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Optimal pricing policy for stock dependent demand with effective investment in preservation technology

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Abstract

In this paper, we suggested an optimal pricing policy for deteriorating items. To reduce the rate of deterioration, we apply a preservation technology and calculate the optimal preservation technology investment. The demand function is dependent on time, stock and selling price. Shortages are allowed in our consideration, and two cases are studied, first complete back-ordering and the second one is partially back-ordering. Our main objective is to find the optimal cycle length, ordering frequency the optimal preservation technology investment and the optimal selling price that maximizes the total profit. This model proves that the total profit is a concave function of the selling price, ordering frequency, preservation technology investment and time cycle. Numerical examples are provided to illustrate the features and advances of the model. A sensitivity analysis is performed in order to assess the stability of the proposed model.

Keywords: Dynamic pricing, Time and stock dependent demand, Preservation technology investment. Controllable deterioration rate, Optimization
MSC: 90B05, 90B30, 90B50

1. Introduction

In classical inventory models, it is assumed that the items can be preserved for an infinite time without any change of their physical status. However, in reality, many items become partially or totally unusable a certain time period. In real life, the deterioration phenomenon is observed for inventory items such as fruits, vegetables, pharmaceuticals, volatile liquids, and others. In general, it

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is found that products always deteriorate continuously with respect to time, but deterioration can be controlled by applying some preservation technology. So in this paper, an optimal pricing policy is developed for deteriorating products by using the preservation technology that reduce the economic losses and improve the customer service.

The fundamental result in the development of economic order quantity model with deterioration is that of Ghare and Schrader [3] who considered an exponentially decaying inventory for a constant demand. However, as evident by chemical and basic sciences, the rate of deterioration especially with regard to perishable food items is seldom constant. Goyal and Giri [4] presented several tendencies of the modeling of deteriorating inventory.

Zauberman *et al.* [21] presented a method for color retention of Litchi fruits with SO2 fumigation. In order to reduce the deterioration rate and to extend the expiration date of the product, preservation technologies like procedural changes and specialized equipment acquisition have been mathematically modeled by many researchers. In recent years, deteriorating inventory problems have been widely studied by many researchers. As presented by Wee [16], deterioration was defined as decay, damage, spoilage, evaporation, obsolescence, pilferage, loss of utility or loss of marginal value of commodities that result in decreasing usefulness. Yang *et al.* [17] developed the trade off between preservation technology investment and the optimal dynamic trade credit for a deteriorating inventory model. Hsu *et al.* [5] designed a deteriorating inventory model considering constants deterioration and demand rates, where in preservation technology also included. Moreover, that model also considered the fixed cost of preservation technology which was independent of replenishment cycle length.

Although most of the researchers visualized that deterioration begins as soon as the items are produced or those are received by retailers, the reality reveals something different. In practice, most of the items begin deteriorate after a certain time, which is termed as 'non-instantaneous deterioration'. For example, fresh fruits or vegetables do not deteriorate during the early stage of storage. The time period after which deterioration would start plays an important role while setting optimal strategies. Therefore, many enterprises have studied the causes of deterioration and developed preservation technologies to control it and to increase the profit. However, the deterioration rate of inventories mentioned above is viewed as an exogenous variable, which is not subject to control. In practice, the deterioration rate of products can be controlled and reduced through various effects like procedural changes and specialized equipment acquisition.

Nevertheless, The deterioration rate in the research papers mentioned previously is addressed an exogenous variable, either constant or varying thro time, which is not under control. In fact, several firms have recognized that it is not worthy to manage and control the items deterioration rate and have implemented several activity. Furthermore, thro an effective investment in decreasing the deterioration rate, the firms avoid unnecessary waste and reduce obviously economic losses and hence improve their business competitiveness.

Optimization of the product portfolio has been recognized as a critical problem in industry, management, economy and so on. It aims at the selection of an optimal mix of the products to offer in the target market. So while solving some problems a multi objective integer non-linear constraint model was developed by Ahmadi and Nikabadi [1]. Having taken some realistic problem many researcher as Nadjafikhah [12]; Ezzati *et al.* [2]; Kazemi and Asl [6] presented some special model. Wu *et al.* [15] derived an optimal replenishment policy for items with non-instantaneous deterioration, stock-dependent demand and partial backlogging. Zhang *et al.* [19] proposed a pricing policies for deteriorating items with preservation technology investment without shortage and stock. Pal *et al.* [13] derived a deteriorating inventory model with stock and price-sensitive demand, where they assumed inflation and delay in payment. Shah and Shah [14] attempt the same problem. However, they could not prove the existence of the optimal solution analytically. Moreover, they considered deterioration to start from the very beginning of replenishment time. Mishra [11] developed an inventory model with controllable deterioration rate under time-dependent demand and time-varying holding cost. Liu *et al.* [9] provided joint dynamic pricing and investment strategy for foods perishing at a constant rate with price and quality dependent demand. Zhang *et al.* [20] studied the integrated supply chain model for deteriorating items, in this model both manufacturer and retailer cooperatively invest in preservation technology in order to reduce their deterioration cost under different realistic scenarios.

Table 1: A comparison of the present article with the existing literature on perservation technology

Authors	Demand pattern	Shortage	Deterioration	Preservation technology
Zhang et al. (2014)	Price dependent	No	Constant	Considered
Pal et al. (2014)	Price dependent	Yes	Constant	Not considered
Shah et al. (2014)	Price and stock dependent	Yes	Constant	Considered
Mishra et al. (2014)	Time-dependent	Yes	Non-insta -ntanous	Not considered
Liu et al. (2015)	Price dependent	No	Constant	Considered
Zhang et al. (2016)	Price dependent	No	Constant	Considered
Lu et al. (2016)	Price and stock dependent	No	Constant	Not considered
Khedlekar et al. (2016)	Price dependent	No	Constant	Considered
Mishra et al. (2017)	Price and stock dependent	Yes	Constant	Considered
This work	Price, stock and time dependent	Yes	Constant	Considered

Thereafter, Lu *et al.* [8] presented an inventory model, in which they suggested the joint dynamic pricing and replenishment policy for a deteriorating item under limited capacity. Khedlekar *et al.* [7] established an inventory model with declining demand under preservation technology investment.

In this paper, we have considered the constant deterioration rate and allowed the retailer to invest in preservation technology to reduce the deterioration. We have also considered the stock, time and price dependent demand scenario with linear form of dependence. Shortages are allowed and two cases are studied first is complete back-ordering and second is partial back-ordering as Mishra *et al.* [10]. The contribution of the present work with respect to the existing literature is shown in Table 1. The rest of the paper is designed as follows. Section 2 provides the notation and assumption used to formulate the model. The model is formulated and analyzed in section 3. Section 4 derives theoretical result and optimal solution. The proposed model is illustrated though some numerical in section 5. Section 6 does a sensitivity analysis. Finally, conclusions are made and future research direction is suggested in section 7.

2. Assumptions & Notations

We have designed the proposed model by using the following assumptions and notations.

Assumptions

1. This model is developed for a single item, a single supplier, a single manufacturer,
2. Shortage is allowed, and is mixture of partially backlogging and lost sales,
3. Lead time is zero,
4. The demand function $B(p,t)$ is a function of instantaneous stock level $I(t)$, time t and selling price p, which is defined as

$$B(p,t) = \begin{cases} a + b_1 t - bp + \beta I(t), & I(t) > 0 \\ a + b_1 t - bp, & I(t) \leq 0. \end{cases}$$

   where $0 \leq \beta \leq 1$ is stock dependent consumption rate parameter,
   a, $b > 0$, b, is real and $p > 0$,
5. The original rate of deterioration is constant, and is θ, where $0 \leq \theta \leq 1$,
6. The deterioration rate is reduced through investment in preservation technology α. The proportion of reduced deterioration rate is $\lambda(\alpha) = \lambda_0 e^{-\delta\alpha}$, where $\delta > 0$, and $\alpha > 0$ and $\lambda(\alpha)$ is a continuous, convex, decreasing function, where $\lambda(0) = \lambda_0$, $\lim_{\alpha\to\infty}\lambda(\alpha) = 0$. Note that $\lambda^{'}(\alpha) < 0$ and $\lambda^{''}(\alpha) > 0$,
7. The preservation technology is used for controlling the deterioration rate,
8. The planning horizon is finite.

Notations

The following notation are used throughout the paper

Parameters

$I(t)$ – On-hand inventory of product at time t,
t_1 – The time at which the inventory level reaches zero,
$\lambda(\alpha)$ – Reduced deterioration rate due to use of preservation technology,
λ_0 – Deterioration rate without preservation technology investment,
δ – Parameter of exponential distribution,
β – Stock dependent consumption rate parameter,
$B(p,t)$ – Demand rate function is a function of stock level $I(t)$, selling price p, and time t,
a – Demand scale parameter,
b – Price sensitive parameter,
b_1 – Time sensitive parameter,
c – The purchase cost per unit,
d – Deterioration cost per unit,
h – The inventory holding cost per unit per time unit,
η – Backordering parameter,

c_1 – The cost of lost sales per unit,
s – Shortage cost unit per unit time,
A – The ordering cost per order,
$S(\frac{T}{n})$ – Maximum shortage level for complete backordering,
$B_1(t)$ – Backorder level at any time t for partial partial backordering,
$L(t)$ – Number of lost sales at any time t,
$B_1(\frac{T}{n})$ – Maximum backorder level for partial backordering,
$\widehat{TP}$ – Total profit of the selling season,
Q – Ordering quantity,
D_τ – Total deteriorated items in interval $[0, t_1]$.

Decision variables

n – Ordering frequency,
p – Unit selling price,
α – Preservation technology cost per unit time for reducing the deterioration rate,
T – Inventory cycle length.

3. Mathematical Formulation

Given the assumptions and notations mentioned before, this inventory model has two cases: (i) with shortage and complete backordering and (ii) with shortages and partial backordering. The detailed derivation of both cases are given below.

Case I: The Inventory Model with Complete Backordering

In this section, we mathematically model the proposed inventory control problem. The inventory fluctuation over time is shown in figure 1. The figure also delineates the effect of preservation technology investment. The one curve indicates inventory level when preservation technology investment has applied while other one curve is for without preservation technology investment. Here, $I(t)$ represents the inventory level at any time in the interval $0 \leq t \leq \frac{T}{n}$. It is important to remark that the inventory level depleted due to demand and deterioration of product within the interval $0 \leq t \leq t_1$. At the time t_1 inventory level reaches at zero. Subsequently, shortage start considered to happen and total demand in the interval $[t_1, \frac{T}{n}]$ is entirely backordered. The variation of inventory with time t thus be described by the following differential equation:

$$\frac{dI(t)}{dt} = \begin{cases} -\big(a + b_1 t - bp + \beta I(t)\big) - \lambda(\alpha) I(t), & \text{if } 0 \leq t \leq t_1. \\ -\big(a + b_1 t - bp\big), & \text{if } t_1 \leq t \leq \frac{T}{n}. \end{cases} \tag{3.1}$$

Considering the boundary condition $I(t) = 0$ at $t = t_1$. The solution of Eq.(3.1) is

$$I(t) = \begin{cases} \frac{1}{\beta+\lambda(\alpha)^2}\Bigg[\bigg((\lambda(\alpha)+\beta)\big(a - bp + b_1 t_1\big) - b_1\bigg) e^{(t_1 - t)(\lambda(\alpha)+\beta)} \\ \qquad + \bigg(b_1 - (\lambda(\alpha)+\beta)\big(a - bp + b_1 t_1\big)\bigg) e^{(t - t_1)(\lambda(\alpha)+\beta)}\Bigg], & \text{if } 0 \leq t \leq t_1 \\ -\bigg(a + \frac{b_1}{2} - bp\bigg)\bigg(t^2 - t_1^2\bigg), & \text{if } t_1 \leq t \leq \frac{T}{n}. \end{cases} \tag{3.2}$$

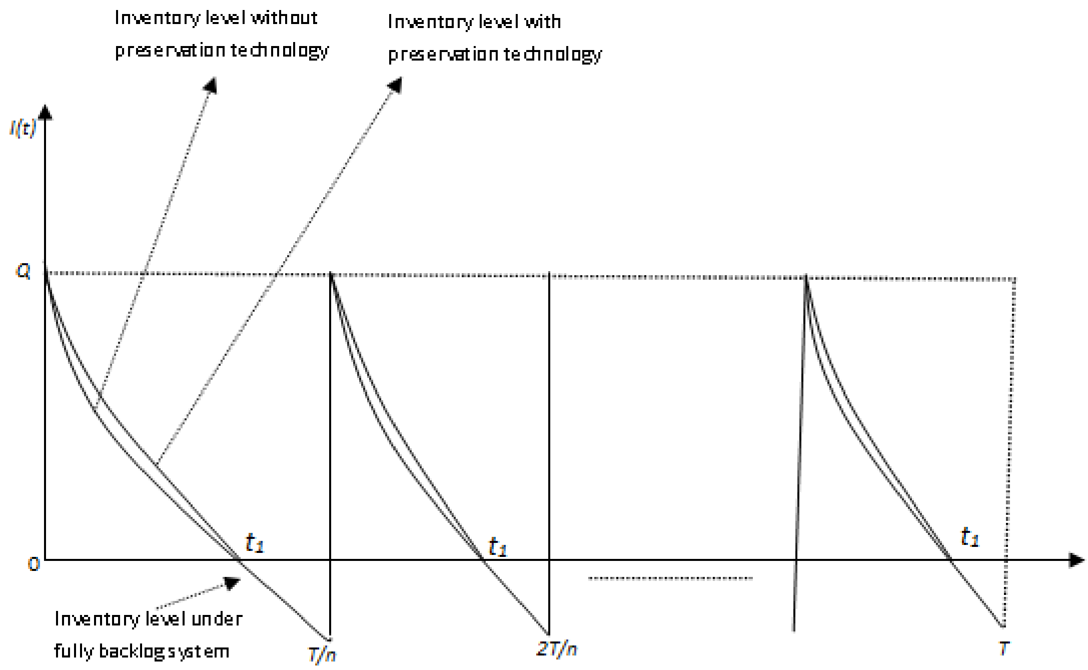

Figure 1: Logistic diagram of inventory system with complete backordering

The initial stock (S) for each cycle is calculated with

$$S = I(0) = \frac{1}{\beta + \lambda(\alpha)^2}\Bigg[\Big((\lambda(\alpha)+\beta)\big(a-bp+b_1t_1\big)-b_1\Big)e^{t_1(\lambda(\alpha)+\beta)} + \Big(b_1-(\lambda(\alpha)+\beta)\big(a-bp+b_1t_1\big)\Big)e^{-t_1(\lambda(\alpha)+\beta)}\Bigg] \tag{3.3}$$

Total number of items deteriorated during $[0, t_1]$

$$D_\tau = -t_1\big(a-bp+\frac{b_1}{2}t_1\big) + \frac{1}{\beta+\lambda(\alpha)^2}\Bigg[\Big((\lambda(\alpha)+\beta)\big(a-bp\big)+b_1\Big) + \Big((\lambda(\alpha)+\beta)\big(a-bp+b_1t_1\big)+b_1\Big)e^{t_1(\lambda(\alpha)+\beta)}\Bigg] \tag{3.4}$$

The order quantity per cycle is determined with

$$Q = \frac{T}{n^2}\big(an-bpn+\frac{b_1}{2}T\big) + \frac{1}{\beta+\lambda(\alpha)^2}\Bigg[\Big((\lambda(\alpha)+\beta)\big(a-bp\big)+b_1\Big) + \Big((\lambda(\alpha)+\beta)\big(a-bp+b_1t_1\big)+b_1\Big)e^{t_1(\lambda(\alpha)+\beta)}\Bigg] - t_1\big(a-bp+\frac{b_1}{2}t_1\big) \tag{3.5}$$

Now, to calculate the profit function, we calculate the following terms:
Sales Revenue

$$\widehat{SR} = pT\big(a-bp+\frac{b_1}{2n}T\big) \tag{3.6}$$

Purchase Cost

$$\begin{aligned}\widehat{PC} =& ncQ = \frac{cT}{n}\big(an - bpn + \frac{b_1}{2}T\big) + \frac{nh}{\beta+\lambda(\alpha)^2}\Bigg[\Big((\lambda(\alpha)+\beta)\big(a-bp\big)+b_1\Big)\\ &+\Big((\lambda(\alpha)+\beta)\big(a-bp+b_1t_1\big)+b_1\Big)e^{t_1(\lambda(\alpha)+\beta)}\Bigg] - nct_1\big(a-bp+\frac{b_1}{2}t_1\big)\end{aligned} \tag{3.7}$$

Holding Cost

$$\widehat{HC} = nh\int_0^{t_1} I(t)dt$$

$$\begin{aligned}\widehat{HC} =& \frac{nh}{(\lambda(\alpha)+\beta)^2}\Bigg[\big(bp-a\big)\Big\{1+(\lambda(\alpha)+\beta)t_1\Big\}+b_1(t_1-\frac{1}{2}t_1^2)\\ &\frac{1}{\beta+\lambda(\alpha)}\Big\{\Big((\lambda(\alpha)+\beta)\big(a-bp+b_1t_1\big)-b_1\Big)e^{t_1(\lambda(\alpha)+\beta)}\Big\}\Bigg]\end{aligned} \tag{3.8}$$

Shortage Cost

$$\widehat{SC} = ns\int_{t_1}^{\frac{T}{n}} I(t)dt \tag{3.9}$$

$$\widehat{SC} = -\frac{1}{6n^2}\Bigg[s(T-nt_1)^2\Big(3n(a-bp)+b_1(T+2nt_1)\Big)\Bigg] \tag{3.10}$$

Deterioration Cost

$$\begin{aligned}\widehat{DC} =& nd\Bigg[(\frac{T}{n}-t_1)\big(a-bp\big)+\frac{b_1}{2}(\frac{T}{n}-t_1)+\frac{1}{\beta+\lambda(\alpha)^2}\Big\{\Big(b_1-(\lambda(\alpha)+\beta)\big(a-bp\big)\Big)\\ &+\Big((\lambda(\alpha)+\beta)\big(a-bp+b_1t_1\big)-b_1\Big)e^{t_1(\lambda(\alpha)+\beta)}\Big\}\\ &+\frac{1}{\beta+\lambda(\alpha)^3}\Big\{2\beta\Big(\big(\lambda(\alpha)+\beta\big)\big(a-bp\big)-b_1\Big)e^{t_1(\lambda(\alpha)+\beta)}\\ &+\Big(2(a-bp)+b_1t_1\Big)\big\{\ \lambda(\alpha)t_1\big(\beta^2+2\beta\lambda(\alpha)+\lambda(\alpha)^2\big)\big\}+2\beta^2(bp-a)\Big\}\Bigg]\end{aligned} \tag{3.11}$$

Ordering Cost

$$\widehat{OC} = nA \tag{3.12}$$

Preservation Technology Cost

$$\widehat{PTC} = \alpha T \tag{3.13}$$

Therefore, the total profit of the inventory system is expressed by

$$\widehat{TP} = \widehat{SR} - \Big(\widehat{PC}+\widehat{HC}+\widehat{SC}+\widehat{DC}+\widehat{OC}+\widehat{PTC}\Big) \tag{3.14}$$

Using $t_1 = \frac{\gamma T}{n}$, $0 < \gamma < 1$, and for the small value of x, the Taylor series that the exponential function has a approximation of $e^x \approx 1 + x + \frac{x^2}{2!}$. Using this result and simplifying, we obtained the following profit function:

$$\widehat{TP}(n,\alpha,p,T) = -\frac{1}{6n^2}\Big[6An^3 + T\Big\{6n^2\alpha + 3bnp\Big(2n(p-c) + T\big(s - 2s\gamma - (h-s+c\beta)\gamma^2\big)\Big) + 2an\Big(2n(c-p) - sT + 2sT\gamma + T(h-s+c\beta)\gamma^2\Big)\Big\} + b_1T\Big\{3n(c-p) - sT + 3sT\gamma^2 + T\big(3h-2s+3c\beta\big)\gamma^3\Big\} + 3d\Big\{T\Big(-n(\gamma-1)\big(2n(a-bp) + b_1T(1+\gamma)\big)\Big) + T\gamma^2\big(n(a-bp)+b_1T\gamma\big)\lambda(\alpha)\Big\} + 3cT^2\gamma^2\big(n(a-bp)+b_1T\gamma\big)\lambda(\alpha)\Big] \tag{3.15}$$

Case II: The Inventory Model with Partial Backordering

In this case the differential equations which represent the inventory are expressed below;

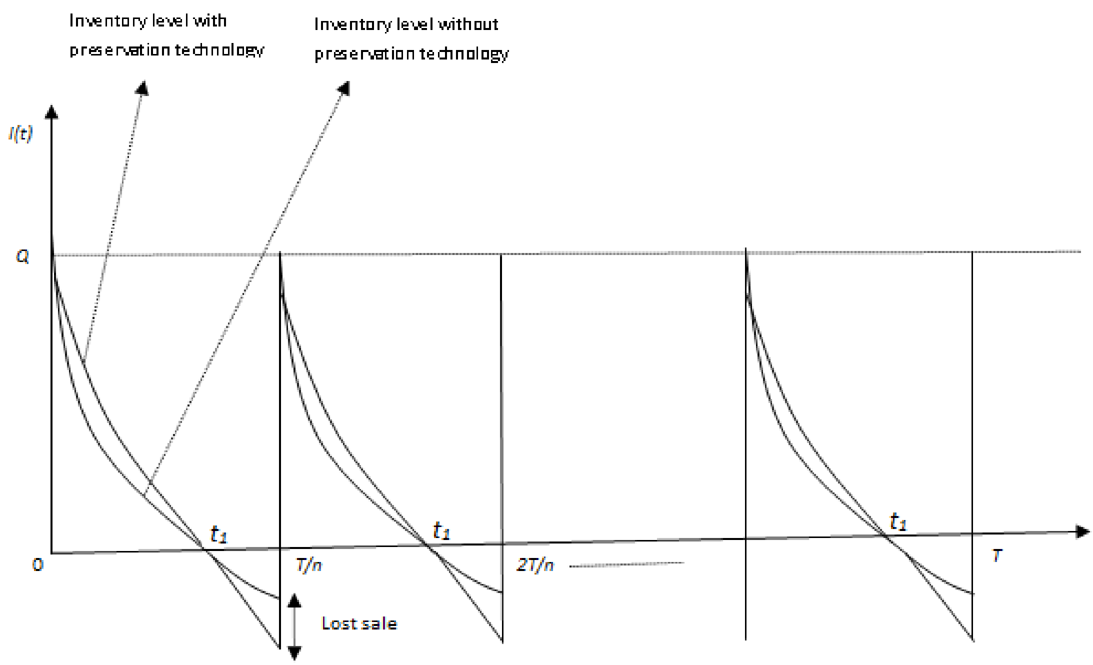

Figure 2: Logistic diagram of inventory system with partial backordering

$$\frac{dI(t)}{dt} = \begin{cases} -\big(a + b_1t - bp + \beta I(t)\big) - \lambda(\alpha)I(t), & \text{if } 0 \le t \le t_1 \\ \big(a + b_1t - bp\big)e^{-\eta(\frac{T}{n}-t)}, & \text{if } t_1 \le t \le \frac{T}{n}. \end{cases} \tag{3.16}$$

In this case the first differential equation representing inventory level during period $[0, t_1]$ is similar to case I.

Now, we solve the second differential equation:

$$\frac{dI(t)}{dt} = \big(a + b_1t - bp\big)e^{-\eta(\frac{T}{n}-t)}, \quad t_1 \le t \le \frac{T}{n}. \tag{3.17}$$

with the boundary condition $I(t_1) = 0$. Solving the differential equation, we get the inventory level as follows:

$$I(t) = \frac{1}{\eta^2}\left[\left(b_1 - \eta\big(a - bp + b_1 t\big)\right)e^{(t-\frac{T}{n})\eta} + \left(\eta\big(a - bp + b_1 t_1\big) - b_1\right)e^{(t_1-\frac{T}{n})\eta}\right] \tag{3.18}$$

The number of lost sales at time t is given by

$$L(t) = \int_{t_1}^{t} \big(a + b_1 t - bp\big)\left\{1 - e^{(t-\frac{T}{n})\eta}\right\}dt \tag{3.19}$$

$$L(t) = \frac{1}{\eta^2}\left[\left(\eta\big(bp - a - b_1 t\big) + b_1\right)e^{(t-\frac{T}{n})\eta} + \left(\eta\big(a - bp - b_1 t_1\big) - b_1\right)e^{(t_1-\frac{T}{n})\eta} + b_1\big(t^2 - t_1^2\big)\eta^2\right] \tag{3.20}$$

Putting $t = \frac{T}{n}$ into Eq. (3.24), the maximum backorder level per cycle is

$$I(\frac{T}{n}) = \frac{1}{\eta^2}\left[\left(b_1 - \eta\big(a - bp + b_1\frac{T}{n})\right) + \left(\eta\big(a - bp + b_1 t_1\big) - b_1\right)e^{(t_1-\frac{T}{n})\eta}\right] \tag{3.21}$$

Hence, the order quantity over the replenishment cycle is determined

$$Q = S + I\left(\frac{T}{n}\right) \tag{3.22}$$

$$\begin{aligned} Q =& \frac{1}{\beta + \lambda(\alpha)^2}\left[\left((\lambda(\alpha) + \beta)\big(a - bp + b_1 t_1\big) - b_1\right)e^{t_1(\lambda(\alpha)+\beta)}\right.\\ &\left. + \left(b_1 - (\lambda(\alpha) + \beta)\big(a - bp + b_1 t_1\big)\right)e^{-t_1(\lambda(\alpha)+\beta)}\right] + \frac{1}{\eta^2}\left[\left(b_1 - \eta\big(a - bp + b_1\frac{T}{n})\right)\right.\\ &\left. + \left(\eta\big(a - bp + b_1 t_1\big) - b_1\right)e^{(t_1-\frac{T}{n})\eta}\right] \end{aligned} \tag{3.23}$$

The shortage Cost

$$\widehat{SC} = ns\int_{t_1}^{\frac{T}{n}} I(t)dt \tag{3.24}$$

$$\begin{aligned} \widehat{SC} = \frac{s}{\eta^3}&\left[\left\{\eta n\big(t_1^2\eta + T - 2t_1\big) + 2n + t\eta - n\eta\big(1 + T - t_1\eta\big)\big(a - bp\big)\right\}e^{(t_1-\frac{T}{n})\eta}\right.\\ &\left. + \big(a - bp\big)n\eta + b_1\big(T\eta - 2n\big)\right] \end{aligned} \tag{3.25}$$

The Lost Sale Cost

$$\widehat{LSC} = nc_1\int_{t_1}^{\frac{T}{n}} \big(a + b_1 t - bp\big)\left\{1 - e^{(t-\frac{T}{n})\eta}\right\}dt$$

$$\begin{aligned} \widehat{LSC} = \frac{c_1}{2n\eta^3}&\left[\left\{2\eta n^2\big(a - bp\big) - b_1\big(1 - t_1\eta\big)\right\}e^{(t_1-\frac{T}{n})\eta}\right.\\ &\left. 2n\eta\big(a - bp\big)\left\{n - \eta(1 - n)\right\} - b_1\left\{2n(T\eta - n) - \eta^2(T^2 - t_1^2)\right\}\right] \end{aligned} \tag{3.26}$$

Therefore, the total profit of the inventory system is expressed by

$$\widehat{TP} = \widehat{SR} - \left(\widehat{PC} + \widehat{LSC} + \widehat{HC} + \widehat{SC} + \widehat{DC} + \widehat{OC} + \widehat{PTC}\right) \tag{3.27}$$

Using $t_1 = \frac{\gamma T}{n}$, $0 < \gamma < 1$, and for the small value of x, the Taylor series establishes that the exponential function has a approximation of $e^x \approx 1 + x + \frac{x^2}{2!}$. Using this result and amplifying, it is obtained the following profit function:

$$\begin{aligned}\widehat{TP}(n,\alpha,p,T) = \frac{1}{4n^5\eta^2}\Bigg[& n\eta\Bigg\{ -4n^4sT^2(a-bp)(\gamma-1) - 2n^3\eta\Big(2An^2 + T\big(2n\alpha - 2(a-bp) \\ & (2sT + np - cn) + 4sT\gamma(a-bp) + T\gamma^2(a-bp)(h+c\beta)\big)\Big) - 2n(a-bp)T^2 \\ & (4sT + c_1n(\gamma-1))(\gamma-1)\eta^2 - 4nsT^4(\gamma-1)\eta^2 - sT^5\eta^4(a-bp)(\gamma-1)\Bigg\} \\ & - b_1T\Bigg\{4n^4sT\eta(4+(\gamma-5)\gamma) - 8n^5s(\gamma-1) + 2n^3T\big(c(n+T\beta\gamma^3) + T(h\gamma^3 \\ & + 4s(2(\gamma-3)\gamma))\big)\eta^2 + 2n^2T^2(\gamma-1)^2(4sT + c_1n\gamma)\eta^3 + 2nsT^4(1+\gamma(2\gamma-3))\eta^4 \\ & + sT^5(\gamma-1)\gamma\eta^5\Bigg\} - 2n^3\eta^2\Bigg\{d\Big(T\big(-n(\gamma-1)(2an - 2bnp + b_1T(1+\gamma)\big) \\ & + T\gamma^2(an - bnp + b_1T\gamma)\lambda(\alpha)\Big) + cT^2\gamma^2(an - bnp + b_1T\gamma)\lambda(\alpha)\Bigg\}\Bigg]\end{aligned} \tag{3.28}$$

4. Theoretical Result and Optimal Solution

This section determines the optimal values of selling price p, preservation technology cost α, ordering frequency n and inventory cycle length T which maximizes the total profit $\widehat{TP}(n,\alpha,p,T)$. We also established some theorems which shows the concavity of retailer's total profit $\widehat{TP}(n,\alpha,p,T)$.

4.1. The EOQ Inventory Model with Complete Backordering

Theorem 4.1. *When preservation technology α, selling price p and inventory cycle length T are fixed, then the profit function $\widehat{TP}(n,\alpha,p,T)$ is concave with respect to ordering frequency n.*

Theorem 4.2. *There exists a unique value of selling price p that maximizes profit function $\widehat{TP}(n,\alpha,p,T)$ for fixed values of preservation technology α, inventory cycle length T and ordering frequency n.*

Proof. The first and second order partial derivatives of the total profit function $\widehat{TP}(n,\alpha,p,T)$ given by Eq. (3.15) with respect to p are given below:

$$\begin{aligned}\frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial p} = & \frac{T}{2n}\Bigg[2an + b_1T + b(c+d)T\gamma^2\lambda(\alpha) + b\Bigg\{2n(c+d-2p) - sT - 2dn\gamma \\ & + 2sT\gamma + (T(h-s+c\beta)\gamma^2\Bigg\}\Bigg]\end{aligned} \tag{4.1}$$

Set $\frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial p} = 0$, and solve it for the optimal p^*

$$p^* = \frac{1}{4bn}\Bigg[2an + b_1T + b(c+d)T\gamma^2\lambda(\alpha) + b\Big\{2n(c+d) - sT - 2dn\gamma + 2sT\gamma + T(h-s+c\beta)\gamma^2\Big\}\Bigg] \tag{4.2}$$

$$\frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial p^2} = -2bT < 0 \tag{4.3}$$

Hence, p^* is global optimal that optimizes the profit function $\widehat{TP}(n,\alpha,p,T)$ given by Eq. (3.15) for fixed values of preservation technology α, inventory cycle length T and ordering frequency n.

This completes the proof of Th. (4.2).

Theorem 4.3. *When ordering frequency n, selling price p and inventory cycle length T are fixed, then the profit function $\widehat{TP}(n,\alpha,p,T)$ is concave with respect to preservation technology α.*

Proof. The second order partial derivative of the total profit function $\widehat{TP}(n,\alpha,p,T)$ given by Eq. (3.15) with respect to α, is given below:

$$\frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial \alpha^2} = -\frac{(c+d)e^{-\alpha\delta}T\gamma^2\big(an - bnp + b_1T\gamma\big)\delta^2\lambda_0}{2n^2} < 0 \tag{4.4}$$

Set $\frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial \alpha} = 0$ and solve it for the optimal α^*
clearly, from Eq. (4.5) it is concluded that the profit function given Eq. (3.15), is concave with respect to α.

Theorem 4.4. *When preservation technology α, selling price p and ordering frequency n are fixed, then the profit function $\widehat{TP}(n,\alpha,p,T)$ is concave with respect to inventory cycle length T.*

Proof. The second order partial derivative of the total profit function $\widehat{TP}(n,\alpha,p,T)$ given by Eq. (3.15) with respect to T, is given below:

$$\frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial T^2} = \frac{e^{-\alpha\delta}}{n^2}\Bigg[e^{\alpha\delta}\Bigg\{n(a-bp)\big(s\big((\gamma-1)^2 - (h+c\beta)\gamma^2\big) + b_1\Bigg(sT - n(c+d-p) + (dn-3sT)\gamma^2 + T(2s-3h-3c\beta)\gamma^3\Bigg)\Bigg\} - (c+d)\gamma^2(an - bnp + 3b_1T\gamma)\lambda_0\Bigg] \tag{4.5}$$

Now, we check with help of Mathematica for any n that,

$$\frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial T^2} < 0 \tag{4.6}$$

Therefore, it is concluded that the profit function given by Eq. (3.15), is concave function in T.

This completes the proof of Th. (4.4).

Theorem 4.5. *When ordering frequency n is fixed, the profit function $\widehat{TP}(n,\alpha,p,T)$ is concave if the corresponding Hessian matrix H of expected profit function is negative definite. Where*

$$H=\begin{pmatrix} \frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial p^2} & \frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial p\partial\alpha} & \frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial p\partial T} \\ \frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial\alpha\partial p} & \frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial\alpha^2} & \frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial\alpha\partial T} \\ \frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial T\partial p} & \frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial T\partial\alpha} & \frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial T^2} \end{pmatrix}$$

Proof. Differentiating Eq. (3.15) with respect to p, α, T, we have

$$\begin{aligned} \frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial p} =& \frac{T}{2n}\Bigg[2an + b_1T + b(c+d)T\gamma^2\lambda(\alpha) + b\Big\{2n(c+d-2p) - sT - 2dn\gamma \\ & + 2sT\gamma + (T(h-s+c\beta)\gamma^2\Big\}\Bigg] \\ \frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial\alpha} =& \frac{T}{2n^2}\Big[(c+d)e^{-\alpha\delta}T\gamma^2\big(an - bnp + b_1T\gamma\big)\delta\lambda_0 - 2n^2\Big] \\ \frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial T} =& \frac{e^{-\alpha\delta}}{2n^2}\Bigg[e^{\alpha\delta}\Big\{-2n^2\alpha + 2nbp\big(n(c+d-p) - sT - dn\gamma + 2sT\gamma \\ & + T(h-s+c\beta)\gamma^2\big) + b_1T\Big(sT - 2n(c+d-p) + (2nd-3sT)\gamma^2 \\ & + T(2s-3h-3c\beta)\gamma^3\Big) + 2an\Big(np + dn(\gamma-1) + T\big(s(\gamma-1)^2 - h\gamma^2\big) \\ & - c\big(n + T\beta\gamma^2\big)\Big)\Big\} - (c+d)T\gamma^2(2an - 2bnp + 3b_1T\gamma)\lambda_0\Bigg] \end{aligned}$$

Using the first-order condition of classical optimization, we solve p, α, and T from the equations $\frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial p} = 0$, $\frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial\alpha} = 0$, and $\frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial T} = 0$

Now, $\frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial p} = 0$ gives

$$\begin{aligned} p^* =& \frac{1}{4bn}\Bigg[2an + b_1T + b(c+d)T\gamma^2\lambda(\alpha) + b\Big\{2n(c+d) - sT - 2dn\gamma \\ & + 2sT\gamma + T(h-s+c\beta)\gamma^2\Big\}\Bigg] \end{aligned} \tag{4.7}$$

substituting the value of p in the equations $\frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial\alpha}$, and $\frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial T}$ and then solving them, we get the solution of decision variable p, α, T of the model.
The solutions will be optimal if the second-order condition of optimization method will be satisfied. Now we find the second order derivatives and putting the value in Hessian matrix the solution will be optimal if the corresponding Hessian matrix of profit function is negative definite, i.e. if all the eigenvalues of the Hessian matrix, are negative then the profit function is concave.

We verified the above Th. (4.5) in Ex. (1).

4.2. The EOQ Inventory Model with Partial Backordering

Theorem 4.6. *When preservation technology α, selling price p and inventory cycle length T are fixed, then the profit function $\widehat{TP}(n,\alpha,p,T)$ is concave with respect to ordering frequency n.*

Theorem 4.7. *There exists a unique value of selling price p that maximizes profit function $\widehat{TP}(n,\alpha,p,T)$ for fixed values of preservation technology α, inventory cycle length T and ordering frequency n.*

Proof. The second order partial derivative of the total profit function $\widehat{TP}(n,\alpha,p,T)$ given by Eq. (3.28) with respect to p, is given below: Set $\frac{\partial(\widehat{TP}(n,\alpha,p,T))}{\partial p} = 0$ and solve it for the optimal p^*

$$\frac{\partial^2(\widehat{TP}(n,\alpha,p,T))}{\partial p^2} = -2bT < 0 \tag{4.8}$$

Hence, p^* is global optimal that optimizes the profit function $\widehat{TP}(n,\alpha,p,T)$ given by Eq. (3.28) for fixed values of preservation technology α, inventory cycle length T and ordering frequency n.

This completes the proof of Th. (4.2).

5. Illustrative Example

In this section, we presented some numerical examples to illustrate the mathematical formulation. The numerical examples corresponds to complete and partial backordering considering a demand function.

5.1. Illustrative Example for Completely Backordering

Example 1. We consider the values of the parameters in appropriate units as follows: $A = 400$ per order, $a = 300$ units, $b = 5$, $c = 10$ per unit, $d = 5$ per unit, $h = 0.2$ per unit/week, $s = 2$ per unit/week, $\beta = 0.5$, $\gamma = 0.9$, $\lambda_0 = 0.001$, $\delta = 0.8$, $b_1 = 20$. For different value of n, the optimal profit is computed. Note that when $n = 5$, then the optimal results for the model are $\widehat{TP} = 116451$, $Q^* = 8351$, $\alpha^* = 2.39$, $p* = 50.53$, $T = 59.85$ weeks. The above results are optimal as the eigenvalues of the Hessian matrix are -644.79, -91.13, -47.84. So the profit

function is concave and unimodal function.

Example 2. Consider the following parameter values $T = 80$ weeks, $A = 400$ per order, $a = 30$ units, $b = 0.3$ $c = 15$ per unit, $d = 5$ per unit, $h = 0.2$ per unit/week, $s = 2$ per unit/week, $\beta = 0.5$, $\gamma = 0.9$, $\lambda_0 = 0.002$, $\delta = 0.9$, $b_1 = 5$. For distinct value of 9, the optimal profit is computed. As it shown in Table 2, the profit function is concave with respect to ordering frequency n. Note that when $n = 8$, the profit function attains its maximum. Hence the maximum value for profit is $\widehat{TP} = 58960.1$. Figure 3 schemes the total profit with respect to preservation cost α and selling price p. It is easy to see that the profit function $\widehat{TP}$ is jointly concave in preservation cost α and selling price p.

Table 2: The optimal solution is a global maximum at $n = 9$

n	p^*	α^*	Q^*	$\widehat{TP}^*$
6	134.04	2.52	32284	50313.50
7	123.14	2.25	11924	55743.30
8	114.97	1.97	5578	58131.20
9	**108.61**	**1.75**	**3062**	**58960.10**
10	103.53	1.56	1882	58951.54
11	99.37	1.38	1257	58481.60
12	95.90	1.32	894	57754.50

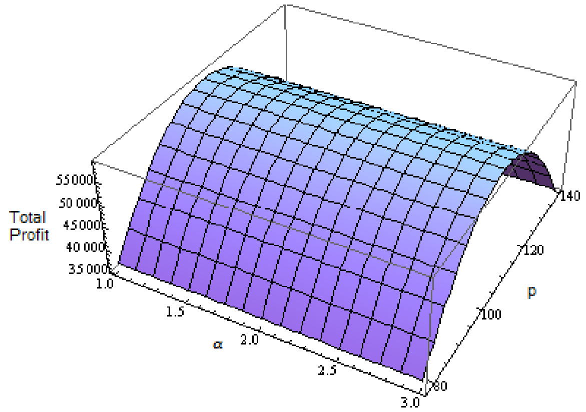

Figure 3: The total profit function with respect to preservation cost α and selling price p

5.2. Illustrative Example for Partial Backordering

Example 3. Consider the the following parameter values $T = 80$ weeks, $A = 400$ per order, $a = 70$ units, $b = 0.9$ $c = 10$ per unit, $d = 5$ per unit, $h = 0.02$ per unit/week, $s = 2$ per unit/week, $\beta = 0.5$, $\gamma = 0.9$, $\lambda_0 = 0.001$, $\delta = 0.9$, $b_1 = 20$, $\eta = 0.1$, $c_1 = 25$. For distinct value of n, the optimal profit is computed. As it shown in Table 3, the profit function is concave with respect to ordering frequency n. Notice that when $n = 9$, the profit function attains its maximum value. Hence the maximum value for profit is $\widehat{TP} = 66017.2$. Figure 4 shows the total profit with respect to preservation cost α and selling price p. It is easy to see that the profit function $\widehat{TP}$ is jointly concave in preservation cost α and selling price p.

6. Sensitive Analysis

The total profit function represents real solutions in which the system parameters are considered as static values. We observed the sensitive of the key parameters which help to decision makers to take the best decisions with respect to the situation at hand and how the objects change. For this purpose, we studies the effects of changes in the inventory system parameters (a, b, c) having changed each of the parameters by $\pm 30\%$ and $\pm 15\%$ taking one parameter at a time and keeping the remaining parameters unchanged.

Based on the result of table 4, the following managerial insights are provided

(a) When the value of scaling parameter a increases and other parameters values are fixed, then it can be observed that the optimal total profit per unit time $\widehat{TP}^*$, the optimal selling price p^*, the optimal order quantity Q^* and optimal the optimal preservation cost α^* increase. This shows that when the scaling factor a increase, then the market demand rate will increase, making that the firms increases the lot size per replenishment cycle. Furthermore, the firm will place the selling price higher to obtain more profit. The firm will also invest more capital into the improvement of preservation technology to reduce the deterioration rate.

(b) When the price elasticity b increases, then it can be observed that the optimal total profit per unit time $\widehat{TP}^*$, the optimal selling price p^*, the optimal order quantity Q^* and optimal the optimal preservation cost α^* decreases. It suggests that as price elasticity b increases, the firm will bring down the selling price to avoid the demand rate decreases dramatically. By reducing the selling price, the order quantity will shrink and due to this, the total profit unit per time

Table 3: Effect of changes of parameters on optimal solution for Ex. 2 for complete backloging

Parameters	% Change	p^*	α^*	Q^*	$\widehat{TP}^*$
a	–30	93.63	1.61	2575	17232
	–15	101.11	1.68	2818	36745.9
	+15	116.11	1.81	3305	83874.6
	+30	123.61	1.88	3548	111489
b	–30	145.91	1.81	3272	133526
	–15	123.97	1.78	3167	89367.8
	+15	97.26	1.72	2957	36922.2
	+30	88.53	1.69	2852	20357
c	–30	102.31	1.52	3269	91249.5
	–15	105.46	1.64	3165	74865.9
	+15	111.76	1.84	2959	43531.7
	+30	114.91	1.92	2856	28580.37

will also decrease significantly. Further, the firm has an incentive to increase the preservation technology investment.

(c) With the increment in the value of the buying cost c, then the preservation cost α^*, the optimal selling price p^* increases, while the optimal order quantity Q^* and optimal profit per unit time $\widehat{TP}^*$, decrease, and the preservation cost is also increasing. This implies that when the buying cost and initial deterioration rate are large then the retailer must spend more in order to reduce the deterioration cost.

7. Concluding Remarks and Future Research indications

This model presented an optimal pricing policy for non-instantaneously deteriorating item with preservation technology investment with selling price, time and stock depend demand. Here, some useful properties characterizing the optimal solution are formulated. The proposed model is illustrated through some numerical examples and the optimal cycle length, selling price and investment for preservation are obtained. Our sensitivity analysis reveals that when the cost of preservation technology increases then the profit of the system also increases accordingly. Furthermore, it has been shown that if the deterioration rate increases, the amount of investment also increases. If the deterioration is marginal or less in value at the initial level. The numerical result demonstrates that spending in preservation technology substantially aids decision makers in developing a competitive advantage and increases their total profit. The properties provide managerial insight towards determining optimal policies with a changed market scenario.

Two proposed economic order quantity inventory models help the manufacturer and retailers in determining the optimal ordering frequency, pricing, preservation technology investment, cycle length, order quantity and profit.

The present model may be extended in various ways. Regarding further research, we may take care of multiple items instead of a single item under stochastic demand. For addressing a realistic situation, one can design an extended model by introducing warehouse, storage facility, stochastic inflation, quantity and price discounts. Also one can formulate the model by considering time-dependent deterioration rate and uncertainty of demand.

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