,z) - H(s,t,r,x',y',z') \mid \leqslant \mu\big(|x-x'| + |y-y'| + |z-z'|\big)$
$\mid H(s,t,r,x,y,z) - H(s',t',r',x,y,z) \mid \leqslant \nu\big(|(s-s'| + |t-t'| + |r-r'|\big)$
$\forall (s,t,r), (s',t',r'), (x,y,z), (x',y',z') \in I$

(iii) $\mid \varphi(u) - \varphi(v) \mid < \alpha \mid u - v \mid, \forall u, v \in \mathbb{R}.$

Lemma 3.2. *Consider the iterative procedure 3.3. Under all assumptions of Theorem 3.1 and the conditions (i)-(iii), the functions $H(s,t,,r,x,y,z)\varphi(U_m(x,y,z))$ are Lipschitzian.*

Proof . Using the conditions (ii) and (iii) we obtain

$$\begin{aligned}
& \mid H(s,t,r,x,y,z)\varphi(U_m(x,y,z)) - H(s,t,r,x',y',z')\varphi(U_m(x',y',z')) \mid \\
\leq & \mid H(s,t,r,x,y,z)\varphi(U_m(x,y,z)) - H(s,t,r,x,y,z)\varphi(U_m(x',y',z')) \mid \\
+ & \mid H(s,t,r,x,y,z)\varphi(U_m(x',y',z')) - H(s,t,r,x',y',z')\varphi(U_m(x',y',z')) \mid \\
= & \mid H(s,t,r,x,y,z) \mid\mid \varphi(U_m(x,y,z)) - \varphi(U_m(x',y',z')) \mid \\
+ & \mid H(s,t,r,x,y,z) - H(s,t,r,x',y',z') \mid\mid \varphi(U_m(x',y',z')) \mid \\
\leq & M_H \mid \varphi(U_m(x,y,z)) - \varphi(U_m(x',y',z')) \mid + B_m\mu\big(|x-x'| + |y-y'| + |z-z'|\big) \\
\leq & M_H\alpha \mid U_m(x,y,z) - U_m(x',y',z') \mid + \mu M\big(|x-x'| + |y-y'| + |z-z'|\big),
\end{aligned}$$

where

$$B_k = \sup\{\mid \varphi\big(U_k(s,t,r)\big) \mid, a \leq s \leq b, c \leq t \leq d, e \leq r \leq h\}, \quad M = \max_{i=1,2,\ldots,m}\{B_i\},$$

and $\forall m \geq 1$

$$\begin{aligned}
& \mid U_m(x,y,z) - U_m(x',y',z') \mid \\
\leq & \beta\big(|x-x'| + |y-y'| + |z-z'|\big) + \lambda(b-a)(d-c)(h-e)B_{m-1}\nu\big(|x-x'| + |y-y'| + |z-z'|\big) \\
\leq & (\beta + \lambda(b-a)(d-c)(h-e)M\nu)\big(|x-x'| + |y-y'| + |z-z'|\big).
\end{aligned}$$

Then, for any $x,\ x' \in [a,b],\ \ y,\ y' \in [c,d],\ \ z,\ z' \in [e,h]$ we have

$$\begin{aligned}
& \mid H(s,t,r,x,y,z)\varphi(U_m(x,y,z)) - H(s,t,r,x',y',z')\varphi(U_m(x',y',z')) \mid \\
\leq & \big(M_H\alpha(\beta + \lambda(b-a)(d-c)(h-e)M\nu) + M\mu\big)\big(|x-x'| + |y-y'| + |z-z'|\big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \mid H(s,t,r,x,y,z)\varphi(U_0(x,y,z)) - H(s,t,r,x',y',z')\varphi(U_0(x',y',z')) \mid \\
\leq & M_H\alpha \mid f(x,y,z) - f(x',y',z') \mid + M_0\mu\big(|x-x'| + |y-y'| + |z-z'|\big) \\
\leq & M_H\alpha\beta\big(|x-x'| + |y-y'| + |z-z'|\big) + M_0\mu\big(|x-x'| + |y-y'| + |z-z'|\big) \\
\leq & (M_H\alpha\beta + M_0\mu)\big(|x-x'| + |y-y'| + |z-z'|\big).
\end{aligned}$$

Supposing

$$L = \max\{\alpha M_H(\beta + \lambda(b-a)(d-c)\nu M) + \mu M, M_H\alpha\beta + M_0\mu\},$$

we have

$$\begin{aligned}
\mid H(s,t,r,x,y,z)\varphi(U_m(x,y,z)) - H(s,t,r,x',y',z')\varphi(U_m(x',y',z')) \mid \leq \\
L\big(|x-x'| + |y-y'| + |z-z'|\big)
\end{aligned}$$

Thus, the function $H(s,t,,r,x,y,z)\varphi(U_m(x,y,z))$ for all m are Lipschitzian. □

Theorem 3.3. *Consider the 3D-NFIEs (1.1) with the hypotheses of Theorem 3.1. If $\sigma < 1$, then the iterative procedure (3.7) converges to the unique solution of Eq. (1.1), U^*, and its error estimate is as follows*

$$d(U^*, \overline{U}_m) \leq \frac{\sigma^{m+1}}{\rho(1-\sigma)} M_0 + \frac{(\delta_x + \delta_y + \delta_z)LP}{(1-\sigma)}$$

where

$$L = \max\{\alpha M_H(\beta + \lambda(b-a)(d-c)\nu M) + \mu M, M_H \alpha\beta + M_0 \mu\},$$

and

$$P = (b-a)(d-c)(h-e). \tag{3.8}$$

Proof. Using (3.6) we have

$$d(U^*, \overline{U}_m) \leq d(U^*, U_m) + d(U_m, \overline{U}_m) \leq \frac{\sigma^{m+1}}{\rho(1-\sigma)} M_0 + d(U_m, \overline{U}_m), \tag{3.9}$$

therefore, we shall to obtain the estimates for $\mid U_m(s,t,r) - \overline{U}_m(s,t,r) \mid$. From (2.5) and (3.3) we get

$$\begin{aligned} U_0(s,t,r) &= f(s,t,r) \\ U_m(s,t,r) &= f(s,t,r) + \delta_x\delta_y\delta_z \sum_{k=1}^{2N}\sum_{j=1}^{2N}\sum_{i=1}^{2N} H\big(s,t,r,a+\frac{\delta_x}{2}(2i-1), c+\frac{\delta_y}{2}(2j-1), e+\frac{\delta_z}{2}(2k-1)\big) \\ &\quad \varphi(U_{m-1}(a+\frac{\delta_x}{2}(2i-1), c+\frac{\delta_y}{2}(2j-1), e+\frac{\delta_z}{2}(2k-1))) + E_m(s,t,r). \end{aligned} \tag{3.10}$$

with

$$\mid E_m(s,t,r) \mid \leq L(b-a)(d-c)(h-e)\big(\delta_x + \delta_y + \delta_z\big) = (\delta_x + \delta_y + \delta_z)LP. \tag{3.11}$$

Form (3.7),(3.10) and (3.11), for $m = 1$, we obtain

$$\mid U_1(s,t,r) - \overline{U}_1(s,t,r) \mid \leq \mid E_1(s,t,r) \mid \leq (\delta_x + \delta_y + \delta_z)LP \tag{3.12}$$

Using (3.7) and (3.10) we obtain

$$\begin{aligned} &\big|U_m(s,t,r) - \overline{U}_m(s,t,r)\big| \leq \\ &\big|E_m(s,t,r)\big| + \delta_x\delta_y\delta_z \sum_{k=1}^{2N}\sum_{j=1}^{2N}\sum_{i=1}^{2N} \rho\big|H(s,t,r,a+\frac{\delta_x}{2}(2i-1), c+\frac{\delta_y}{2}(2j-1), e+\frac{\delta_z}{2}(2k-1))\big| \\ &\big|U_{m-1}(a+\frac{\delta_x}{2}(2i-1), c+\frac{\delta_y}{2}(2j-1), e+\frac{\delta_z}{2}(2k-1)) \\ &-\overline{U}_{m-1}\big(a+\frac{\delta_x}{2}(2i-1), c+\frac{\delta_y}{2}(2j-1), e+\frac{\delta_z}{2}(2k-1)\big)\big| \end{aligned} \tag{3.13}$$

Now, from (3.12) and (3.13) for $m = 2$ it follows that

$$\begin{aligned}
& \mid U_2(s,t,r) - \overline{U}_2(s,t,r) \mid \\
\leq\ & (\delta_x + \delta_y + \delta_z)LP \\
+\ & \lambda\rho M_H \delta_x\delta_y\delta_z \sum_{k=1}^{2N}\sum_{j=1}^{2N}\sum_{i=1}^{2N} \mid U_1\big(a + \frac{\delta_x}{2}(2i-1), c + \frac{\delta_y}{2}(2j-1), e + \frac{\delta_z}{2}(2k-1)\big) \\
-\ & \overline{U}_1\big(a + \frac{\delta_x}{2}(2i-1), c + \frac{\delta_y}{2}(2j-1), e + \frac{\delta_z}{2}(2k-1)\big) \mid \\
\leq\ & (\delta_x + \delta_y + \delta_z)LP + \rho\lambda M_H \delta_x\delta_y\delta_z \sum_{k=1}^{2N}\sum_{j=1}^{2N}\sum_{i=1}^{2N} (\delta_x + \delta_y + \delta_z)LP \\
=\ & \big(1 + \rho\lambda M_H (b-a)(d-c)(h-e)\big)(\delta_x + \delta_y + \delta_z)LP.
\end{aligned}$$

By induction, for $m \in N$, $m \geq 3$, we obtain

$$\begin{aligned}
& \mid U_m(s,t,r) - \overline{U}_m(s,t,r) \mid \\
& \leq [1 + \lambda\rho M_H(b-a)(d-c)(h-e) + ... + (\lambda\rho M_H(b-a)(d-c)(h-e))^{m-1}](\delta_x + \delta_y + \delta_z)LP \\
& = \frac{1 - \big(\lambda\rho M_H(b-a)(d-c)(h-e)\big)^m}{1 - \lambda\rho M_H(b-a)(d-c)(h-e)}(\delta_x + \delta_y + \delta_z)LP \\
& \leq \frac{(\delta_x + \delta_y + \delta_z)LP}{1 - \lambda\rho M_H(b-a)(d-c)(h-e)} = \frac{(\delta_x + \delta_y + \delta_z)LP}{(1-\sigma)}.
\end{aligned} \tag{3.14}$$

Hence, from (3.9), (3.12 and (3.14) we conclude that

$$d(U^*, \overline{U}_m) \leq \frac{\sigma^{m+1}}{\rho(1-\sigma)}M_0 + \frac{(\delta_x + \delta_y + \delta_z)LP}{(1-\sigma)}$$

Remark 9. Since $\sigma < 1$, it is easy to see that

$$\lim_{\substack{m\to\infty \\ \delta_x,\delta_y,\delta_z\to 0}} d(U^*, \overline{U}_m) = 0.$$

3.3. The numerical stability analysis

With the purpose of studying the numerical stability of the iterative method (3.7), considering the small change in the first iteration, another first iteration term $V_0(s,t,r) = g(s,t,r) \in C(I,R)$ is considered in such a way that there exists $\varepsilon > 0$ for which $\left|V_0(s,t,r) - U_0(s,t,r)\right| < \varepsilon, \forall s,t,r \in I$. The new sequence of successive approximations is:

$$V_m(s,t,r) = g(s,t,r) + \lambda \int_e^h \int_c^d \int_a^b H(s,t,r,x,y,z)\varphi(V_{m-1}(x,y,z))dxdydz, \quad m \geq 1. \tag{3.15}$$

Applying the same numerical method (2.4) to solve (1.1) we have

$$\begin{aligned}
\overline{V}_0(s,t,r) &= V_0(s,t,r) \\
\overline{V}_m(s,t,r) &= g(s,t,r) + \delta_x\delta_y\delta_z \sum_{k=1}^{2N}\sum_{j=1}^{2N}\sum_{i=1}^{2N} H\left(s,t,r,a + \frac{\delta_x}{2}(2i-1), c + \frac{\delta_y}{2}(2j-1), e + \frac{\delta_z}{2}(2k-1)\right) \\
& \varphi(\overline{V}_{m-1}(a + \frac{\delta_x}{2}(2i-1), c + \frac{\delta_y}{2}(2j-1), e + \frac{\delta_z}{2}(2k-1))).
\end{aligned} \tag{3.16}$$

Theorem 3.4. *Let the conditions of Theorem 3.3 are fulfilled. Then the iterative approach (3.7) is numerically stable with respect to the selection of the first iteration.*

Proof . We reintroduce the proof of Theorem (3.3) and we obtain,

$$\mid V_m(s,t,r) - \overline{V}_m(s,t,r) \mid \leq \frac{(\delta_x+\delta_y+\delta_z)L^{'}P}{(1-\sigma)}. \tag{3.17}$$

where

$$L^{'} = max\{\alpha M_H(\beta + \lambda M^{'}(b-a)(d-c)\nu) + M^{'}\mu, M_H\alpha\beta + M_0^{'}\mu\},$$

and

$$B_k^{'} = \sup\{\mid \varphi\big(V_k(s,t,r)\big) \mid, a \leq s \leq b, c \leq t \leq d, e \leq r \leq h\}, \quad M^{'} = \max_{i=1,2,\dots,m}\{B_i^{'}\},$$
$$M_0^{'} = \sup\{\mid \varphi\big(V_0(s,t,r)\big) \mid, a \leq s \leq b, c \leq t \leq d, e \leq r \leq h\},$$

we have

$$\begin{aligned}
\mid \overline{U}_m(s,t,r) - \overline{V}_m(s,t,r) \mid &\leqslant \mid \overline{U}_m(s,t,r) - U_m(s,t,r) \mid + \mid U_m(s,t,r) - V_m(s,t) \mid \\
&+ \mid V_m(s,t) - \overline{V}_m(s,t) \mid \\
&\leqslant \mid U_m(s,t) - V_m(s,t) \mid + \frac{(\delta_x+\delta_y+\delta_z)LP}{(1-\sigma)} + \frac{(\delta_x+\delta_y+\delta_z)L^{'}P}{(1-\sigma)}.
\end{aligned}$$

We have

$$\mid U_0(s,t,r) - V_0(s,t,r) \mid < \varepsilon, \qquad \forall (s,t,r) \in D,$$

and

$$\begin{aligned}
\big|U_1(s,t,r) - V_1(s,t,r)\big| &\leq \Big|U_0(s,t,r) + \lambda \int_e^h \int_c^d \int_a^b H(s,t,r,x,y,z)\varphi\big(U_0(x,y,z)\big)dxdydz \\
&\quad - V_0(s,t,r) - \lambda \int_e^h \int_c^d \int_a^b H(s,t,r,x,y,z)\varphi\big(V_0(x,y,z)\big)dxdydz\Big| \\
&< \varepsilon + \rho\lambda M_H \int_e^h \int_c^d \int_a^b \mid U_0(s,t,r) - V_0(s,t,r) \mid dxdy \\
&< (1 + \rho\lambda M_H(b-a)(d-c)(h-e))\varepsilon = (1+\sigma)\varepsilon,
\end{aligned}$$

for $m \geq 2$, by induction, we have

$$\begin{aligned}
\Big|U_m(s,t,r) - V_m(s,t,r)\Big| &\leq \big|U_0(s,t,r) - V_0(s,t,r)\big| \\
&+ \lambda \int_e^h \int_c^d \int_a^b \big|H(s,t,r,x,y,z)\varphi\big(U_{m-1}(x,y,z)\big) \\
&- H(s,t,r,x,y,z)\varphi\big(V_{m-1}(x,y,z)\big)\big| dxdydz \\
&< \varepsilon + \rho\lambda M_H \int_e^h \int_c^d \int_a^b \mid U_{m-1}(s,t,r) - V_{m-1}(s,t,r) \mid dxdy \\
&< (1 + \sigma + ... + \sigma^m)\varepsilon,
\end{aligned}$$

for all $(s,t,r) \in I$ and $m \geq 0$. Then,

$$d(U_m(s,t,r), V_m(s,t,r)) < \frac{1}{1-\sigma}\varepsilon,$$

Now, we get,

$$\mid \overline{U}_m(s,t) - \overline{V}_m(s,t) \mid < \frac{1}{1-\sigma}\varepsilon + \frac{(\delta_x + \delta_y + \delta_z)(L + L^{'})P}{1-\sigma}$$

□

Remark 3.5. *Since $\sigma < 1$, it is easy to see that*

$$\lim_{\delta_x, \delta_y, \delta_z \varepsilon \to 0} d(\overline{U}_m, \overline{V}_m) = 0.$$

this shows the stability of the method.

Remark 3.6. *The posteriori error estimate is fruitful to get the stopage criterion.*
For given $\varepsilon^{'} > 0$ (previously chosen) there is determined the first natural number m for which

$$\mid \overline{U}_m(s,t,r) - \overline{U}_{m-1}(s,t,r) \mid < \varepsilon^{'}$$

and we stop to this m retaining the approximations $\overline{U}_m(s,t,r)$ of the solution. We can give a short proof of this criterion as follows:

$$\begin{aligned}
d(U^*, \overline{U}_m) &\leq d(U^*, U_m) + d(U_m, \overline{U}_m) \\
&\leq \frac{\sigma}{1-\sigma} d(U_m, U_{m-1}) + \frac{(\delta_x + \delta_y + \delta_z)LP}{(1-\sigma)}
\end{aligned}$$

and

$$\begin{aligned}
d(U_m, U_{m-1}) &\leq d(U_m - \overline{U}_m) + d(\overline{U}_m, \overline{U}_{m-1}) + d(\overline{U}_{m-1}, U_{m-1}) \\
&\leq \frac{2(\delta_x + \delta_y + \delta_z)LP}{(1-\sigma)} + d(\overline{U}_m, \overline{U}_{m-1})
\end{aligned}$$

So,

$$d(U^*,\overline{U}_m) \le \frac{\sigma}{1-\sigma} d(\overline{U}_m,\overline{U}_{m-1}) + \frac{2\sigma}{1-\sigma}\frac{(\delta_x+\delta_y+\delta_z)LP}{(1-\sigma)} + \frac{(\delta_x+\delta_y+\delta_z)LP}{(1-\sigma)}$$

and therefore, in order to obtain $\mid U^*(s,t,r) - \overline{U}_m(s,t,r) \mid < \varepsilon$ *we require*

$$\frac{\sigma+1}{(1-\sigma)^2}(\delta_x+\delta_y+\delta_z)LP < \frac{1}{2}\varepsilon \tag{3.18}$$

and

$$\frac{\sigma}{1-\sigma} d(\overline{U}_m,\overline{U}_{m-1}) < \frac{\varepsilon}{2}$$

We can select the least natural number $\mathbb{N}$*, for which inequality (3.18) is kept. Finally, we find the lowest natural number* $m \in \mathbb{N}$ *for which,*

$$d(\overline{U}_m - \overline{U}_{m-1}) < \frac{\varepsilon}{2}.\frac{1-\sigma}{\sigma} = \varepsilon^{'}.$$

With these, the inequality $\mid \overline{U}_m(s,t,r) - \overline{U}_{m-1}(s,t,r) \mid < \varepsilon^{'}$ *ended up with* $\mid U^*(s,t,r) - \overline{U}_m(s,t,r) \mid < \varepsilon$*, and the desired accuracy* ε *is achieved.*

3.4. Algorithm of the method

The iterative procedure 3.7 gives the following algorithm of computation for the solution of Eq.(1.1) :

Step 0: Input the values $a, b, c, d, e, \delta_x, \delta_y, \delta_z, \lambda, \varepsilon^{'}, N$ and the functions H, f, φ.
Step 1: For $p_1 = \overline{0, 2N}$, $p_2 = \overline{0, 2N}$ and $p_3 = \overline{0, 2N}$ set $\overline{U}_0(s_{p_1}, t_{p_2}, r_{p_3}) = f(s_{p_1}, t_{p_2}, r_{p_3})$.
Step 2: For $p_1 = \overline{0, 2N}$, $p_2 = \overline{0, 2N}$, $p_3 = \overline{0, 2N}$ compute $\overline{U}_m(s_{p_1}, t_{p_2}, r_{p_3})$ by (3.7).
Step 3: If $\mid \overline{U}_m(s_{p_1}, t_{p_2}, r_{p_3}) - \overline{U}_{m-1}(s_{p_1}, t_{p_2}, r_{p_3}) \mid < \varepsilon^{'}$, Print " m " and Print $\overline{U}_m(s_{p_1}, t_{p_2}, r_{p_3})$, $p_1 = \overline{0, 2N}$, $p_2 = \overline{0, 2N}$ and $p_3 = \overline{0, 2N}$. STOP.; otherwise, set $m = m + 1$ and go to step 2.

Remark 3.7. *The above algorithm has practical stoppage criterion explained in Remark 3.6. Also, according to the Remark 3.6, the inequality* $\mid \overline{U}_m(s_{p_1}, t_{p_2}, r_{p_3}) - \overline{U}_{m-1}(s_{p_1}, t_{p_2}, r_{p_3}) \mid < \varepsilon^{'}$ *, leads to* $\mid U^*(s_{p_1}, t_{p_2}, r_{p_3}) - \overline{U}_m(s_{p_1}, t_{p_2}, r_{p_3}) \mid < \varepsilon, \forall p_1 = \overline{0, 2N}$*,* $p_2 = \overline{0, 2N}$ *,* $p_3 = \overline{0, 2N}$*, and the desired accuracy* ε *is achieved.*

4. Numerical experiments

The presented iterative method was evaluted on some examples. The evluated results confirmed the accuracy and the convergence of the method as well as theoretical conclusions. We introduce the following notation in order to interpret the error of the method:

$$\|e_N\|_\infty := \|U^* - \overline{U}_m^{(N)}\|_\infty = \max\{\mid U^*(s_{p_1}, t_{p_2}, r_{p_3}) - \overline{U}_m(s_{p_1}, t_{p_2}, r_{p_3}) \mid p_1, p_2, p_3 = \overline{0, 2N}\} \tag{4.1}$$

The experimental rate of convergence for the following examples is also calculated which is difined as (Chapter 2, [7]):

$$Ratio = \frac{\|U^* - \overline{U}_m^{(N)}\|_\infty}{\|U^* - \overline{U}_m^{(2N)}\|_\infty},$$

. where, U^* is the exact solution and $\overline{U}_m$ is the approximate solution of the Eq. (1.1) , which is computed by the numerical method described in Section 3.
Moreover, the following formula can be used to estimate the order of convergence of the values R_i [31]:

$$L_i = log\left(\frac{R_{i-2} - R_{i-1}}{R_{i-1} - R_i}\right) \Big/ log(2) \tag{4.2}$$

where $R_i = \|e_N\|_\infty$ with step size h_i. In other words, the approximate solutions R_i have error with higher order in relation to h than $F_i = F(h_i)$. Here $F(h)$ denote the value obtained by any numerical method with step size h. In *Table 4* and *Table 6* the J is the value of resolution (see Section 2, $N = 2^J$). Here, we suppose that $[a, b] \times [c, d] \times [e, h] = [0, 1]^3$ and $\lambda = 1$.

Example 4.1. *Consider the following three-dimensional nonlinear Fredholm integral equation [12]:*

$$U(s,t,r) - \int_0^1\int_0^1\int_0^1 H(s,t,r,x,y,z)(U(x,y,z))^2 dxdydz = f(s,t,r), \qquad (s,t,r) \in [0,1]^3, \tag{4.3}$$

where

$$f(s,t,r) = s^2t^2r - \frac{1}{16800}s^2r - \frac{1}{12000}s^2tr,$$
$$H(s,t,r,x,y,z) = \frac{1}{100}s^2x(y^2+t)zr,$$

the exact solution is given by

$$U(s,t,r) = s^2t^2r.$$

Applying the numerical method for $2N = 8$, *and* $\varepsilon' = 10^{-20}$, *the number of iterations is* $m = 7$ *and the results* $e_{p_1,p_2,p_3} = |U^*(s_{p_1}, t_{p_2}, r_{p_3}) - \overline{U}_7(s_{p_1}, t_{p_2}, r_{p_3})|$, *for* $p_1, p_2, p_3 = \overline{0, 10}$ *can be viewed in Table 1. In order to illustrate the numerical stability, in the fifth column we include the differences between the effective computed values* $d_{p_1,p_2,p_3} = |\overline{U}_7(s_{p_1}, t_{p_2}, r_{p_3}) - \overline{V}_7(s_{p_1}, t_{p_2}, r_{p_3})|$ *for* $p_1, p_2, p_3 = \overline{0, 10}$, *where the perturbation of the first term of the sequence of successive approximations is 0.1* ($f(s,t,r) := f(s,t,r) + 0.1$).
In order to test the convergence, we put $2N = 16$, $\varepsilon' = 10^{-15}$ *and we can see how* e_{p_1,p_2,p_3}, *decrease when* $\delta_x, \delta_y, \delta_z$ *decreases. The number of iterations is* $m = 7$ *and the results are presented in Table 2. For* $2N = 32$, $\varepsilon' = 10^{-15}$, *we have* $m = 8$ *iterations and the results are listed in Table 3. For* $\varepsilon' = 10^{-20}$ *and* $N \in \{2, 4, 8, 16, 32, 64\}$ *we test the rate of convergence and the numerical results are in Table 4. The order of convergence given in last column of the Table 4 is equal to 2 (see Eq. (4.2)). Note that the errors* $\|e_N\|_\infty$ *are decreasing by factor of approximately 4 whenever N is doubled, which is consistent with (2.5).*

Table 1. The results for Example 4.1 with $2N = 8$ ***or*** $\delta_x = \delta_y = \delta_z = \frac{1}{8}$.

$(s_{p_1},t_{p_2},r_{p_3})$	$U^*(s_{p_1},t_{p_2},r_{p_3})$	$\overline{U}_7(s_{p_1},t_{p_2},r_{p_3})$	e_{p_1,p_2,p_3}	d_{p_1,p_2,p_3}
(0.0,0.0,0.0)	0.	0.	0.	0.100000000
(0.1,0.1,0.1)	0.00001	0.000009996486574	$3.513425711 \times 10^{-9}$	0.100000051
(0.2,0.2,0.2)	0.00032	0.000319969243491	$3.075653759 \times 10^{-8}$	0.100000480
(0.3,0.3,0.3)	0.00243	0.002429887255854	$1.127441345 \times 10^{-7}$	0.100001843
(0.4,0.4,0.4)	0.01024	0.010239711561588	$2.884384112 \times 10^{-7}$	0.100004897
(0.5,0.5,0.5)	0.03125	0.031249395251045	$6.047489575 \times 10^{-7}$	0.100010595
(0.6,0.6,0.6)	0.07776	0.077758883467240	$1.116532759 \times 10^{-6}$	0.100010595
(0.7,0.7,0.7)	0.16807	0.168068113405800	$1.886594200 \times 10^{-6}$	0.100020088
(0.8,0.8,0.8)	0.32768	0.327677014314946	$2.985685054 \times 10^{-6}$	0.100054765
(0.9,0.9,0.9)	0.59049	0.590485507495503	$4.492504497 \times 10^{-6}$	0.100083882
(1.0,1.0,1.0)	1.00000	0.999993506300903	$6.493699097 \times 10^{-6}$	0.100123167

Table 2. The results for Example 4.1 with $2N = 16$.

$(s_{p_1},t_{p_2},r_{p_3})$	$U^*(s_{p_1},t_{p_2},r_{p_3})$	$\overline{U}_7(s_{p_1},t_{p_2},r_{p_3})$	e_{p_1,p_2,p_3}
(0.0,0.0,0.0)	0.	0.	0.
(0.1,0.1,0.1)	0.00001	0.000009999106814	$8.93185232 \times 10^{-10}$
(0.2,0.2,0.2)	0.00032	0.000319992184165	$7.81583453 \times 10^{-9}$
(0.3,0.3,0.3)	0.00243	0.002429971359118	$2.86408818 \times 10^{-8}$
(0.4,0.4,0.4)	0.01024	0.010239926747680	$7.32523191 \times 10^{-8}$
(0.5,0.5,0.5)	0.03125	0.031249846454803	$1.53545196 \times 10^{-7}$
(0.6,0.6,0.6)	0.07776	0.077759716574378	$2.83425621 \times 10^{-7}$
(0.7,0.7,0.7)	0.16807	0.168069521189240	$4.78810760 \times 10^{-7}$
(0.8,0.8,0.8)	0.32768	0.327679242371163	$7.57628837 \times 10^{-7}$
(0.9,0.9,0.9)	0.59049	0.590488860180866	$1.13981913 \times 10^{-6}$
(1.0,1.0,1.0)	1.00000	0.999998352668009	$1.64733199 \times 10^{-6}$

Table 3. The results for Example 4.1 with $2N = 32$.

$(s_{p_1},t_{p_2},r_{p_3})$	$U^*(s_{p_1},t_{p_2},r_{p_3})$	$\overline{U}_8(s_{p_1},t_{p_2},r_{p_3})$	e_{p_1,p_2,p_3}
(0.0,0.0,0.0)	0.	0.	0.
(0.1,0.1,0.1)	0.00001	0.00000999977577	$2.242335569 \times 10^{-10}$
(0.2,0.2,0.2)	0.00032	0.00031999803803	$1.961964776 \times 10^{-9}$
(0.3,0.3,0.3)	0.00243	0.00242999281104	$7.188956207 \times 10^{-9}$
(0.4,0.4,0.4)	0.01024	0.01023998161474	$1.838525935 \times 10^{-8}$
(0.5,0.5,0.5)	0.03125	0.03124996146478	$3.853521470 \times 10^{-8}$
(0.6,0.6,0.6)	0.07776	0.07775992887254	$7.112745169 \times 10^{-8}$
(0.7,0.7,0.7)	0.16807	0.16806987984511	$1.201548887 \times 10^{-7}$
(0.8,0.8,0.8)	0.32768	0.32767980988526	$1.901147331 \times 10^{-7}$
(0.9,0.9,0.9)	0.59049	0.59048971399151	$2.860084813 \times 10^{-7}$
(1.0,1.0,1.0)	1.00000	0.99999958665808	$4.133419186 \times 10^{-7}$

Table 4. Rate of convergence and order of convergence for iteration (3.7), in example 4.1.

J	N	$2N$	m	$R_i = \|e_N\|_\infty$	Ratio	L_i
1	2	4	7	2.45×10^{-5}	–	–
2	4	8	7	6.49×10^{-6}	3.775	–
3	8	16	7	1.65×10^{-6}	3.933	1.896
4	16	32	8	4.13×10^{-7}	3.995	1.968
5	32	64	9	1.02×10^{-7}	4.043	1.992
6	64	128	9	2.51×10^{-8}	4.076	2.013

Example 4.2. *The 3D-NFIEs (1.1) with*

$$f(s,t,r) = tr\cos^2(s) + \frac{1}{140}s^2cos(r)(cos^7(1)-1)$$
$$H(s,t,r,x,y,z) = s^2ycos(r)sin(x),$$
$$\varphi(\beta) = \beta^3,$$

has the exact solution

$$U(s,t,r) = tr\cos^2(s).$$

or $2N = 8, 2N = 16, 2N = 32$ *and* $\varepsilon^{'} = 10^{-25}$ *we test the convergence and the results are listed in Table 5. Also, for* $N \in \{2,4,8,16,32,64\}$ *we test the rate of convergence, the order of convergence and computational results are given in Table 6.*

Table 5. The results for Example 4.2 with $2N = 8, 2N = 16, 2N = 32$.

$(s_{p_1}, t_{p_2}, r_{p_3})$	$U^*(s_{p_1}, t_{p_2}, r_{p_3})$	$e_{p_1,p_2,p_3}, 2N = 8$	$e_{p_1,p_2,p_3}, 2N = 16$	$e_{p_1,p_2,p_3}, 2N = 32$
(0.0,0.0,0.0)	0.	0.	0.	0.
(0.1,0.1,0.1)	0.0099003	$1.7928993932 \times 10^{-7}$	$4.4948148910 \times 10^{-8}$	$1.12448859 \times 10^{-8}$
(0.2,0.2,0.2)	0.0384212	$7.0639333343 \times 10^{-7}$	$1.7709344350 \times 10^{-7}$	$4.43042846 \times 10^{-8}$
(0.3,0.3,0.3)	0.0821401	$1.5492799370 \times 10^{-6}$	$3.8840587214 \times 10^{-7}$	$9.71692908 \times 10^{-8}$
(0.4,0.4,0.4)	0.1357365	$2.6554577436 \times 10^{-6}$	$6.6572564270 \times 10^{-7}$	$1.66547658 \times 10^{-7}$
(0.5,0.5,0.5)	0.1925378	$3.9532931057 \times 10^{-6}$	$9.9109413431 \times 10^{-7}$	$2.47946596 \times 10^{-7}$
(0.6,0.6,0.6)	0.2452244	$5.3538242244 \times 10^{-6}$	$1.3422085443 \times 10^{-6}$	$3.35786510 \times 10^{-7}$
(0.7,0.7,0.7)	0.2866419	$6.7530339998 \times 10^{-6}$	$1.6929916925 \times 10^{-6}$	$4.23543550 \times 10^{-7}$
(0.8,0.8,0.8)	0.3106561	$8.0345394645 \times 10^{-6}$	$2.0142662641 \times 10^{-6}$	$5.03918293 \times 10^{-7}$
(0.9,0.9,0.9)	0.3129831	$9.0726469376 \times 10^{-6}$	$2.2745207406 \times 10^{-6}$	$5.69027358 \times 10^{-7}$
(1.0,1.0,1.0)	0.2919266	$9.7357147854 \times 10^{-6}$	$2.4407524459 \times 10^{-6}$	$6.10614311 \times 10^{-7}$

Table 6. Rate of convergence and order of convergence for iteration (3.7), in example 4.2.

J	N	2N	m	$R_i = \|e_N\|_\infty$	Ratio	L_i
1	2	4	6	3.850×10^{-5}	–	–
2	4	8	6	9.736×10^{-6}	3.955	–
3	8	16	6	2.441×10^{-6}	3.978	1.979
4	16	32	6	6.106×10^{-7}	3.997	1.995
5	32	64	7	1.526×10^{-7}	4.001	2.009
6	64	128	8	3.814×10^{-8}	4.000	2.000

5. Conclusions

A numerical method of the successive approximations based on the Haar wavelet methods is investigated to obtain the numerical solution of 3D-NFIEs. The proposed method is simple, involves lower computations. In Theorem 3.1 sufficient conditions for the existence and uniqueness solution of the 3D-NFIEs are presented. Proof of the convergence and the error estimation of the proposed method in terms of Lipschitz condition are provided in Theorem 3.3. The method merely requires Lipschitz properties for the convergence and smoothness conditions are not necessary. Analysis of numerical stability of the iterative method with repect to selecting the first iteration was verified in Theorem 3.4.

References

[1] M. A. Abdelkawy, E. A. Doha, A. H. Bhrawy, and A. Z. M. Amin, *Efficient pseudospectral scheme for 3D integral equations*, Roman. Acad. Ser. A. Math. Phys. Tech. Sci. Inf. Sci. 18(3) (2017) 199–206.

[2] M. Abdou, A. Badr and M. Soliman, *On a method for solving a two-dimensional nonlinear integral equation of the second kind*, J. Comput. Appl. Math. 235(12) (2011) 3589–3598.

[3] R. P. Agarwal, N. Hussain, and M. A. Taoudi, *Fixed point theorems in ordered banach spaces and applications to nonlinear integral equations*, Abst. Appl. Anal. 2012 (2012).

[4] M. Almousa, *The solutions of three dimensional Fredholm integral equations using Adomian decomposition method*, AIP Conference Proceedings, 020053, (2016) 020053-1-020053-2.

[5] B. Asady, F. Hakimzadegan, and R. Nazarlue, *Utilizing artificial neural network approach for solving two-dimensional integral equations*, Math. Sci. 8(1) (2014) 1–9.

[6] M. Asgari and R. Ezzati, *Using operational matrix of two-dimensional bernstein polynomials for solving two-dimensional integral equations of fractional order*, Appl. Math. Comput. 307 (2017) 290–298.

[7] K. Atkinson and E. Kendall, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge: Cambridge University Press, 2011.

[8] K. Atkinson and F. Potra, *Projection and iterated projection methods for nonlinear integral equations*, SIAM J. Numerical Anal. 24 (1987) 1352–1373.

[9] I. Aziz, F. Khan, et al, *A new method based on haar wavelet for the numerical solution of two-dimensional nonlinear integral equations*, J. Comput. Appl. Math. 272 (2014) 70–80.

[10] I. Aziz, S. Islam and W. Khan, *Quadrature rules for numerical integration based on Haar wavelets and hybrid functions*, Comput. Math. Appl. 61(9) (2011) 2770–2781.

[11] M. Bakhshi, M. Asghari-Larimi and M. Asghari-Larimi, *Three dimensional differential transform method for solving nonlinear threedimensional Volterra integral equations*, J. Math. Comput. Sci. 4(2) (2012) 246–256.

[12] M. Basseem, *Degenerate kernel method for three dimension nonlinear integral equations of the second kind*, Universal J. Integ. Eq. 3 (2015) 61–66.

[13] A. M. Bica , M. Curila and S. Curila, *About a numerical method of successive interpolations for functional Hammerstein integral equations*, J. Comput. Appl. Math. 236(2) (2012) 2005–2024.

[14] A. Boggess and F. J. Narcowich, *First Course in Wavelets with Fourier Analyis*, Prentice Hall, 2001.

[15] A. H. Borzabadi and O. S. Fard, *A numerical scheme for a class of nonlinear fredholm integral equations of the second kind*, J. Comput. Appl. Math. 232(2) (2009) 449–454.

[16] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, 2004.

[17] Z. Cheng, *Quantum effects of thermal radiation in a Kerr nonlinear blackbody*, J. Optical Soc. Amer. B 19 (2002) 1692–1705.

[18] W. C. Chew, M. S. Tong and B. Hu, *Integral Equation Methods for Electromagnetic and Elastic Waves*, Synthesis Lectures on Comput. Elect. 3(1) (2008) 1–241.

[19] M. Erfanian and H. Zeidabadi, *Solving two-dimensional nonlinear mixed Volterra Fredholm integral equations by using rationalized Haar functions in the complex plane*, J. Math. Mod. 7(4) (2019) 399–416.

[20] M. Esmaeilbeigi, F. Mirzaee and D. Moazami, *Radial basis functions method for solving three-dimensional linear Fredholm integral equations on the cubic domains*, Iran. J. Numerical Anal. Opt. 7(2) (2017) 15–37.

[21] S. Fazeli, G. Hojjati and H. Kheiri, *A piecewise approximation for linear two-dimensional volterra integral equation by chebyshev polynomials*, Int. J. Nonlinear Sci. 16(3) (2013) 255–261.

[22] A. Haar. *Zur Theories der orthogonalen Funktionensystem*, Math. Annal. 69 (1910) 331–371.

[23] G. Han and R. Wang, *Richardson extrapolation of iterated discrete galerkin solution for two-dimensional fredholm integral equations*, J. Comput. Appl. Math. 139(1) (2002) 49–63.

[24] G. Hursan and M. S. Zhdanov, *Contraction integral equation method in three-dimensional electromagnetic modeling*, Radio Sci. 6(37) (2002) 1–13.

[25] S. Islam, I. Aziz and F. Haq, *A comparative study of numerical integration based on Haar wavelets and hybrid functions*, Comput. Math. Appl. 59(6) (2010) 2026–2036.

[26] M. Kazemi and R. Ezzati, *Numerical solution of two-dimensional nonlinear integral equations via quadrature rules and iterative method*, Adv. Diff. Eq. Control Proc. 17 (2016) 27–56.

[27] M. Kazemi and R. Ezzati, *Existence of solutions for some nonlinear Volterra integral equations via Petryshyn's fixed point theorem*, *Int. J. Nonlinear Anal. Appl.* 9 (2018) 1–12.

[28] M. Kazemi and R. Ezzati, *Existence of solution for some nonlinear two-dimensional volterra integral equations via measures of noncompactness*, Appl. Math. Comput. 275 (2016) 165–171.

[29] F. Khan, T. Arshad, A. Ghaffar, K.S. Nisar and D. Kumar, *Numerical solutions of 2D Fredholm integral equation of first kind by discretization technique*, AIMS Math. 5(3) (2020) 2295–2306.

[30] Ülo. Lepik, *Application of the Haar wavelet transform to solving integral and differential equations*, Proc. Estonian Acad. Sci. Phys. Math. 56(1) (2007) 28–46.

[31] J. Majak , B.S. Shvartsman, M. Kirs, M. Pohlak and H. Herranen, *Convergence theorem for the Haar wavelet based discretization method*, Comp. Struct. 126 (2015) 227–232.

[32] K. Maleknejad, R. Mollapourasl and K. Nouri, *Study on existence of solutions for some nonlinear functional–integral equations*, Nonlinear Anal. 69 (2008) 2582–2588.

[33] K. Maleknejad, J. Rashidinia and T. Eftekhari, *Numerical solution of three-dimentional Voltera- Fredholm integral equations of the first and second kinds based on Bernstein's approximation*, Appl. Math. Comput. 339 (2018) 272–285.

[34] D. S. Mohamed, *Shifted Chebyshev polynomials for solving three-dimensional Volterra integral equations of the second kind*, *arXiv preprint arXiv:1609.08539,* 2016.

[35] F. Mirzaee, E. Hadadiyan and S. Bimesl, *Numerical solution for three-dimensional nonlinear mixed Volterra-Fredholm integral equations via three-dimensional block-pulse functions*, Appl. Math. Comput. 237 (2014) 168–175.

[36] F. Mirzaee and E. Hadadiyan, *A computational method for nonlinear mixed Volterra-Fredholm integral equations*, Caspian J. Math. Sci. 2 (2) (2014) 113–123.

[37] F. Mirzaee and E. Hadadiyan, *Three-dimensional triangular functions and their applications for solving nonlinear mixed Volterra-Fredholm integral equations*, Alexandria Engin. J. 3(55) (2016) 2943–2952.

[38] B. G. Pachpatte, *Multidimensional Integral Equations and Inequalities*, Springer Science, Business Media, 2011.

[39] K. Sadri, A. Amini and C. Cheng, *Low cost numerical solution for three-dimensional linear and nonlinear integral equations via three-dimensional Jacobi polynomials*, J. Comput. Appl. Math. 319 (2017) 493–513.

[40] Qi. Tang and D. Waxman, *An integral equation describing an asexual population in a changing environment*, Nonlinear Anal. Theo. Meth. Appl. 53(5) (2003) 683–699.

[41] S. M. Torabi and A. Tari, *Two-step collocation methods for two-dimensional Volterra integral equations of the second kind*, J. Appl. Anal. 25(1) (2019) 1–11.

[42] A. Ziqan, S. Armiti and I. Suwan, *Solving three- dimensional Volterra integral equation by the reduced differential transform method*, Int. J. Appl. Math. Res. 5(2) (2016) 103–106.