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Existence of solutions for a quasilinear elliptic system with variable exponent

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Abstract

We consider the following quasilinear elliptic system in a Sobolev space with variable exponent:

$$-\mathrm{div}(a(|Du|)Du) = f,$$

where a is a C^1-function and $f \in W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$. We use the theory of Young measures and weak monotonicity conditions to obtain the existence of solutions.

Keywords: Quasilinear elliptic systems, Sobolev spaces with variable exponent, Weak solutions, Young measures.
2010 MSC: 35Jxx, 46E30, 35D30.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with Lipschitz boundary $\partial\Omega$. Consider the following quasilinear elliptic system:

$$\begin{cases} -\mathrm{div}(a(|Du|)Du) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where a is a C^1-function defined from $[0,+\infty)$ to $[0,+\infty)$ and f belongs to Sobolev space with variable exponent $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$. When $a(\xi) = |\xi|^{p-2}$, problem (1.1) is the well known p-Laplace system. In recent years, there have been a large number of papers on the existence and regularity of solutions of the p-Laplace system (see [13, 19, 21] and the references therein). In the case of

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degenerate p-Laplacian system where $a(\xi) = |\xi - \Theta(u)|^{p-2}$, $\Theta : \mathbb{R}^m \to \mathbb{M}^{m\times n}$, we have proved in [2] existence result by using the theory of Young measures and without assuming any conditions of Leray-Lions type. Here, $\mathbb{M}^{m\times n}$ is the space of $m \times n$ matrices equipped with the inner product $\xi : \eta = \sum_{i=1}^{m}\sum_{j=1}^{n} \xi_{ij}\eta_{ij}$. The extension of [2] to the case of exponent variable $p(x)$ can be found in [3]. For the $p(x)$-Laplace equations, Cianchi and Maz'ya [11] established the Lipschitz continuity of solutions to Dirichlet and Neumann cases. In [1], Acerbi and Mingione proved Caldéron and Zygmund type estimates for a class of $p(x)$-Laplacian system whose right-hand side is under the divergence form.

Problems of the form (1.1) were studied in [11, 12] under some conditions on the function a. Moreover, they treated the corresponding Neumann case. Dirichlet problems of the form (1.1) are the main objective of the present paper. When the right-hand side of (1.1) belongs to $W^{-1,p'}(\Omega;\mathbb{R}^m)$, we have proved in [7] the existence of weak solutions based on the Galerkin approximation and the theory of Young measures.

Here and after, Du denotes the gradient of a function $u : \Omega \to \mathbb{R}^m$, and is a matrix-valued function, i.e., $Du \in \mathbb{M}^{m\times n}$. The function $a : [0, +\infty) \to [0, +\infty)$ is assumed to be of class $C^1([0, +\infty))$, and to fulfill

$$-1 \le i_a \le s_a < \infty \tag{1.2}$$

where

$$i_a = \inf_{t>0} \frac{ta'(t)}{a(t)} \quad \text{and} \quad s_a = \sup_{t>0} \frac{ta'(t)}{a(t)}. \tag{1.3}$$

The function a satisfies the following growth and coercivity conditions: For all $\xi \in \mathbb{M}^{m\times n}$, some constants $c_1, c_2 > 0$ and $l(x) \in L^1(\Omega)$,

$$\big|a(|\xi|)\xi\big| \le c_1|\xi|^{p(x)-1}, \tag{1.4}$$

$$a(|\xi|)\xi : \xi \ge c_2|\xi|^{p(x)} - l(x). \tag{1.5}$$

Moreover, we assume that a satisfies one of the following conditions:

(H0) There exists a convex and C^1-function $b : \mathbb{M}^{m\times n} \to \mathbb{R}$ such that

$$a(|\xi|)\xi = \frac{\partial b(\xi)}{\partial \xi} := D_\xi b(\xi).$$

(H1) For $\overline{\lambda} = \langle \nu_x, id\rangle$ where $\nu = \{\nu_x\}_{x\in\Omega}$ is any family of Young measures generated by a sequence in $L^{p(x)}(\Omega;\mathbb{M}^{m\times n})$ and not a Dirac measure for almost every $x \in \Omega$, we have

$$\int_{\mathbb{M}^{m\times n}} \big(a(|\lambda|)\lambda - a(|\overline{\lambda}|)\overline{\lambda}\big) : (\lambda - \overline{\lambda})d\nu(\lambda) > 0.$$

Our objective in this paper is to study the existence of solutions for (1.1) in the framework of Sobolev spaces with variable exponent under the above conditions and to extend the result of [7] to the variable exponent spaces. Moreover, we will use a Galerkin method to construct the approximating solutions and the theory of Young measures to identify weak limits and in the passage to the limit. We refer the reader to see [4, 5, 6, 8] where the theory of Young measures finds its applications in different nonlinear elliptic systems.

Note that, (1.3) and (1.4) will serve us to prove that the function a is monotone. The condition (H0) allows to take a potential b, which is only convex but not strictly convex to avoid the use of the well known classical monotone operator theory, and to consider (1.1) with $a(|\xi|)\xi = \partial b/\partial \xi$. Assumption (H1) may be called strictly $p(x)$-quasimonotone as in the framework $W^{1,p}(\Omega;\mathbb{R}^m)$ (see [18]).

A weak solution for (1.1) is a function $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ such that

$$\int_\Omega a(|Du|)Du : D\varphi dx = \langle f, \varphi\rangle \quad \text{for all } \varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m).$$

Here $\langle .,. \rangle$ denotes the duality pairing of $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ and $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

The principal result of this paper reads as follows:

Theorem 1.1. *Under assumptions* (1.3)-(1.5), *(H0) and (H1), problem* (1.1) *has a weak solution* $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

2. Preliminaries

We recall some necessary notations, definitions and properties for our function spaces (see [14, 20]) and an overview about Young measures (see [9, 15, 17]).

For each open bounded subset Ω of $\mathbb{R}^n$ $(n \geq 2)$, we denote $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}),\ p(x) > 1 \text{ for any } x \in \Omega\}$. We define for every $p \in C^+(\overline{\Omega})$,

$$p^- = \inf_{x\in\Omega} p(x) \quad \text{and} \quad p^+ = \sup_{x\in\Omega} p(x).$$

The Sobolev space $W^{1,p(x)}(\Omega;\mathbb{R}^m)$ consists of all functions u in the Lebesgue space

$$L^{p(x)}(\Omega;\mathbb{R}^m) = \Big\{u : \Omega \to \mathbb{R}^m \text{ measurable} : \int_\Omega |u(x)|^{p(x)}dx < \infty\Big\},$$

such that $Du \in L^{p(x)}(\Omega;\mathbb{M}^{m\times n})$. The space $L^{p(x)}(\Omega;\mathbb{R}^m)$ is endowed with the norm

$$\|u\|_{p(x)} = \inf\Big\{\beta > 0,\ \int_\Omega \Big|\frac{u(x)}{\beta}\Big|^{p(x)}dx \leq 1\Big\}.$$

It is a Banach space. Moreover, it is reflexive if and only if $1 < p^- \leq p^+ < \infty$. Its dual is defined by $L^{p'(x)}(\Omega;\mathbb{R}^m)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega;\mathbb{R}^m)$ and $v \in L^{p'(x)}(\Omega;\mathbb{R}^m)$, the generalized Hölder inequality

$$\Big|\int_\Omega uv dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p^+}\Big)\|u\|_{p(x)}\|v\|_{p'(x)}$$

holds true. The space $W^{1,p(x)}(\Omega;\mathbb{R}^m)$ is endowed with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|Du\|_{p(x)}.$$

Proposition 2.1 ([16]). *We denote* $\rho(u) = \int_\Omega |u|^{p(x)}dx$, $\forall u \in L^{p(x)}(\Omega;\mathbb{R}^m)$. *If* $u_k, u \in L^{p(x)}(\Omega;\mathbb{R}^m)$ *and* $p^+ < \infty$, *then:*

(i) $\|u\|_{p(x)} < 1$ $(= 1; > 1) \Leftrightarrow \rho(u) < 1$ $(= 1; > 1)$.

(ii) $\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq \rho(u) \leq \|u\|_{p(x)}^{p^+}$; $\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq \rho(u) \leq \|u\|_{p(x)}^{p^-}$.

(iii) $\|u_k\|_{p(x)} \to 0 \Leftrightarrow \rho(u_k) \to 0$; $\|u_k\|_{p(x)} \to +\infty \Leftrightarrow \rho(u_k) \to +\infty$.

We denote by $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ the closure of $C_0^\infty(\Omega;\mathbb{R}^m)$ in $W^{1,p(x)}(\Omega;\mathbb{R}^m)$ and $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ is its dual space. We denote $p^*(x) = \frac{np(x)}{n-p(x)}$ for $p(x) < n$; $= \infty$ for $p(x) > n$.

Proposition 2.2 ([16]). *(i) Under the assumption $1 < p^-$, the spaces $W^{1,p(x)}(\Omega;\mathbb{R}^m)$ and $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ are separable and reflexive Banach spaces.*
(ii) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then $W^{1,p(x)}(\Omega;\mathbb{R}^m) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega;\mathbb{R}^m)$ is compact and continuous. In particular, we have $W_0^{1,p(x)}(\Omega;\mathbb{R}^m) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega;\mathbb{R}^m)$ is compact and continuous.
(iii) There exists a constant $c_3 > 0$, such that

$$\|u\|_{p(x)} \leq c_3 \|Du\|_{p(x)} \qquad \forall u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m),$$

hence $\|Du\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are two equivalent norms on $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

Weak convergence is a basic tool of modern nonlinear analysis because it has the same compactness properties as the convergence in finite-dimensional spaces (see [15]). But, this convergence sometimes does not behave as one desire with respect to nonlinear functionals and operators. To overcome this difficulty, one can use the technics of Young measures.

By $C_0(\mathbb{R}^m)$ we denote the set of functions $g \in C(\mathbb{R}^m)$ satisfying $\lim_{|\lambda|\to\infty} g(\lambda) = 0$. Its dual can be identified with the space of signed Radon measures with finite mass denoted by $\mathcal{M}(\mathbb{R}^m)$. The related duality pairing is given by

$$\langle \nu, g \rangle = \int_{\mathbb{R}^m} g(\lambda) d\nu(\lambda) \quad \text{for } \nu : \Omega \to \mathcal{M}(\mathbb{R}^m).$$

Lemma 2.3 ([15]). *Let $(z_k)_k$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exists a subsequence (still denoted z_k) and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for almost each $g \in C(\mathbb{R}^m)$ we have*

$$\varphi(z_k) \rightharpoonup^* \overline{\varphi} \quad \textit{weakly in } L^\infty(\Omega;\mathbb{R}^m),$$

where $\overline{\varphi}(x) = \langle \nu_x, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu_x(\lambda)$ for a.e. $x \in \Omega$.

Definition 2.4. *The family $\nu = \{\nu_x\}_{x\in\Omega}$ is called Young measure associated with the subsequence $(z_k)_k$.*

The fundamental theorem of Young measures can be stated in the following lemma:

Lemma 2.5 ([9]). *Let Ω be Lebesgue measurable, let $K \subset \mathbb{R}^m$ be closed, and let $z_k : \Omega \to \mathbb{R}^m$, $k \in \mathbb{N}$, be a sequence of Lebesgue measurable functions satisfying $z_k \to K$ in measure as $k \to \infty$, i.e., given any open neighborhood U of K in $\mathbb{R}^m$*

$$\lim_{k\to\infty} |\{x \in \Omega : u_k(x) \notin U\}| = 0.$$

Then there exist a subsequence denoted also $(z_k)_k$ and a family $\{\nu_x\}_{x\in\Omega}$ of positive measures on $\mathbb{R}^m$, such that

(a) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} := \int_{\mathbb{R}^m} d\nu_x(\lambda) \leq 1$ *for a.e.* $x \in \Omega$.

(b) $\operatorname{supp}\nu_x \subset K$ *for a.e.* $x \in \Omega$, *and*

(c) $\varphi(z_k) \rightharpoonup^* \langle \nu_x, \varphi\rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu_x(\lambda)$ *in* $L^\infty(\Omega)$ *for each* $\varphi \in C_0(\mathbb{R}^m)$.

Suppose further that (z_k) *satisfies the boundedness condition*

$$\forall R > 0: \quad \lim_{L\to\infty} \sup_{k\in\mathbb{N}} \left|\{x \in \Omega \cap B_R(0): \ |z_k(x)| \geq L\}\right| = 0. \tag{2.1}$$

Then $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ *for a.e.* $x \in \Omega$, *and for any measurable* $\Omega' \subset \Omega$ *there holds* $\varphi(z_k) \rightharpoonup \overline{\varphi} = \langle \nu_x, \varphi\rangle$ *weakly in* $L^1(\Omega')$ *for any continuous function* $\varphi : \mathbb{R}^m \to \mathbb{R}$ *provided the sequence* $\varphi(z_k)$ *is weakly precompact in* $L^1(\Omega')$.

Lemma 2.5 has usefull applications, in particular in non-linear PDE theory. The following properties build the basic tools used in the sequel, and can be seen as the applications of Lemma 2.5 (see [9, 17]):
If $|\Omega| < \infty$ (finite measure), then there holds

$$z_k \longrightarrow z \text{ in measure} \Leftrightarrow \nu_x = \delta_{z(x)} \text{ for a.e. } x \in \Omega, \tag{2.2}$$

where $\delta_{z(x)}$ is the Young measure associated to z_k. Let $\varphi : \mathbb{M}^{m\times n} \to \mathbb{R}$ be a continuous function and $z_k : \Omega \to \mathbb{R}^m$ a sequence of measurable functions such that Dz_k generates the Young measure ν_x, with $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for a.e. $x \in \Omega$, then

$$\liminf_{k\to\infty} \int_\Omega \varphi(Dz_k)dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \varphi(\lambda) d\nu_x(\lambda)dx, \tag{2.3}$$

provided that the negative part $\varphi^-(Dz_k)$ is equiintegrable.

3. Approximating solutions

Now, as mentioned in the introduction, we will use the Galerkin method to construct the approximating solutions. To this purpose, we consider the mapping $T : W_0^{1,p(x)}(\Omega;\mathbb{R}^m) \to W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ defined for $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ as

$$\langle T(u), \varphi\rangle = \int_\Omega a(|Du|)Du : D\varphi dx - \langle f, \varphi\rangle. \tag{3.1}$$

As a first remark, the problem (1.1) is equivalent to find such $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ which satisfy $\langle T(u), \varphi\rangle = 0$ for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. In the sequel, we will use a positive constant c which may change values from line to line.

Lemma 3.1. *The mapping* T *satisfies the following properties:*

(i) T *is linear, well defined and bounded.*

(ii) The restriction of T *to a finite linear subspace of* $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ *is continuous.*

(iii) T *is coercive.*

Proof . (i) For arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, $T(u)$ is trivially linear. We have by (1.4)

$$\int_\Omega \big|a(|Du|)Du\big|^{p'(x)}dx \le c\int_\Omega |Du|^{p(x)}dx < \infty$$

where c is a positive constant. It follows by Hölder's inequality that

$$\begin{aligned}\big|\langle T(u),\varphi\rangle\big| &\le c\|Du\|_{p(x)}^{p^+-1}\|D\varphi\|_{p(x)} + c\|f\|_{-1,p'(x)}\|\varphi\|_{1,p(x)}\\ &\le c\|D\varphi\|_{p(x)},\end{aligned}$$

thus T is well defined and bounded.
(ii) Let W be a finite linear subspace of $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ such that $\dim W = r$. For simplicity, we denote T as the restriction $T|_W$ of T to W. Let $(u_k = \alpha_{ki}w_i)$ a sequence in W which converges to $u = \alpha_i w_i$ in W (with conventional summation). Here $\{w_1,..,w_r\}$ is a basis of W. We have in one hand, $Du_k \to Du$ almost everywhere and the continuity of the function a gives

$$a(|Du_k|)Du_k \to a(|Du|)Du \quad \text{almost everywhere.}$$

On the other hand, since $u_k \to u$ strongly in W,

$$\int_\Omega |Du_k - Du|^{p(x)}dx \longrightarrow 0 \quad \text{as } k\to\infty.$$

According to [10, Chap IV, Sec 3, Theorem 3] there exists a subsequence still denoted (Du_k) and $g \in L^1(\Omega)$ such that $|Du_k - Du|^{p(x)} \le g$. We know that for $1 < p$, $|A+B|^p \le 2^{p-1}(|A|^p + |B|^p)$, thus

$$|Du_k|^{p(x)} = |Du_k - Du + Du|^{p(x)} \le 2^{p^+-1}(g + |Du|^{p(x)}).$$

This implies that $\|Du_k\|_{p(x)}$ is bounded by a constant c. Now, in order to apply the Vitali Convergence Theorem, we choose $\Omega' \subset \Omega$ to be a measurable subset and by Hölder's inequality

$$\int_{\Omega'} \big|a(|Du_k|)Du_k : D\varphi\big|dx \le \underbrace{c\|Du_k\|_{p(x)}^{p^+-1}}_{\le c}\Big(\int_{\Omega'}|D\varphi|^{p(x)}dx\Big)^{\frac{1}{p(x)}}.$$

If we choose the measure of Ω' to be small enough, then $\int_{\Omega'}|D\varphi|^{p(x)}dx$ is arbitrary small, hence $\big(a(|Du_k|)Du_k : D\varphi\big)$ is equiintegrable. By vertue of the Vitali Convergence Theorem, we get $\lim_{k\to\infty}\langle T(u_k),\varphi\rangle = \langle T(u),\varphi\rangle$.
(iii) From Eq. (1.5), it follows that

$$\begin{aligned}\langle T(u),u\rangle &= \int_\Omega a(|Du|)Du : Du dx - \langle f,u\rangle\\ &\ge c_2\int_\Omega |Du|^{p(x)}dx - \int_\Omega l(x)dx - c\|f\|_{-1,p'(x)}\|u\|_{1,p(x)}.\end{aligned}$$

Hence

$$\frac{\langle T(u),u\rangle}{\|u\|_{1,p(x)}} \ge c\|Du\|_{p(x)}^{p(x)-1} - \frac{\|l\|_{L^1}}{\|u\|_{1,p(x)}} - c \longrightarrow \infty \quad \text{as } \|u\|_{1,p(x)}\to\infty.$$

□

Now, in order to find $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ such that $\langle T(u),\varphi\rangle = 0$, we consider $W_1 \subset W_2 \subset ... \subset W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ a sequence of finite dimensional subspaces such that $\underset{k\geq 1}{\cup} W_k$ is dense in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. The sequence (W_k) exists since $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is separable. Fix k and assume that $\dim W_k = r$ and $w_1,..,w_k$ is a basis of W_k. We define the map

$$S:\mathbb{R}^r \to \mathbb{R}^r, \qquad \alpha \mapsto \Big(\langle T(\alpha_i w_i), w_j\rangle\Big)_{j=1,..,r}$$

for $\alpha = (\alpha_1,..,\alpha_r) \in \mathbb{R}^r$. The properties of S allow us to construct the desired approximate solutions. Remark that for $u = \alpha_i w_i$ (with conventional summation), we have

$$S(\alpha).\alpha = \langle T(u), u\rangle. \tag{3.2}$$

Hence S is continuous and $S(\alpha).\alpha \to \infty$ as $\|\alpha\|_{\mathbb{R}^r} \to \infty$ by Lemma 3.1, because $\|\alpha\|_{\mathbb{R}^r} \to \infty$ is equivalent to $\|u\|_{1,p(x)} \to \infty$. Therefore, there exists $R > 0$ such that $S(\alpha).\alpha > 0$ for all $\alpha \in \partial B_R(0) \subset \mathbb{R}^r$. By vertue of the topological arguments (see e.g. [23, Proposition 2.8]), it follows that

$$S(x) = 0 \quad \text{has a solution } x \in B_R(0).$$

Hence, for all $k \in \mathbb{N}$, there exists $u_k \in W_k$ such that

$$\langle T(u_k),\varphi\rangle = 0 \quad \text{for all } \varphi \in W_k. \tag{3.3}$$

Corollary 3.2. *The sequence constructed in* (3.3) *is uniformly bounded in* $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$*, i.e., there is* $R > 0$ *such that*

$$\|u_k\|_{1,p(x)} \leq R \quad \textit{for all } k \in \mathbb{N}.$$

Proof . We have $\langle T(u),u\rangle \to \infty$ as $\|u\|_{1,p(x)} \to \infty$ by Lemma 3.1. Hence, there exists $R > 0$ with the property that $\langle T(u),u\rangle > 1$ whenever $\|u\|_{1,p(x)} > R$. This gives a contradiction with the Galerkin approximations u_k which satisfies (3.3). Therefore, (u_k) is uniformly bounded. □

Now, as (u_k) is uniformly bounded in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, it follows by Lemma 2.3 the existence of a Young measure ν_x generated by Du_k in $L^{p(x)}(\Omega;\mathbb{M}^{m\times n})$. Before we proceed in the proof of the main result, we still need some properties on the Young measure ν_x. The proof of the following lemma can be found in [3], but for completeness of this work, we will present its proof.

Lemma 3.3. *The Young measure* ν_x *generated by* Du_k *in* $L^{p(x)}(\Omega;\mathbb{M}^{m\times n})$ *has the following properties:*

(i) ν_x *is a probability measure, i.e.* $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ *for a.e.* $x \in \Omega$.

(ii) The weak L^1*-limit of* Du_k *is given by* $\langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)$.

(iii) ν_x *satisfies* $\langle \nu_x, id\rangle = Du(x)$ *for a.e.* $x \in \Omega$.

Proof . (i) It is sufficient to show that Du_k satisfies Eq. (2.1) in Lemma 2.5. By vertue of Corollary 3.2, there is a positive constant c such that for any $R > 0$

$$\begin{aligned} c \geq \int_\Omega |Du_k|^{p(x)}dx &\geq \int_{\{x\in\Omega\cap B_R(0):\,|Du_k|\geq L\}} |Du_k|^{p(x)}dx \\ &\geq L^{p^-}\big|\{x \in \Omega\cap B_R(0):\ |Du_k| \geq L\}\big|, \end{aligned}$$

which gives

$$\sup_{k\in\mathbb{N}} \big|\{x \in \Omega \cap B_R(0) : |Du_k| \geq L\}\big| \leq \frac{c}{L^{p^-}} \longrightarrow 0 \quad \text{as } L \to \infty.$$

Hence, ν_x is a probability measure by Lemma 2.5.

(ii) We have $1 < p^- \leq p(x)$ and $\mathbb{M}^{m\times n}$ can be regarded as $\mathbb{R}^{mn}$, then $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$ is reflexive. Then by Corollary 3.2, there is a subsequence (still denoted by Du_k) weakly convergent in $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$, hence weakly convergent in $L^1(\Omega; \mathbb{M}^{m\times n})$ since $L^{p(x)}(\Omega; \mathbb{M}^{m\times n}) \subset L^1(\Omega; \mathbb{M}^{m\times n})$. By vertue of Lemma 2.5 and take $\varphi \equiv id$, we deduce that

$$Du_k \rightharpoonup \langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) \quad \text{weakly in } L^1(\Omega; \mathbb{M}^{m\times n}).$$

(iii) Corollary 3.2 allows to deduce that a subsequence of (u_k) is converging weakly in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ to an element denoted by $u \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$. Therefore, $Du_k \rightharpoonup Du$ in $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$ (for a subsequence). The uniqueness of limit implies that

$$\langle \nu_x, id\rangle = Du(x) \quad \text{for a.e. } x \in \Omega.$$

□

To pass to the limit in the approximating equations, we will use the following usefull lemmas, which can be seen as the key ingredient in the proof of the main result.

Lemma 3.4. *The Young measure ν_x generated by Du_k satisfies the following inequality*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(a(|\lambda|)\lambda - a(|Du|)Du\big) : (\lambda - Du) d\nu_x(\lambda) dx \leq 0.$$

Proof . Consider the sequence

$$\begin{aligned} A_k &:= \big(a(|Du_k|)Du_k - a(|Du|)Du\big) : (Du_k - Du) \\ &= a(|Du_k|)Du_k : (Du_k - Du) - a(|Du|)Du : (Du_k - Du) \\ &= A_{k,1} + A_{k,2}. \end{aligned}$$

Since

$$\int_\Omega \big|a(|Du|)Du\big|^{p'(x)} dx \leq c \int_\Omega |Du|^{p(x)} dx < \infty$$

for arbitrary $u \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$, $a(|Du|)Du \in L^{p'(x)}(\Omega; \mathbb{M}^{m\times n})$. Therefore

$$\liminf_{k\to\infty} \int_\Omega A_{k,2} dx = \int_\Omega a(|Du|)Du : \Big(\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - Du\Big) dx = 0 \tag{3.4}$$

by Lemma 3.3. We have $\big(a(|Du_k|)Du_k : Du\big)^-$ is equiintegrable (see the proof of Lemma 3.1 if necessary). The sequence $\big(a(|Du_k|)Du_k : Du_k\big)$ is easily seen to be equiintegrable. Indeed, by Eq. (1.5), we have

$$a(|Du_k|)Du_k : Du_k \geq c_2 |Du_k|^{p(x)} - l(x),$$

which implies

$$\int_{\Omega'} \big|\min\big(a(|Du_k|)Du_k : Du_k, 0\big)\big| dx \leq c_2 \int_{\Omega'} |Du_k|^{p(x)} dx + \int_{\Omega'} |l(x)| dx < \infty$$

by the boundedness of (u_k). Now, by applying the equation (2.3) to the sequence $\big(a(|Du_k|)Du_k : (Du_k - Du)\big)$, we get

$$A := \liminf_{k\to\infty} \int_\Omega A_k dx = \liminf_{k\to\infty} \int_\Omega A_{k,1} dx \\ \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : (\lambda - Du)d\nu_x(\lambda)dx.$$

If we arrive at $A \leq 0$, then the needed result follows immediately. Using the Mazur's theorem (see [22, Theorem 2, page 120]), it follows the existence of $\varphi_k \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ such that $\varphi_k \to u$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, where each φ_k is a convex linear combination of $\{u_1, .., u_k\}$, that means $\varphi_k \in W_k$. By taking $u_k - \varphi_k$ as a test function in (3.3), we obtain

$$\int_\Omega a(|Du_k|)Du_k : (Du_k - D\varphi_k)dx = \langle f, u_k - \varphi_k\rangle. \tag{3.5}$$

From the Hölder inequality, it follows that

$$\big|\langle f, u_k - \varphi_k\rangle\big| \leq c\|f\|_{-1,p'(x)}\|u_k - \varphi_k\|_{1,p(x)}.$$

The right hand side of the above inequality vanishes as $k \to \infty$, since by the construction of φ_k we have

$$\|u_k - \varphi_k\|_{1,p(x)} \leq \|u_k - u\|_{1,p(x)} + \|\varphi_k - u\|_{1,p(x)} \longrightarrow 0 \quad \text{as } k \to \infty.$$

Hence, the left hand side in (3.5) tends to zero as $k \to \infty$. Using this result and the fact that $\varphi_k \to u$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, we deduce the following

$$\begin{aligned} A &= \liminf_{k\to\infty} \int_\Omega A_k dx \\ &= \liminf_{k\to\infty} \int_\Omega a(|Du_k|)Du_k : (Du_k - Du)dx \\ &= \liminf_{k\to\infty} \Big(\int_\Omega a(|Du_k|)Du_k : (Du_k - D\varphi_k)dx + \int_\Omega a(|Du_k|)Du_k : (D\varphi_k - Du)dx \Big) \\ &\overset{(3.5)}{=} \liminf_{k\to\infty} \Big(\langle f, u_k - \varphi_k\rangle + \int_\Omega a(|Du_k|)Du_k : (D\varphi_k - Du)dx \Big) \\ &= \liminf_{k\to\infty} \int_\Omega a(|Du_k|)Du_k : (D\varphi_k - Du)dx \\ &\leq \liminf_{k\to\infty} c\big\||a(|Du_k|)Du_k|\big\|_{p'(x)} \|D\varphi_k - Du\|_{p(x)} = 0. \end{aligned}$$

Consequently, $A \leq 0$ together with (3.4) implies the needed result. □

Lemma 3.5. *If a satisfies* (1.2) *and* (1.3)*, then it is monotone, i.e.,*

$$\big(a(|\xi|)\xi - a(|\eta|)\eta\big) : (\xi - \eta) \geq 0 \quad \textit{for all } \xi, \eta \in \mathbb{M}^{m\times n}.$$

Proof . For $\xi, \eta \in \mathbb{M}^{m\times n}$ and $t \in [0,1]$ we set $\theta_t = t\xi + (1-t)\eta$, then

$$\begin{aligned} \big(a(|\xi|)\xi - a(|\eta|)\eta\big) : (\xi - \eta) &= \Big(\int_0^1 \frac{d}{dt}\big(a(|\theta_t|)\theta_t\big) dx \Big) : (\xi - \eta) \\ &= \Big(\int_0^1 \big(a'(|\theta_t|)|\theta_t| + a(|\theta_t|)\big) dx \Big) : (\xi - \eta)^2 \\ &= \Big(\int_0^1 a(|\theta_t|)\big(\frac{a'(|\theta_t|)|\theta_t|}{a(|\theta_t|)} + 1\big) dx \Big) : (\xi - \eta)^2 \geq 0 \end{aligned}$$

by the equations (1.2) and (1.3). □

Lemma 3.6. *The Young measure ν_x generated by Du_k satisfies*

$$\big(a(|\lambda|)\lambda - a(|Du|)Du\big) : (\lambda - Du) = 0 \quad \textit{on } \operatorname{supp}\nu_x.$$

Proof . According to Lemma 3.4, we have

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(a(|\lambda|)\lambda - a(|Du|)Du\big) : (\lambda - Du)d\nu_x(\lambda)dx \leq 0,$$

and by vertue of the monotonicity of the function a in Lemma 3.5, it follows that the above integral is nonnegative, thus must vanish with respect to the product measure $d\nu_x(\lambda) \otimes dx$. Hence

$$\big(a(|\lambda|)\lambda - a(|Du|)Du\big) : (\lambda - Du) = 0 \quad \text{on } \operatorname{supp}\nu_x.$$

□

4. Proof of Theorem 1.1

Now, we have all ingredients to pass to the limit in the approximating equations by considering both conditions (H0) and (H1). Start with the condition (H0). We show first that for almost every $x \in \Omega$, $\operatorname{supp}\nu_x \subset K_x$, where

$$K_x = \big\{\lambda \in \mathbb{M}^{m\times n} : \ b(\lambda) = b(Du) + a(|Du|)Du : (\lambda - Du)\big\}.$$

If $\lambda \in \operatorname{supp}\nu_x$, then by Lemma 3.6

$$(1-\tau)\big(a(|\lambda|)\lambda - a(|Du|)Du\big) : (\lambda - Du) = 0 \quad \text{for all } \tau \in [0,1]. \tag{4.1}$$

It follows by Lemma 3.5 that

$$\begin{aligned} 0 &\leq (1-\tau)\big(a(|\lambda|)\lambda - a(|Du + \tau(\lambda - Du)|)(Du + \tau(\lambda - Du))\big) : (\lambda - Du) \\ &\overset{(4.1)}{=} (1-\tau)\big(a(|Du|)Du - a(|Du + \tau(\lambda - Du)|)(Du + \tau(\lambda - Du))\big) : (\lambda - Du). \end{aligned} \tag{4.2}$$

Remark that, by the monotonicity of the function a, we have

$$\big(a(|Du|)Du - a(|Du + \tau(\lambda - Du)|)(Du + \tau(\lambda - Du))\big) : \tau(\lambda - Du) \leq 0,$$

and since $\tau \in [0,1]$

$$\big(a(|Du|)Du - a(|Du + \tau(\lambda - Du)|)(Du + \tau(\lambda - Du))\big) : (1-\tau)(\lambda - Du) \leq 0. \tag{4.3}$$

From (4.2) and (4.3), we get for $\tau \in [0,1]$ that

$$\big(a(|Du|)Du - a(|Du + \tau(\lambda - Du)|)(Du + \tau(\lambda - Du))\big) : (\lambda - Du) = 0,$$

i.e.,

$$a(|Du|)Du : (\lambda - Du) = a(|Du + \tau(\lambda - Du)|)(Du + \tau(\lambda - Du)) : (\lambda - Du).$$

By integrating the above equality over $[0,1]$ and using the fact that

$$a(|Du + \tau(\lambda - Du)|)(Du + \tau(\lambda - Du)) : (\lambda - Du) = \frac{\partial b}{\partial \tau}(Du + \tau(\lambda - Du)) : (\lambda - Du),$$

we obtain

$$\begin{aligned} b(\lambda) &= b(Du) + \int_0^1 a(|Du|)Du : (\lambda - Du)d\tau \\ &= b(Du) + a(|Du|)Du : (\lambda - Du) \end{aligned}$$

as desired, thus $\lambda \in K_x$, i.e., $\operatorname{supp}\nu_x \subset K_x$. Now, the convexity of the potential b implies that

$$b(\lambda) \geq \underbrace{b(Du) + a(|Du|)Du : (\lambda - Du)}_{=:B(\lambda)} \quad \text{for all } \lambda \in \mathbb{M}^{m\times n}.$$

Since the mapping $\lambda \mapsto b(\lambda)$ is of class C^1, for every $\xi \in \mathbb{M}^{m\times n}$, $\tau \in \mathbb{R}$

$$\frac{b(\lambda+\tau\xi) - b(\lambda)}{\tau} \geq \frac{B(\lambda+\tau\xi) - B(\lambda)}{\tau} \quad \text{if} \quad \tau > 0$$

$$\frac{b(\lambda+\tau\xi) - b(\lambda)}{\tau} \leq \frac{B(\lambda+\tau\xi) - B(\lambda)}{\tau} \quad \text{if} \quad \tau < 0.$$

Hence $Db = DB$, i.e.,

$$a(|\lambda|)\lambda = a(|Du|)Du \quad \text{for all } \lambda \in K_x \supset \operatorname{supp}\nu_x. \tag{4.4}$$

The equiintegrability of $a(|Du_k|)Du_k$ implies that its weak L^1-limit is given by

$$\begin{aligned} \overline{a}(x) &:= \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda d\nu_x(\lambda) \\ &\overset{(4.4)}{=} \int_{\operatorname{supp}\nu_x} a(|Du|)Dud\nu_x(\lambda) \\ &= a(|Du|)Du \underbrace{\int_{\operatorname{supp}\nu_x} d\nu_x(\lambda)}_{=:1} = a(|Du|)Du. \end{aligned} \tag{4.5}$$

Now, consider the continuous function

$$g(\lambda) = \big|a(|\lambda|)\lambda - \overline{a}(x)\big|, \quad \lambda \in \mathbb{M}^{m\times n}.$$

Since $a(|Du_k|)Du_k$ is equiintegrable, then $g_k(x) := g(Du_k)$ is equiintegrable and its weak L^1-limit is given by

$$g_k \rightharpoonup \overline{g} \quad \text{in} \quad L^1(\Omega) \tag{4.6}$$

where

$$\begin{aligned} \overline{g}(x) &= \int_{\mathbb{M}^{m\times n}} \big|a(|\lambda|)\lambda - \overline{a}(x)\big| d\nu_x(\lambda) \\ &\overset{(4.5)}{=} \int_{\operatorname{supp}\nu_x} \big|a(|Du|)Du - \overline{a}(x)\big| d\nu_x(\lambda) = 0. \end{aligned}$$

As a matter of fact, the convergence in (4.6) is strong since $g_k \geq 0$. Therefore

$$\lim_{k\to\infty} \int_\Omega a(|Du_k|)Du_k : D\varphi dx = \int_\Omega a(|Du|)Du : D\varphi dx \quad \forall \varphi \in \bigcup_{k\geq 1} W_k.$$

Now, for the case (H1), we argue by contradiction and suppose that ν_x is not a Dirac measure on a set $x \in \Omega'$ of positive Lebesgue measure $|\Omega'| > 0$. We have $\overline{\lambda} = \langle \nu_x, id \rangle = Du(x)$ for a.e. $x \in \Omega$, thus

$$\begin{aligned}\int_{\mathbb{M}^{m\times n}} a(|\overline{\lambda}|)\overline{\lambda} : (\lambda - \overline{\lambda}) d\nu_x(\lambda) &= \int_{\mathbb{M}^{m\times n}} \Big(a(|\overline{\lambda}|)\overline{\lambda} : \lambda - a(|\overline{\lambda}|)\overline{\lambda} : \overline{\lambda} \Big) d\nu_x(\lambda) \\ &= a(|\overline{\lambda}|)\overline{\lambda} : \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{=:\overline{\lambda}} - a(|\overline{\lambda}|)\overline{\lambda} : \overline{\lambda} \underbrace{\int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda)}_{=:1} \\ &= 0.\end{aligned}$$

By vertue of the strict $p(x)$-quasimonotone in (H1), we obtain then

$$\int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda d\nu_x(\lambda) > \int_{\mathbb{M}^{m\times n}} a(|\lambda|) : \overline{\lambda} d\nu_x(\lambda).$$

Integrating the above inequality over Ω and using Lemma 3.4, we get

$$\begin{aligned}\int_{\Omega} \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda d\nu_x(\lambda) dx &> \int_{\Omega} \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \overline{\lambda} d\nu_x(\lambda) dx \\ &\geq \int_{\Omega} \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda d\nu_x(\lambda) dx,\end{aligned}$$

which is a contradiction. Therefore ν_x is a Dirac measure and we can write $\nu_x = \delta_{h(x)}$. Then

$$h(x) = \int_{\mathbb{M}^{m\times n}} \lambda d\delta_{h(x)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) = Du(x).$$

Hence $\nu_x = \delta_{Du(x)}$. By vertue of the Eq. (2.2), it follows that $Du_k \to Du$ in measure and almost everywhere. The continuity of the function a implies that $a(|Du_k|)Du_k \to a(|Du|)Du$ almost everywhere in Ω. Since $a(|Du_k|)Du_k$ is equiintegrable, the Vitali Theorem gives

$$\int_{\Omega} \Big(a(|Du_k|)Du_k - a(|Du|)Du \Big) : D\varphi dx \to 0 \quad \text{as} \quad k \to \infty.$$

The density of $\underset{k\geq 1}{\cup} W_k$ in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ implies that u is a weak solution of (1.1) and the proof of Theorem 1.1 is finished.

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