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# Projection and multi-projection methods for second kind Volterra-Hammerstein integral equation

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**Abstract**

In this article, we discuss the piecewise polynomial based Galerkin method to approximate the solutions of second kind Volterra-Hammerstein integral equations. We discuss the convergence of the approximate solutions to the exact solutions and obtain the orders of convergence $\mathcal{O}(h^r)$ and $\mathcal{O}(h^{2r})$, respectively, for Galerkin and its iterated Galerkin methods in uniform norm, where $h$, $r$ denotes the norm of the partition and smoothness of the kernel, respectively. We also obtain the superconvergence results for multi-Galerkin and iterated multi-Galerkin methods. We show that iterated multi-Galerkin method has the order of convergence $\mathcal{O}(h^{3r})$ in the uniform norm. Numerical results are provided to demonstrate the theoretical results.

*Keywords:* Volterra-Hammerstein integral equations, Galerkin method, Multi-Galerkin method, Piecewise polynomials, Superconvergence rates.
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## 1. Introduction

Nonlinear Volterra integral equation of the second kind often occur in Hammerstein form,

$$u(x) - \int_0^x \ell(x,\vartheta)\psi(\vartheta, u(\vartheta))\, d\vartheta = g(x), \quad 0 \le x \le 1, \tag{1.1}$$

where the functions $\ell$, $g$ and $\psi$ are known, $u$ is the unknown function to be determined in the Banach space $\mathbb{X}$. Several problems in engineering, biology, and physics are modeled by these type of

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integral equations (1.1). For example, they are models of population dynamics, epidemic diffusion, reaction-diffusion in small cells, and in general of evolutionary phenomena incorporating memory. In general, finding the exact solution of the integral equations by analytical methods is very difficult, and it is usually very useful to find a numerical approximation of the exact solution. There are many projection methods for enhancing the accuracy of the numerical approximate solution. Several authors have investigated on the approximate solution of these type of equations (1.1) (see [2], [3], [4], [8], [15]). G. N. Elnagar et. al. [7] proposed the Chebyshev spectral method for obtaining the approximate solution of integral equations of type (1.1) and they present a numerical experiment, which verify the convergence, applicability and the accuracy of the Chebyshev spectral method. K. Maleknejad et. al. [15] introduced a fixed point method to approximate the solution of integral equations of type (1.1) and obtained error bounds for the corresponding approximation. F. Ghoreishi et. al. [8] proposed a variation of Tau method for the numerical solution of the integral equations of type (1.1) based on arbitrary polynomial basis functions and numerical results were provided using proposed Tau algorithm based on the standard, Legendre and Chebyshev basis functions. Authors proposed ([12], [13]) a new type of collocation method in certain piecewise-polynomial spaces to establish superconvergence results for Fredholm-Hammerstein integral equations and in [2], H. Brunner applied this method to integral equations of type (1.1) and showed that the convergence rate of linear collocation solution is of order $\mathcal{O}(h^{2r})$ at the knots, where $h$, $r$ are the norm of the partition and smoothness of the kernel, respectively. M. Mandal et. al. ([17], [20]) proposed the Galerkin and iterated Galerkin methods to find the approximate solution of the second kind linear Volterra integral equations and obtained the superconvergence results in the infinity norm. In ([18], [20]) these results were enhanced to second kind nonlinear Volterra integral equations for obtaining the same superconvergence results.

In this article, the Galerkin method is applied to solve Volterra-Hammerstein integral equation (1.1) with a smooth kernel in the piecewise polynomial space of degree at most $(m-1)$ and obtain the superconvergence. We will prove that the Galerkin method and iterated Galerkin method has order of convergence $\mathcal{O}(h^m)$ and $\mathcal{O}(h^{2m})$, respectively in uniform norm, $h$ is denoted as the norm of the partition.

In ([6], [9], [11], [14], [16], [22]), multi-projection (multi-Galerkin and multi-collocation) methods were discussed to solve linear and nonlinear Fredholm integral equations of second kind. In ([17], [18]), multi-Galerkin and its iterated multi-Galerkin methods were proposed to solve second kind linear and Volterra-Urysohn integral equations and found the superconvergence results. In this article, we also discuss the multi-Galerkin method and its iterated multi-Galerkin for Volterra-Hammerstein integral equations (1.1) to enhance the order of convergence further. We will show that the presented multi-Galerkin and its iterated version i.e., the iterated multi-Galerkin methods have order of convergence $\mathcal{O}(h^{2m})$ and $\mathcal{O}(h^{3m})$ in uniform norm, respectively, under appropriate assumptions on the right hand side function $g$, the kernel $l(.,.)$ and the solution $u$,. We observe that the iterated Galerkin solutions and multi-Galerkin solutions have the same order of convergence, but the iterated multi-Galerkin solutions improve over the iterated Galerkin and multi-Galerkin solutions. Our theoretical results are demonstrated by numerical results.

This paper is organized as follows. In Sec 2, the Galerkin method and iterated Galerkin method are applied to the equation (1.1) and study the convergence analysis. In Sec 3, multi-Galerkin method and iterated multi-Galerkin methods are discussed to obtain the improved convergence results. In Sec 4, numerical results are provided to demonstrate the theoretical results. We assume that $c$ is a generic constant, throughout this paper.

## 2. Convergence analysis by Galerkin method for Volterra-Hammerstein integral equations

Let $\mathbb{X} = L^\infty[0,1]$. Let us consider the Volterra-Hammerstein integral equation stated by the equation (1.1).
Consider a transformation $\vartheta(.,.) : ([0,1]\times[0,1]) \to [0,1]$, by taking $\vartheta = x\lambda,\ (x,\lambda) \in ([0,1]\times[0,1])$, the Volterra integral equation (1.1) transforms

$$u(x) - \int_0^1 \tilde{\ell}(x,\vartheta(x,\lambda))\psi(\vartheta(x,\lambda), u(\vartheta(x,\lambda)))\, d\lambda = g(x),\ x \in [0,1], \tag{2.1}$$

where $\tilde{\ell}(x,\vartheta(x,\lambda)) = x\ell(x,\vartheta(x,\lambda))$.
Define

$$(\mathcal{L}\psi)(u)(x) = \int_0^1 \tilde{\ell}(x,\vartheta(x,\lambda))\psi(\vartheta(x,\lambda), u(\vartheta(x,\lambda)))\, d\lambda,\ u \in \mathbb{X}.$$

Then we can write the above equation (2.1) in operator form as

$$u - (\mathcal{L}\psi)(u) = g. \tag{2.2}$$

The Fr$\acute{e}$chet derivative $(\mathcal{L}\psi)'(u)$ is stated by

$$((\mathcal{L}\psi)'(u))v(x) = \int_0^1 \tilde{\ell}(x,\vartheta(x,\lambda))\psi^{(0,1)}(\vartheta(x,\lambda), u(\vartheta(x,\lambda)))v(\vartheta(x,\lambda))\, d\lambda, \quad v \in \mathbb{X},$$

where $\psi^{(0,1)}(\vartheta(x,\lambda), u(\vartheta(x,\lambda))) = \frac{\partial}{\partial u}\psi(\vartheta(x,\lambda), u(\vartheta(x,\lambda)))$, the partial derivative of $\psi$ w.r.t. u.
For any $v \in \mathcal{C}^m[0,1]$, denote

$$\|v\|_{m,\infty} = \max\{\|v^{(i)}\|_\infty : 0 \le i \le m\},$$

where $v^{(i)}$ is the $i$-th derivative of $v$.
In the rest of article, we consider the following assumptions on $g$, $\tilde{\ell}(.,.)$ and $\psi(.,u(.))$:

(i) $g \in \mathcal{C}^m[0,1], \quad m \ge 1$.

(ii) $\tilde{\ell}(x,\vartheta(x,\lambda)) \in L^\infty([0,1]\times[0,1]), \qquad \tilde{M} = \|\tilde{\ell}\|_{L^2} = [\int_0^1 |\tilde{\ell}(x,\vartheta(x,\lambda))|^2]^{\frac{1}{2}} < \infty.$

(iii) $\tilde{\ell}(x,\vartheta(x,\lambda)) \in \mathcal{C}^m([0,1]\times[0,1]), \quad m \ge 1$.

(iv) The nonlinear function $\psi(\vartheta(x,\lambda), u(\vartheta(x,\lambda)))$ and the partial derivative $\psi^{(0,1)}(\vartheta(x,\lambda), u(\vartheta(x,\lambda)))$ of $\psi$ w.r.t the second variable exists and both are Lipschitz continuous w.r.t. second variable, i.e.,

$$|\psi(\vartheta(x,\lambda), u_1(\vartheta(x,\lambda))) - \psi(\vartheta(x,\lambda), u_2(\vartheta(x,\lambda)))| \le c_1|u_1 - u_2|,$$

$$|\psi^{(0,1)}(\vartheta(x,\lambda), u_1(\vartheta(x,\lambda))) - \psi^{(0,1)}(\vartheta(x,\lambda), u_2(\vartheta(x,\lambda)))| \le c_2|u_1 - u_2|.$$

Let $\psi^{(0,1)}(\vartheta(x,\lambda), u(\vartheta(x,\lambda)))$ is Lipschitz continuous w.r.t the first and second variables i.e., $\forall$ $\vartheta_1, \vartheta_2, u_1, u_2 \in \mathbb{R},\ \exists\ \ q_1, q_2 > 0$, it follows

$$|\psi^{(0,1)}(\vartheta_1(x,\lambda), u_1(\vartheta(x,\lambda))) - \psi^{(0,1)}(\vartheta_2(x,\lambda), u_2(\vartheta(x,\lambda)))| \le \{q_1|\vartheta_1 - \vartheta_2| + q_2|u_1 - u_2|\}.$$

(v) $d = \sup_{\vartheta(.,.)\in[0,1]} |\psi^{(0,1)}(\vartheta(.,.), u(\vartheta(.,.)))| < \infty.$

(vi) We assume that $\tilde{M}$ and $c_1$ satisfy the condition that $\tilde{M}c_1 < 1$.

Now for any $v \in \mathcal{B} \subseteq L^\infty[0,1]$, $x \in [0,1]$, where $\mathcal{B}$ is the closed unit ball in $L^\infty[0,1]$, consider

$$\begin{aligned}
&|(\mathcal{L}\psi)'(u_0)v(t)| \\
&= \left| \int_0^1 \tilde{\ell}(x,\vartheta(x,\lambda))\psi^{(0,1)}(\vartheta(x,\lambda), u_0(\vartheta(x,\lambda)))v(\vartheta(x,\lambda))\, d\eta \right| \\
&\le \sup_{\vartheta(x,\lambda)\in[0,1]} |\psi^{(0,1)}(\vartheta(x,\lambda), u_0(\vartheta(x,\lambda)))| \left[\int_0^1 |\tilde{\ell}(x,\vartheta(x,\lambda))|^2 d\eta\right]^{\frac{1}{2}} \left[\int_0^1 |v(\vartheta(x,\lambda))|^2 d\eta\right]^{\frac{1}{2}} \\
&\le d \left[\int_0^1 |\tilde{\ell}(x,\vartheta(x,\lambda))|^2\right]^{\frac{1}{2}} \|v\|_{L^2} \\
&\le Md.
\end{aligned}$$

This implies $\|(\mathcal{L}\psi)'(u_0)v\|_\infty = \sup_{x\in[0,1]} |((\mathcal{L}\psi)'(u_0))v(t)| \le \tilde{M}d < \infty$.

Hence

$$\|(\mathcal{L}\psi)'(u_0)\|_\infty = \sup_{v\in\mathcal{B}} \|(\mathcal{L}\psi)'(u_0)v\|_\infty \le \tilde{M}d < \infty, \tag{2.3}$$

The operator $\mathcal{S}$ on $\mathbb{X}$ be defined by

$$\mathcal{S}y := g + (\mathcal{L}\psi)(y), \quad y \in \mathbb{X},$$

then the equation (1.1) becomes

$$u = \mathcal{S}u. \tag{2.4}$$

From the analysis of Theorem 1 of [10], and using assumptions (ii), (iv) and (vi), it implies that the equation (2.4), has a unique solution, say $u_0 \in \mathbb{X}$, i.e., $u_0 = \mathcal{S}u_0$. Note that under the assumptions (i) and (iii), the solution $u_0 \in \mathcal{C}^m[0,1]$.

**Theorem 2.1.** *Let $u_0 \in \mathbb{X}$ be the isolated solution and the kernel $\tilde{\ell}(.,.) \in L^\infty([0,1]\times[0,1])$. Then the linear integral operator $(\mathcal{L}\psi)'(u_0) : \mathbb{X} \to \mathbb{X}$ is a compact operator.*

**Proof .** The proof follows similar way as in ([18], Theorem 1). □

Next, we discuss the Galerkin method to find the approximate solution of the equation (2.2). Let $\Pi_n : 0 = t_0 < t_1 < ... < t_n = 1$, be a partition of $[0,1]$ and the norm of the partitions $h = \max_i h_i = \{t_i - t_{i-1} : 1 \le i \le n\} \to 0$, as $n \to \infty$. Define $\mathbb{X}_n = S^\nu_{r,n}(\Pi_n)$, the space of all piece-wise polynomials of degree $\le r-1$ with $\nu$ $(-1 \le \nu \le r-2)$ continuous derivatives and breakpoints at $t_1, \cdots, t_{n-1}$.

**Orthogonal projection operator:** Let the orthogonal projection $\mathbb{P}_n : L^\infty[0,1] \to \mathbb{X}_n$, be given by

$$\langle \mathbb{P}_n y_1, y_2 \rangle = \langle y_1, y_2 \rangle, \ y_2 \in \mathbb{X}_n, y_1 \in \mathbb{X}, \tag{2.5}$$

where $\langle y_1,\ y_2\rangle = \int_0^1 y_1(x)y_2(x)\, dx$.

Here we state a lemma from Chatelin ([5], Corollary 7.6, p. 328).

**Lemma 2.2.** *Let $\mathbb{P}_n : \mathbb{X} \to \mathbb{X}_n$ be the orthogonal projection given by (2.5). Then*

*i) $\exists$ a constant $P > 0$ such that*

$$||\mathbb{P}_n||_\infty \leq P < \infty. \tag{2.6}$$

*ii)*

$$\|\mathbb{P}_n v - v\|_\infty \to 0, \text{ as } n \to \infty, \ v \in \mathbb{X}. \tag{2.7}$$

*iii) If $v \in \mathcal{C}^m[0,1]$, then*

$$\|(\mathbb{I} - \mathbb{P}_n)v\|_\infty \leq ch^m \|v\|_{m,\infty}, \tag{2.8}$$

*where $c$ denotes a constant.*

The Galerkin method for solving (2.2) is seeking an approximation $u_n \in \mathbb{X}_n$ such that

$$u_n - \mathbb{P}_n(\mathcal{L}\psi)(u_n) = \mathbb{P}_n g. \tag{2.9}$$

Let $\mathcal{S}_n$ be the operator defined by

$$\mathcal{S}_n(v) := \mathbb{P}_n(\mathcal{L}\psi)(v) + \mathbb{P}_n g. \tag{2.10}$$

Then the equation (2.9) can be written as

$$u_n = \mathcal{S}_n u_n. \tag{2.11}$$

The iterated approximate solution is defined as

$$\tilde{u}_n = g + (\mathcal{L}\psi)(u_n). \tag{2.12}$$

Using $\mathbb{P}_n \tilde{u}_n = u_n$, the equation (2.12) can be written as

$$\tilde{u}_n - (\mathcal{L}\psi)(\mathbb{P}_n \tilde{u}_n) = g. \tag{2.13}$$

Letting $\widetilde{\mathcal{S}}_n(v) := (\mathcal{L}\psi)(\mathbb{P}_n v) + g$, $v \in \mathbb{X}$, the equation (2.13) can be written as $\tilde{u}_n = \widetilde{\mathcal{S}}_n \tilde{u}_n$.

Now the existence and uniqueness of approximation $x_n$ be discussed. We quote a theorem from [1] and the definition of $\nu$-convergence. We will use the well known Theorem 2 of Vainikko [21].

**Definition 2.3.** *(**$\nu$-convergence**) Let $\mathcal{S}, \{\mathcal{S}_n\} \in BL(\mathbb{X})$, then $\{\mathcal{S}_n\}$ is said to be $\nu$ convergent to $\mathcal{S}$ if $\|\mathcal{S}_n\| \leq C, \|(\mathcal{S}_n - \mathcal{S})\mathcal{S}\| \to 0$ and $\|(\mathcal{S}_n - \mathcal{S})\mathcal{S}_n\| \to 0, \text{ as } n \to \infty$.*

Now we prove the following lemma which will help us to prove the existence and order of convergence of approximate solution $u_n$ in Galerkin method.

**Lemma 2.4.** *Let $\forall\ v_1, v_2 \in \mathbb{X}$, the following estimates hold*

$$\|(\mathcal{L}\psi)(v_1) - (\mathcal{L}\psi)(v_2)\|_\infty \leq \tilde{M}c_1\|v_1 - v_2\|_\infty,$$

$$\|(\mathcal{L}\psi)'(v_1) - (\mathcal{L}\psi)'(v_2)\|_\infty \leq \tilde{M}c_2\|v_1 - v_2\|_\infty.$$

**Proof .** Using the assumption (iv) and for any $v_1, v_2, v_3 \in \mathbb{X}$, we obtain

$$\begin{aligned}
&\|((\mathcal{L}\psi)(v_1) - (\mathcal{L}\psi)(v_2))v_3\|_\infty \\
&= \sup_{x\in[0,1]} |((\mathcal{L}\psi)(v_1) - (\mathcal{L}\psi)(v_2))v_3(x)| \\
&= \sup_{x\in[0,1]} \left| \int_0^1 \tilde{\ell}(x, \vartheta(x,\lambda))[\psi(\vartheta(x,\lambda), v_1(\vartheta(x,\lambda))) - \psi(\vartheta(x,\lambda), v_2(\vartheta(x,\lambda)))]v_3(\vartheta(x,\lambda))\, d\lambda \right| \\
&\le c_1 \sup_{\vartheta(x,\eta)\in[0,1]} |v_3(\vartheta(x,\eta))| \|\tilde{\ell}\|_{L^2} \|v_1 - v_2\|_{L^2} \\
&\le \tilde{M} c_1 \|v_1 - v_2\|_\infty \|v_3\|_\infty.
\end{aligned}$$

This implies

$$\|(\mathcal{L}\psi)(v_1) - (\mathcal{L}\psi)(v_2)\|_\infty \le \tilde{M} c_1 \|v_1 - v_2\|_\infty.$$

Similarly using assumption (iv) for $\psi^{(0,1)}(.,.)$, we have

$$\|(\mathcal{L}\psi)'(v_1) - (\mathcal{L}\psi)'(v_2)\|_\infty \le \tilde{M} c_2 \|v_1 - v_2\|_\infty. \tag{2.14}$$

This completes the proof. □

**Theorem 2.5.** *Let $u_0 \in \mathcal{C}^m[0,1]$, $m \ge 1$, be the unique solution of the equation (2.2) and 1 is not an eigenvalue of the operator $(\mathcal{L}\psi)'(u_0)$, which is the the Fréchet derivative of $(\mathcal{L}\psi)(u)$ at $u_0$. Then for some $\delta > 0$, the equation (2.9) has an isolated solution $u_n \in \mathbb{B}(u_0, \delta) = \{u : \|u - u_0\|_\infty < \delta\}$. furthermore, $\exists$ a constant $0 < q < 1$, ind. of $n$ so that*

$$\frac{\alpha_n}{1+q} \le \|u_n - u_0\|_\infty \le \frac{\alpha_n}{1-q},$$

*where $\alpha_n = \|(\mathbb{I} - \mathcal{S}_n'(u_0))^{-1}(\mathcal{S}_n(u_0) - \mathcal{S}(u_0))\|_\infty$. Further, we obtain*

$$\|u_n - u_0\|_\infty = \mathcal{O}(h^m).$$

**Proof .** Since $\mathcal{S}'(u_0) = (\mathcal{L}\psi)'(u_0)$ is compact, using Lemma 2.2, we have

$$\|(\mathbb{I} - \mathbb{P}_n)\mathcal{S}'(u_0)\|_\infty = \|(\mathbb{I} - \mathbb{P}_n)(\mathcal{L}\psi)'(u_0)\|_\infty \to 0, \quad as \quad n \to \infty. \tag{2.15}$$

Hence using estimate (2.15), we have

$$\begin{aligned}
\|\mathcal{S}_n'(u_0) - \mathcal{S}'(u_0)\|_\infty = \|\mathbb{P}_n(\mathcal{L}\psi)'(u_0) - (\mathcal{L}\psi)'(u_0)\|_\infty &= \|(\mathbb{I} - \mathbb{P}_n)(\mathcal{L}\psi)'(u_0)\|_\infty \\
&= \|(\mathbb{I} - \mathbb{P}_n)\mathcal{S}'(u_0)\|_\infty \to 0 \quad as \quad n \quad \to \infty.
\end{aligned}$$

Since 1 is not an eigenvalue of $\mathcal{S}'(u_0)$, then from [1], $\exists\ \mathcal{A}_1 > 0$ so that $\|(\mathbb{I} - \mathcal{S}_n'(u_0))^{-1}\|_\infty \le \mathcal{A}_1 < \infty$. From Lemma 2.4, for any $u \in \mathbb{B}(u_0, \delta)$,

$$\begin{aligned}
\|\mathcal{S}_n'(u_0) - \mathcal{S}_n'(u)\|_\infty &= \|\mathbb{P}_n(\mathcal{L}\psi)'(u_0) - \mathbb{P}_n(\mathcal{L}\psi)'(u)\|_\infty \\
&\le \|\mathbb{P}_n\|_\infty \|(\mathcal{L}\psi)'(u_0) - (\mathcal{L}\psi)'(u)\|_\infty \\
&\le P\tilde{M}c_2\|u_0 - u\|_\infty \le c_2 P\tilde{M}\delta.
\end{aligned} \tag{2.16}$$

Hence, we have

$$\sup_{\|u-u_0\|\le\delta} \|(\mathbb{I} - \mathcal{S}_n'(u_0))^{-1}(\mathcal{S}_n'(u_0) - \mathcal{S}_n'(u))\|_\infty \le \mathcal{A}_1 P c_2 \tilde{M}\delta \le q(say).$$

Choose $\delta$ in such a way that $0 < q < 1$. This proves the equation (4.4) of Theorem 2 of [21]. Taking use of estimate (2.7), we have

$$\begin{aligned}\alpha_n &= \|(\mathbb{I}-\mathcal{S}_n'(u_0))^{-1}(\mathcal{S}_n(u_0)-\mathcal{S}(u_0))\|_\infty\\ &\le \mathcal{A}_1\|\mathcal{S}_n(u_0)-\mathcal{S}(u_0)\|_\infty\\ &\le \mathcal{A}_1\|\mathbb{P}_n(g+(\mathcal{L}\psi)(u_0))-(g+(\mathcal{L}\psi)(u_0))\|_\infty\\ &\le \mathcal{A}_1\|(\mathbb{I}-\mathbb{P}_n)(g+(\mathcal{L}\psi)(u_0))\|_\infty\\ &\le \mathcal{A}_1\|(\mathbb{I}-\mathbb{P}_n)u_0\|_\infty \to 0 \ \ as \ \ n\to\infty. \end{aligned} \tag{2.17}$$

By taking $n$ sufficiently large such that $\alpha_n \le \delta(1-q)$, the equation (4.6) of Theorem 2 of [21] is satisfied, i.e.,

$$\frac{\alpha_n}{1+q} \le \|u_n-u_0\|_\infty \le \frac{\alpha_n}{1-q}.$$

Now using the estimate (2.8), it follows that

$$\|u_n-u_0\|_\infty \le \frac{\alpha_n}{1-q} \le \frac{1}{1-q}\mathcal{A}_1\|(\mathbb{I}-\mathbb{P}_n)u_0\|_\infty \le c\mathcal{A}_1\|(\mathbb{I}-\mathbb{P}_n)u_0\|_\infty = \mathcal{O}(h^m).$$

Hence the proof follows. □ Next, we discuss the existence and convergence of approximation $\tilde{u}_n$ in iterated Galerkin method.

**Theorem 2.6.** *$\widetilde{\mathcal{S}}_n'(u_0)$ is $\nu$-convergent to $\mathcal{S}'(u_0)$ in uniform norm.*

**Proof .** Since

$$\begin{aligned}\widetilde{\mathcal{S}}_n'(u_0) &= (\mathcal{L}\psi)'(\mathbb{P}_nu_0)\mathbb{P}_n = (\mathcal{L}\psi)'(\mathbb{P}_nu_0)\mathbb{P}_n-(\mathcal{L}\psi)'(u_0)\mathbb{P}_n\\ &+ (\mathcal{L}\psi)'(u_0)\mathbb{P}_n,\end{aligned} \tag{2.18}$$

we have

$$\begin{aligned}\|\widetilde{\mathcal{S}}_n'(u_0)\|_\infty &= \|(\mathcal{L}\psi)'(\mathbb{P}_nu_0)\mathbb{P}_n\|_\infty \le \|(\mathcal{L}\psi)'(\mathbb{P}_nu_0)\mathbb{P}_n-(\mathcal{L}\psi)'(u_0)\mathbb{P}_n\|_\infty\\ &+ \|(\mathcal{L}\psi)'(u_0)\|_\infty\|\mathbb{P}_n\|_\infty.\end{aligned} \tag{2.19}$$

From estimate (2.7) and Lemma 2.4, we have

$$\begin{aligned}\|(\mathcal{L}\psi)'(\mathbb{P}_nu_0)-(\mathcal{L}\psi)'(u_0)\|_\infty &\le \tilde{M}c_2\|(\mathbb{I}-\mathbb{P}_n)u_0\|_\infty\\ &\to 0, \ as\ n\to\infty.\end{aligned} \tag{2.20}$$

Hence using estimates (2.3), (2.19) and (2.20), we have

$$\|\widetilde{\mathcal{S}}_n'(u_0)\|_\infty \le \tilde{M}c_2p\|(\mathbb{I}-\mathbb{P}_n)u_0\|_\infty + P\tilde{M}d < \infty, \tag{2.21}$$

i.e., $\|\widetilde{\mathcal{S}}_n'(u_0)\|_\infty$ is uniformly bounded.
Next consider

$$\begin{aligned}&\|[\widetilde{\mathcal{S}}_n'(u_0)-\mathcal{S}'(u_0)]\widetilde{\mathcal{S}}_n'(u_0)\|_\infty\\ &= \|[(\mathcal{L}\psi)'(\mathbb{P}_nu_0)\mathbb{P}_n-(\mathcal{L}\psi)'(u_0)]\widetilde{\mathcal{S}}_n'(u_0)\|_\infty\\ &= \|[(\mathcal{L}\psi)'(\mathbb{P}_nu_0)\mathbb{P}_n-(\mathcal{L}\psi)'(u_0)\mathbb{P}_n+(\mathcal{L}\psi)'(u_0)\mathbb{P}_n-(\mathcal{L}\psi)'(u_0)]\widetilde{\mathcal{S}}_n'(u_0)\|_\infty\\ &\le \|[(\mathcal{L}\psi)'(\mathbb{P}_nu_0)-(\mathcal{L}\psi)'(u_0)]\mathbb{P}_n\widetilde{\mathcal{S}}_n'(u_0)\|_\infty + \|[(\mathcal{L}\psi)'(u_0)(\mathbb{P}_n-\mathbb{I})\widetilde{\mathcal{S}}_n'(u_0)\|_\infty\\ &\le \|\mathbb{P}_n\|_\infty\|\widetilde{\mathcal{S}}_n'(u_0)\|_\infty\|(\mathcal{L}\psi)'(\mathbb{P}_nu_0)-(\mathcal{L}\psi)'(u_0)\|_\infty + \|(\mathcal{L}\psi)'(u_0)\|_\infty\|(\mathbb{I}-\mathbb{P}_n)\widetilde{\mathcal{S}}_n'(u_0)\|_\infty.\end{aligned} \tag{2.22}$$

For the second term of the estimate (2.22), using estimate (2.6), we obtain

$$\begin{aligned}
&\|(\mathbb{I}-\mathbb{P}_n)\widetilde{\mathcal{S}}_n'(u_0)\|_\infty \\
&= \|(\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n\|_\infty \\
&\le \|\mathbb{P}_n\|_\infty \|(\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\|_\infty \\
&\le P\|(\mathbb{I}-\mathbb{P}_n)[(\mathcal{L}\psi)'(\mathbb{P}_n u_0)-(\mathcal{L}\psi)'(u_0)+(\mathcal{L}\psi)'(u_0)]\|_\infty \\
&\le P\{\|(\mathbb{I}-\mathbb{P}_n)[(\mathcal{L}\psi)'(\mathbb{P}_n u_0)-(\mathcal{L}\psi)'(u_0)]\|_\infty + \|(\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)'(u_0)\|_\infty\} \\
&\le P\{(1+\|\mathbb{P}_n\|_\infty)\|(\mathcal{L}\psi)'(\mathbb{P}_n u_0)-(\mathcal{L}\psi)'(u_0)\|_\infty + \|(\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)'(u_0)\|_\infty\} \\
&\le P\{(1+P)\|(\mathcal{L}\psi)'(\mathbb{P}_n u_0)-(\mathcal{L}\psi)'(u_0)\|_\infty + \|(\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)'(u_0)\|_\infty\}. \qquad (2.23)
\end{aligned}$$

Combining the estimates (2.15), (2.20) and (2.23), we get

$$\|(\mathbb{I}-\mathbb{P}_n)\widetilde{\mathcal{S}}_n'(u_0)\|_\infty \to 0, \quad n\to\infty. \qquad (2.24)$$

Hence using the uniform boundedness of $\|\widetilde{\mathcal{S}}_n'(u_0)\|_\infty$, Lemma 2.2 and estimates (2.3), (2.20), (2.24), we obtain

$$\begin{aligned}
\|[\widetilde{\mathcal{S}}_n'(u_0)-\mathcal{S}'(u_0)]\widetilde{\mathcal{S}}_n'(u_0)\|_\infty &\le \tilde{M}c_2P\|\widetilde{\mathcal{S}}_n'(u_0)\|_\infty\|(\mathbb{I}-\mathbb{P}_n)u_0\|_\infty \\
&+ \|(\mathcal{L}\psi)'(u_0)\|_\infty\|(\mathbb{I}-\mathbb{P}_n)\widetilde{\mathcal{S}}_n'(u_0)\|_\infty \to 0, \quad as \quad n\to\infty.
\end{aligned}$$

Similarly, it can be proved that

$$\|(\widetilde{\mathcal{S}}_n'(u_0)-\mathcal{S}'(u_0))\mathcal{S}'(u_0)\|_\infty \to 0, \quad as \quad n\to\infty.$$

This proves the $\nu$-convergence of $\widetilde{\mathcal{S}}_n'(u_0)$ to $\mathcal{S}'(u_0)$ in infinity norm.
Hence the proof follows. □ Now by applying the Theorem 2.6, we prove the following theorem.

**Theorem 2.7.** *Let $u_0 \in \mathbb{X}$ be the unique solution of the equation (2.2) and 1 is not an eigenvalue of the operator $(\mathcal{L}\psi)'(u_0)$. Then for n large enough, the operator $(\mathbb{I}-\widetilde{\mathcal{S}}_n'(u_0))$ is invertible on $\mathbb{X}$ and there exists a constant $L>0$ independent of n such that $\|(\mathbb{I}-\widetilde{\mathcal{S}}_n'(u_0))^{-1}\|_\infty \le L < \infty$.*

Next we discuss the superconvergence result for the iterated Galerkin solution $\tilde{u}_n$ to the exact solution $u_0$.

**Theorem 2.8.** *Let $u_0 \in \mathbb{X}$ be the unique solution of the equation (2.2) and 1 is not an eigenvalue of the operator $(\mathcal{L}\psi)'(u_0)$. Let the orthogonal projection $\mathbb{P}_n : \mathbb{X} \to \mathbb{X}_n$ be given by (2.5). Then the equation (2.13) has an isolated solution $\tilde{u}_n \in \mathbb{B}(u_0,\delta) = \{u : \|u-u_0\|_\infty < \delta\}$ for n large enough and some $\delta > 0$. Moreover, there $\exists\ 0<q<1$, independent of n so that*

$$\frac{\beta_n}{1+q} \le \|\tilde{u}_n - u_0\|_\infty \le \frac{\beta_n}{1-q},$$

*where $\beta_n = \|(\mathbb{I}-\widetilde{\mathcal{S}}_n'(u_0))^{-1}(\widetilde{\mathcal{S}}_n(u_0)-\mathcal{S}(u_0))\|_\infty$, and the following result holds*

$$\|\tilde{u}_n - u_0\|_\infty = \mathcal{O}(h^{2m}). \qquad (2.25)$$

**Proof .** From Theorem 2.7, we have $\|(\mathbb{I}-\widetilde{\mathcal{S}}_n'(u_0))^{-1}\|_\infty \leq L < \infty$.
For any $u \in \mathbb{B}(u_0,\delta)$ and using estimate (2.6) and Lemma 2.4, we obtain

$$\begin{aligned} \|\widetilde{\mathcal{S}}_n'(u)-\widetilde{\mathcal{S}}_n'(u_0)\|_\infty &= \|(\mathcal{L}\psi)'(\mathbb{P}_n u)\mathbb{P}_n-(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n\|_\infty \\ &\leq \|(\mathcal{L}\psi)'(\mathbb{P}_n u)-(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\|_\infty\|\mathbb{P}_n\|_\infty \\ &\leq \tilde{M}c_2\|\mathcal{P}_n(u-u_0)\|_\infty\|\mathbb{P}_n\|_\infty \\ &\leq \tilde{M}P^2c_2\|u-u_0\|_\infty \\ &\leq \tilde{M}P^2c_2\delta. \end{aligned} \tag{2.26}$$

Thus we have

$$\sup_{\|u-u_0\|\leq\delta}\|(\mathbb{I}-\widetilde{\mathcal{S}}_n'(u_0))^{-1}(\widetilde{\mathcal{S}}_n'(u)-\widetilde{\mathcal{S}}_n'(u_0))\|_\infty \leq \tilde{M}LP^2c_2\delta \leq q(say).$$

Here we take $\delta$ in such a way that $0<q<1$. This proves the equation (4.4) of Theorem 2 of [21]. Now using the estimate (2.7) and Lemma 2.4, we have

$$\begin{aligned} \|\widetilde{\mathcal{S}}_n(u_0)-\mathcal{S}(u_0)\|_\infty \leq \|(\mathcal{L}\psi)(\mathbb{P}_n u_0)-(\mathcal{L}\psi)(u_0)\|_\infty &\leq \tilde{M}c_1\|(\mathbb{I}-\mathbb{P}_n)u_0\|_\infty \\ &\to 0, \quad as \quad n\to\infty. \end{aligned} \tag{2.27}$$

Hence

$$\begin{aligned} \beta_n = \|(\mathbb{I}-\widetilde{\mathcal{S}}_n'(u_0))^{-1}(\widetilde{\mathcal{S}}_n(u_0)-\mathcal{S}(u_0))\|_\infty &\leq L\|\widetilde{\mathcal{S}}_n(u_0)-\mathcal{S}(u_0)\|_\infty \\ &\to 0, \quad as \quad n\to\infty. \end{aligned} \tag{2.28}$$

By choosing $n$ large enough such that $\beta_n \leq \delta(1-q)$, the equation (4.6) of Theorem 2 of [21] is satisfied. Hence by applying Theorem 2 of [21], we obtain

$$\frac{\beta_n}{1+q} \leq \|\tilde{u}_n-u_0\|_\infty \leq \frac{\beta_n}{1-q},$$

where $\beta_n = \|(\mathbb{I}-\widetilde{\mathcal{S}}_n'(x_0))^{-1}(\widetilde{\mathcal{S}}_n(u_0)-\mathcal{S}(x_0))\|_\infty$.
Now we consider

$$\begin{aligned} \|\tilde{u}_n-u_0\|_\infty \leq \frac{\beta_n}{1-q} &\leq \frac{1}{1-q}\|(\mathbb{I}-\widetilde{\mathcal{S}}_n'(u_0))^{-1}(\widetilde{\mathcal{S}}_n(u_0)-\mathcal{S}(u_0))\|_\infty \\ &\leq c\|(\mathbb{I}-\widetilde{\mathcal{S}}_n'(u_0))^{-1}\|_\infty\|\widetilde{\mathcal{S}}_n(u_0)-\mathcal{S}(u_0)\|_\infty \\ &\leq cL\|(\mathcal{L}\psi)(\mathbb{P}_n u_0)-(\mathcal{L}\psi)(u_0)\|_\infty. \end{aligned} \tag{2.29}$$

Denote $f(x,\vartheta(x,\lambda),u_0(\vartheta(x,\lambda)),u(\vartheta(x,\lambda)),\theta_1) = \tilde{\ell}(x,\vartheta(x,\lambda))\psi^{(0,1)}(\vartheta(x,\lambda),u_0(\vartheta(x,\lambda))+\theta_1(u-u_0)(\vartheta(x,\lambda))),$ 0
$\theta_1<1$ and $f_x(\vartheta(x,\lambda)) = \tilde{\ell}(x,\vartheta(x,\lambda))\psi^{(0,1)}(\vartheta(x,\lambda),u_0(\vartheta(x,\lambda)))$.

Using the mean-value theorem, we have

$$
\begin{aligned}
&|[(\mathcal{L}\psi)(\mathbb{P}_n u_0) - (\mathcal{L}\psi)(u_0)](x)| \\
&= \left| \int_0^1 \tilde{\ell}(x,\vartheta(x,\lambda))[\psi(\vartheta(x,\lambda), \mathbb{P}_n u_0(\vartheta(x,\lambda))) - \psi(\vartheta(x,\lambda)), u_0(\vartheta(x,\lambda)))]\, d\lambda \right| \\
&= \left| \int_0^1 \tilde{\ell}(x,\vartheta(x,\lambda))\psi^{(0,1)}(\vartheta(x,\lambda), u_0(\vartheta(x,\lambda)) + \theta_1(\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda)))(\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))\, d\lambda \right| \\
&= \left| \int_0^1 f(x,\vartheta(x,\lambda), u_0(\vartheta(x,\lambda)), \mathbb{P}_n u_0(\vartheta(x,\lambda)), \theta_1)(\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))\, d\lambda \right| \\
&= \left| \int_0^1 [f(x,\vartheta(x,\lambda), u_0(\vartheta(x,\lambda)), \mathbb{P}_n u_0(\vartheta(x,\lambda)), \theta_1) - f_x(\vartheta(x,\lambda)) + f_x(\vartheta(x,\lambda))](\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))\, d\lambda \right| \\
&\leq \left| \int_0^1 [f(x,\vartheta(x,\lambda), u_0(\vartheta(x,\lambda)), \mathbb{P}_n u_0(\vartheta(x,\lambda)), \theta_1) - f_x(\vartheta(x,\lambda))](\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))\, d\lambda \right| \\
&+ \left| \int_0^1 f_x(\vartheta(x,\lambda))(\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))\, d\lambda \right|, \qquad (2.30)
\end{aligned}
$$

where $0 < \theta_1 < 1$.
For the first term of the above estimate (2.30), we have

$$
\begin{aligned}
&\left| \int_0^1 [f(x,\vartheta(x,\lambda), u_0(\vartheta(x,\lambda)), \mathbb{P}_n u_0(\vartheta(x,\lambda)), \theta_1) - f_x(\vartheta(x,\lambda))](\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))\, d\lambda \right| \\
&= | \int_0^1 \tilde{\ell}(x,\vartheta(x,\lambda))[\psi^{(0,1)}(\vartheta(x,\lambda), u_0(\vartheta(x,\lambda)) + \theta_1(\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))) - \psi^{(0,1)}(\vartheta(x,\lambda), u_0(\vartheta(x,\lambda)))] \\
&\qquad (\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))\, d\lambda| \\
&\leq \int_0^1 |\tilde{\ell}(x,\vartheta(x,\lambda))||\psi^{(0,1)}(\vartheta(x,\lambda), u_0(\vartheta(x,\lambda)) + \theta_1(\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))) - \psi^{(0,1)}(\vartheta(x,\lambda), u_0(\vartheta(x,\lambda)))| \\
&\qquad |(\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))|\, d\lambda \\
&\leq c_2 \int_0^1 |\tilde{\ell}(x,\vartheta(x,\lambda))||(\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))|^2\, d\lambda \\
&\leq c_2 \|\tilde{\ell}\|_{L^2} \left[ \int_0^1 |(\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))|^4\, d\lambda \right]^{\frac{1}{2}} \\
&\leq \tilde{M} c_2 \|\mathbb{P}_n u_0 - u_0\|_{L^2}^2 \\
&\leq \tilde{M} c_2 \|\mathbb{P}_n u_0 - u_0\|_\infty^2 = \mathcal{O}(h^{2r}). \qquad (2.31)
\end{aligned}
$$

For the second term of (2.30), using the orthogonality of the projection $\mathbb{P}_n$ and estimate (2.8), we obtain

$$
\begin{aligned}
\left| \int_0^1 f_x(\vartheta(x,\lambda))(\mathbb{P}_n u_0 - u_0)(\vartheta(x,\lambda))\, d\lambda \right| &= |\langle f_x(\vartheta(x,.)), (\mathbb{I} - \mathbb{P}_n) u_0(\vartheta(x,.))\rangle| \\
&= |\langle (\mathbb{I} - \mathbb{P}_n) f_x(\vartheta(x,.)), (\mathbb{I} - \mathbb{P}_n) u_0(\vartheta(x,.))\rangle| \\
&\leq \|(\mathbb{I} - \mathbb{P}_n) f_x\|_\infty \|(\mathbb{I} - \mathbb{P}_n) u_0\|_\infty \\
&\leq c h^{2m} \|f_x^{(m)}\|_\infty \|u_0^{(m)}\|_\infty. \qquad (2.32)
\end{aligned}
$$

Thus using estimates (2.30), (2.31) and (2.32), we get

$$
\|(\mathcal{L}\psi)(\mathbb{P}_n u_0) - (\mathcal{L}\psi)(u_0)\|_\infty = \mathcal{O}(h^{2m}). \qquad (2.33)
$$

Hence from estimates (2.29) and (2.33), it follows that

$$
\|\tilde{u}_n - u_0\|_\infty = \mathcal{O}(h^{2m}). \qquad (2.34)
$$

This completes the proof. □

In this section, we discuss the multi-Galerkin and its iterated version to enhance the convergence rates obtained in Galerkin method and iterated Galerkin method.

## 3. Superconvergence results by iterated multi Galerkin method

Here, for solving the equation (1.1), the multi-Galerkin (M-Galerkin) method and with its iterated version (iterated M-Galerkin) are discussed (see [6], [9], [11], [14], [18]) and find the superconvergence results. The multi-Galerkin operator $(\mathcal{L}_n^M\psi)$ be defined by

$$(\mathcal{L}_n^M\psi)(u) := \mathbb{P}_n(\mathcal{L}\psi)(u) + (\mathcal{L}\psi)(\mathbb{P}_n u) - \mathbb{P}_n(\mathcal{L}\psi)(\mathbb{P}_n u). \tag{3.1}$$

The approximate solution $u_n^M \in \mathbb{X}$ for the equation (2.2) in M-Galerkin method can be found such that

$$u_n^M - (\mathcal{L}_n^M\psi)(u_n^M) = g. \tag{3.2}$$

Iterated M-Galerkin approximate solution is given by

$$\tilde{u}_n^M = (\mathcal{L}\psi)(u_n^M) + g. \tag{3.3}$$

To solve the equation (3.2), applying $\mathbb{P}_n$ and $(\mathbb{I}-\mathbb{P}_n)$ to the equation, we have

$$\begin{aligned} \mathbb{P}_n u_n^M &= \mathbb{P}_n(\mathcal{L}\psi)(u_n^M) + \mathbb{P}_n g. & (3.4)\\ (\mathbb{I}-\mathbb{P}_n)u_n^M &= (\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)(\mathbb{P}_n u_n^M) + (\mathbb{I}-\mathbb{P}_n)g. & (3.5)\\ \Rightarrow u_n^M &= \mathbb{P}_n u_n^M + (\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)(\mathbb{P}_n u_n^M) + (\mathbb{I}-\mathbb{P}_n)g. & (3.6) \end{aligned}$$

Substituting (3.6) into (3.4),we get

$$\mathbb{P}_n u_n^M = \mathbb{P}_n(\mathcal{L}\psi)(\mathbb{P}_n u_n^M + (\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)(\mathbb{P}_n u_n^M) + (\mathbb{I}-\mathbb{P}_n)g) + \mathbb{P}_n g. \tag{3.7}$$

Let

$$\mathbb{W}_n^M = \mathbb{P}_n u_n^M,$$

then we can find $\mathbb{W}_n^M \in \mathbb{X}_n$ from the equation

$$\mathbb{W}_n^M = \mathbb{P}_n(\mathcal{L}\psi)(\mathbb{W}_n^M + (\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)(\mathbb{W}_n^M) + (\mathbb{I}-\mathbb{P}_n)g) + \mathbb{P}_n g, \tag{3.8}$$

and obtain

$$u_n^M = \mathbb{W}_n^M + (\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)(\mathbb{W}_n^M) + (\mathbb{I}-\mathbb{P}_n)g. \tag{3.9}$$

Define

$$\mathbb{F}_n(v) = v - \mathbb{P}_n(\mathcal{L}\psi)(v + (\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)(v) + (\mathbb{I}-\mathbb{P}_n)g) - \mathbb{P}_n g. \tag{3.10}$$

The Fréchet derivative of $\mathbb{F}_n$ is given by

$$\mathbb{F}_n'(v)y = y - \mathbb{P}_n(\mathcal{L}\psi)'(v + (\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)(v) + (\mathbb{I}-\mathbb{P}_n)g)(\mathbb{I} + (\mathbb{I}-\mathbb{P}_n)(\mathcal{L}\psi)'(v))y.$$

Equation (3.10) is equivalent to

$$\mathbb{F}_n(\mathbb{W}_n^M) = 0,$$

and it is iteratively solved by applying the Newton-Kantorovich method.
Let $\mathcal{L}_n^M(v) = (\mathcal{L}_n^M\psi)(v) + g, v \in \mathbb{X}$, then the equation (3.2) becomes

$$u_n^M = \mathcal{L}_n^M(u_n^M). \tag{3.11}$$

The Fréchet derivative of $\mathcal{L}_n^M(u)$ at $u_0$ is a linear operator and is defined as

$$\begin{aligned}\mathcal{L}_n^{M\prime}(u_0) = (\mathcal{L}_n^M\psi)'(u_0) &= \mathbb{P}_n(\mathcal{L}\psi)'(u_0) + (\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n - \mathbb{P}_n(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n \\ &= \mathbb{P}_n(\mathcal{L}\psi)'(u_0) + (\mathbb{I} - \mathbb{P}_n)(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n.\end{aligned}$$

Now existence and convergence of the multi-Galerkin approximate solution $u_n^M$ be discussed.

**Theorem 3.1.** *Let $u_0 \in \mathcal{C}^m[0,1]$, $m \geq 1$, be the unique solution of the equation (2.2) and $\tilde{\ell}(.,.) \in \mathcal{C}^m([0,1]\times[0,1])$ and 1 is not an eigenvalue of the operator $(\mathcal{L}\psi)'(u_0)$. Then for some $\delta > 0$, the equation (3.2) has an isolated solution $u_n^M \in \mathbb{B}(u_0,\delta) = \{u : \|u - u_0\|_\infty < \delta\}$. Moreover, there exists a constant $0 < q < 1$, independent of $n$ such that*

$$\frac{\alpha_n}{1+q} \leq \|u_n^M - u_0\|_\infty \leq \frac{\alpha_n}{1-q},$$

*where $\alpha_n = \|(\mathbb{I} - \mathcal{S}_n^{M\prime}(u_0))^{-1}(\mathcal{S}_n^M(u_0) - \mathcal{S}(u_0))\|_\infty$. Further, we obtain*

$$\|u_n^M - u_0\|_\infty = \mathcal{O}(h^{2m}).$$

**Proof .** Consider

$$\begin{aligned}\|\mathcal{S}_n^{M\prime}(u_0) - \mathcal{S}'(u_0)\|_\infty &= \|\mathbb{P}_n(\mathcal{L}\psi)'(u_0) + (\mathbb{I} - \mathbb{P}_n)(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n - (\mathcal{L}\psi)'(u_0)\|_\infty \\ &= \|(\mathbb{I} - \mathbb{P}_n)(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n + (\mathbb{P}_n - \mathbb{I})(\mathcal{L}\psi)'(u_0)\|_\infty \\ &\leq \|(\mathbb{I} - \mathbb{P}_n)(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n\|_\infty + \|(\mathbb{I} - \mathbb{P}_n)(\mathcal{L}\psi)'(u_0)\|_\infty. \end{aligned} \tag{3.12}$$

Note that $\widetilde{\mathcal{S}}_n'(u_0) = (\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n$, hence using estimates (2.15) and (2.24), we have

$$\|(\mathbb{I} - \mathbb{P}_n)(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n\|_\infty = \|(\mathbb{I} - \mathbb{P}_n)\widetilde{\mathcal{S}}_n'(u_0)\|_\infty \to 0, \quad as \quad n \to \infty,$$

and

$$\|(\mathbb{I} - \mathbb{P}_n)(\mathcal{L}\psi)'(u_0)\|_\infty = \|(\mathbb{I} - \mathbb{P}_n)\mathcal{S}'(u_0)\|_\infty \to 0, \quad as \quad n \to \infty.$$

This implies that

$$\|\mathcal{S}_n^{M\prime}(u_0) - \mathcal{S}'(u_0)\|_\infty \to 0, \quad as \quad n \to \infty. \tag{3.13}$$

Since 1 is not an eigenvalue of $\mathcal{S}'(u_0)$, i.e., $(\mathbb{I} - \mathcal{S}'(u_0))$ is invertible on $\mathbb{X}$. Then $\exists$ a constant $L_1 > 0$ such that $\|(\mathbb{I} - \mathcal{S}_n^{M\prime}(u_0))^{-1}\|_\infty \leq L_1 < \infty$, for some sufficiently large $n$.

Using the Lemma 2.4 and estimate (2.6), for any $u \in \mathbb{B}(u_0,\delta)$, we obtain

$$\begin{aligned}&\|\mathcal{S}_n^{M\prime}(u_0) - {\mathcal{S}_n}^{M\prime}(u)\|_\infty \\ &= \|(\mathcal{L}_n^M\psi)'(u_0) - (\mathcal{L}_n^M\psi)'(u)\|_\infty \\ &= \|\mathbb{P}_n(\mathcal{L}\psi)'(u_0) + (\mathbb{I} - \mathbb{P}_n)(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n - \mathbb{P}_n(\mathcal{L}\psi)'(u) - (\mathbb{I} - \mathbb{P}_n)(\mathcal{L}\psi)'(\mathbb{P}_n u)\mathbb{P}_n\|_\infty \\ &= \|\mathbb{P}_n[(\mathcal{L}\psi)'(u_0) - (\mathcal{L}\psi)'(u)] + (\mathbb{I} - \mathbb{P}_n)[(\mathcal{L}\psi)'(\mathbb{P}_n u_0)\mathbb{P}_n - (\mathcal{L}\psi)'(\mathbb{P}_n u)\mathbb{P}_n]\|_\infty \\ &\leq \|\mathbb{P}_n\|_\infty\|(\mathcal{L}\psi)'(u_0) - (\mathcal{L}\psi)'(u)\|_\infty + (1 + \|\mathbb{P}_n\|_\infty)\|[(\mathcal{L}\psi)'(\mathbb{P}_n u_0) - (\mathcal{L}\psi)'(\mathbb{P}_n u)]\mathbb{P}_n\|_\infty \\ &\leq pMc_2\|u - u_0\|_\infty + cMc_2\|\mathbb{P}_n(u_0 - u)\|_\infty\|\mathbb{P}_n\|_\infty \\ &\leq M(pc_2 + cc_2p^2)\|u_0 - u\|_\infty \\ &\leq M(pc_2 + cc_2p^2)\delta.\end{aligned}$$

Thus we obtain

$$\sup_{\|u-u_0\|\leq\delta} \|(\mathbb{I}-\mathcal{S}_n{}^{M'}(u_0))^{-1}(\mathcal{S}_n{}^{M'}(u_0)-\mathcal{S}_n{}^{M'}(u))\|_\infty \leq L_1 M(pc_2+cc_2p^2)\delta \leq q(say).$$

Here we choose $\delta$ in such a way that $0<q<1$. This proves the estimate (4.4) of Theorem 2 of [21]. Hence using the Lemma 2.4 and estimate (2.7), we obtain

$$\begin{aligned} \alpha_n &= \|(\mathbb{I}-\mathcal{S}_n{}^{M'}(u_0))^{-1}(\mathcal{S}_n{}^{M}(u_0)-\mathcal{S}(u_0))\|_\infty \\ &\leq L_1\|\mathcal{S}_n{}^{M}(u_0)-\mathcal{S}(u_0))\|_\infty \\ &\leq L_1\|(\mathbb{I}-\mathbb{P}_n)((\mathcal{L}\psi)(\mathbb{P}_nu_0)-(\mathcal{L}\psi)(u_0))\|_\infty \\ &\leq L_1(1+\|\mathbb{P}_n\|_\infty)\|(\mathcal{L}\psi)(\mathbb{P}_nu_0)-(\mathcal{L}\psi)(u_0)\|_\infty \\ &\leq cMc_1L_1\|(\mathbb{P}_n-\mathbb{I})u_0\|_\infty \to 0 \quad as \quad n \quad \to\infty. \end{aligned} \tag{3.14}$$

By selecting $n$ sufficiently large so that $\alpha_n\leq\delta(1-q)$, the estimate (4.6) of Theorem 2 of [21] is satisfied, i.e.,

$$\frac{\alpha_n}{1+q}\leq\|u_n^M-u_0\|_\infty\leq\frac{\alpha_n}{1-q}.$$

Hence from estimate (2.33), it follows

$$\begin{aligned} \|u_n^M-u_0\|_\infty\leq\frac{\alpha_n}{1-q}\leq\frac{1}{1-q}L_1\|\mathcal{S}_n{}^{M}(u_0)-\mathcal{S}(u_0))\|_\infty &\leq cL_1\|(\mathcal{L}\psi)(\mathbb{P}_nu_0)-(\mathcal{L}\psi)(u_0)\|_\infty \\ &= \mathcal{O}(h^{2m}). \end{aligned}$$

Hence the proof follows. □

**Remark 3.2.** *We can see from Theorem 2.8 and 3.1, the iterated Galerkin method $\tilde{u}_n$ and multi-Galerkin method $u_n^M$ converge with the same rate $\mathcal{O}(h^{2m})$. However, using the Theorem 3.1, we will find the superconvergence result for the iterated multi-Galerkin method.*

Next we discuss the superconvergence results for the iterated multi-Galerkin solution $\tilde{u}_n^M$.

**Theorem 3.3.** *Let $u_0\in\mathbb{X}$ be the unique solution of the equation (2.2) and let $\mathbb{P}_n:\mathbb{X}\to\mathbb{X}_n$ be the orthogonal projection stated by (2.5). Let the iterated multi-Galerkin approximation of $u_0$ be denoted by $\tilde{u}_n^M$ and defined by (3.3). Then the following holds*

$$\begin{aligned} \|\tilde{u}_n^M-u_0\|_\infty &\leq C_1\|u_n^M-u_0\|_\infty^2 \\ &+ (1+M_1p)\|(\mathcal{L}\psi)'(u_0)(\mathbb{I}-\mathbb{P}_n)\|_\infty\|(\mathcal{L}\psi)(\mathbb{P}_nu_0)-(\mathcal{L}\psi)(u_0)\|_\infty, \end{aligned} \tag{3.15}$$

*where $C_1=(c_2+M_1M_2)$. Further, we obtain*

$$\|\tilde{u}_n^M-u_0\|_\infty=\mathcal{O}(h^{3m}). \tag{3.16}$$

**Proof .** The proof

$$\|\tilde{u}_n^M-u_0\|_\infty\leq C_1\|u_n^M-u_0\|_\infty^2+(1+M_1p)\|(\mathcal{L}\psi)'(u_0)(\mathbb{I}-\mathbb{P}_n)\|_\infty\|(\mathcal{L}\psi)(\mathbb{P}_nu_0)-(\mathcal{L}\psi)(u_0)\|_\infty \tag{3.17}$$

can be done easily following the same steps as in Lemma 5 of [18].
Now using orthogonality of $\mathbb{P}_n$ and the estimate (2.8), we have

$$\begin{aligned} |(\mathcal{L}\psi)'(u_0)(\mathbb{I}-\mathbb{P}_n)v(x)| &= \left|\int_0^1 \tilde{\ell}(x,\vartheta(x,\eta)))\psi^{(0,1)}(\vartheta(x,\eta),u_0(\vartheta(x,\eta)))(\mathbb{I}-\mathbb{P}_n)v(\vartheta(x,\eta))\,d\eta\right| \\ &\leq |\langle(\mathbb{I}-\mathbb{P}_n)f_x(.),(\mathbb{I}-\mathbb{P}_n)v(.)\rangle| \\ &\leq \|(\mathbb{I}-\mathbb{P}_n)f_x\|_\infty\|(\mathbb{I}-\mathbb{P}_n)v\|_\infty\leq ch^m\|f_x^{(m)}\|_\infty\|v\|_\infty, \end{aligned}$$

where $f_x(\vartheta(x,\eta)) = \tilde{\ell}(x,\vartheta(x,\eta)))\psi^{(0,1)}(\vartheta(x,\eta), u_0(\vartheta(x,\eta)))$.
This implies

$$\|(\mathcal{L}\psi)'(u_0)(\mathbb{I}-\mathbb{P}_n)\|_\infty \leq ch^m\|f_x^{(m)}\|_\infty. \tag{3.18}$$

Again from estimate (2.33), we have

$$\|(\mathcal{L}\psi)(\mathbb{P}_n u_0) - (\mathcal{L}\psi)(u_0)\|_\infty = \mathcal{O}(h^{2m}). \tag{3.19}$$

Combining the estimates (3.17), (3.18), (3.19) and Theorem 3.1, we obtain

$$\|\tilde{u}_n^M - u_0\|_\infty = \mathcal{O}(h^{min\{4m,3m\}}) = \mathcal{O}(h^{3m}). \tag{3.20}$$

This completes the proof. □

**Remark 3.4.** *From Theorems 2.5, 2.8 and 3.3, the iterated multi-Galerkin method converges with the order $\mathcal{O}(h^{3m})$ and the Galerkin and iterated-Galerkin methods converge with the order $\mathcal{O}(h^m)$ and $\mathcal{O}(h^{2m})$, respectively. Hence we can show that iterated multi-Galerkin solutions improve over the Galerkin and its iterated version.*

## 4. Numerical results

In this sec., numerical results are presented. For that let $\mathbb{X}_n$ be the subspace of the piecewise polynomials functions. We present the errors of the approximate and iterated approximate solutions of Galerkin and M-Galerkin methods in uniform norm. In Tables 1 and 3, the errors and convergence rates in Galerkin and iterated Galerkin methods are presented. The errors and convergence rate in M-Galerkin and iterated M-Galerkin methods are given in Tables 2 and 4. We denote the Galerkin, iterated Galerkin, multi-Galerkin and iterated multi-Galerkin solutions by $u_n$, $\tilde{u}_n$, $u_n^M$ and $\tilde{u}_n^M$, respectively. Also we denote $\|u-u_n\|_\infty = \mathcal{O}(h^a)$, $\|u-\tilde{u}_n\|_\infty = \mathcal{O}(h^b)$, $\|u-u_n^M\|_\infty = \mathcal{O}(h^c)$, $\|u-\tilde{u}_n^M\|_\infty = \mathcal{O}(h^d)$.
The uniform partition of $[0,1]$ be:

$$0 = x_0 < x_1 < x_2 < ... < x_n = 1$$

where $x_i = \frac{i}{n},\ i = 0,1,2,...,n$.
We select the approximating subspaces of dimension $n$, i.e., the space of piecewise constant functions ($m=1$), . Then for $m=1$, the expected orders of convergence are $a=1$, $b=2$, $c=2$ and $d=3$, which are deliberated in the Tables [1-4] of Example 1 and Example 2, given below.

**Example 4.1.** *Consider the following Volterra integral equation of second kind*

$$u(x) - \int_0^x \ell(x,\vartheta)\psi(\vartheta,u(\vartheta))\,d\vartheta = g(x), \quad x\in[0,1],$$

*with the kernel function $\ell(x,\vartheta)=1$, nonlinear function $\psi(\vartheta,u(\vartheta)) = [u(\vartheta)]^3$ and the function $g(x) = e^x - \frac{1}{3}e^{3x} + \frac{1}{3}$ and the analytical solution is given by $u(x)=e^x$.*
*The transformed equation is*

$$u(x) - \int_0^1 \tilde{\ell}(x,\vartheta(x,\lambda))\psi(\vartheta(x,\lambda),u(\vartheta(x,\lambda)))\,d\lambda = g(x),\ x\in[0,1],$$

*where $\tilde{\ell}(x,\vartheta(x,\lambda)) = x$ and $\psi(\vartheta(x,\lambda),u(\vartheta(x,\lambda))) = [u(x\eta)]^3$, $x,\lambda\in[0,1]$.*

Table 1: Galerkin and iterated Galerkin methods

| $n$ | $\|u-u_n\|_\infty$ | $a$ | $\|u-\tilde{u}_n\|_\infty$ | $b$ |
|---|---|---|---|---|
| 2 | $1.76931256598\times10^{-1}$ | - | $4.125966812\times10^{-3}$ | |
| 4 | $1.002369854125\times10^{-1}$ | 0.83 | $1.158036954\times10^{-3}$ | 2.14 |
| 8 | $5.254899664572\times10^{-2}$ | 0.87 | $2.002548966\times10^{-4}$ | 1.86 |
| 16 | $3.005693884114\times10^{-2}$ | 0.98 | $6.256365481\times10^{-5}$ | 1.98 |
| 32 | $1.158964785540\times10^{-2}$ | 0.99 | $2.012584566\times10^{-5}$ | 1.81 |
| 64 | $7.014586544012\times10^{-3}$ | 1.01 | $5.482369401\times10^{-6}$ | 1.84 |
| 128 | $1.125896654701\times10^{-3}$ | 1.06 | $1.124582142\times10^{-6}$ | 1.88 |

Table 2: multi-Galerkin and iterated multi-Galerkin methods

| $n$ | $\|u-u_n^M\|_\infty$ | $c$ | $\|u-\tilde{u}_n^M\|_\infty$ | $d$ |
|---|---|---|---|---|
| 2 | $2.102250124512\times10^{-3}$ | - | $3.122003665401\times10^{-4}$ | - |
| 4 | $4.947896522301\times10^{-4}$ | 2.00 | $2.992589645541\times10^{-5}$ | 3.26 |
| 8 | $1.112596874012\times10^{-4}$ | 2.16 | $3.812025563341\times10^{-6}$ | 2.85 |
| 16 | $3.001452635848\times10^{-5}$ | 1.99 | $5.123365489564\times10^{-7}$ | 3.04 |
| 32 | $7.035642187012\times10^{-6}$ | 2.03 | $5.825698844544\times10^{-8}$ | 3.00 |
| 64 | $2.023689544778\times10^{-6}$ | 2.02 | $7.145589665044\times10^{-9}$ | 3.05 |

**Example 4.2.** *Consider the following Volterra integral equation of second kind*

$$u(x)-\int_0^x \ell(x,\vartheta)\psi(\vartheta,u(\vartheta))\,d\vartheta = g(x),\ \ x\in[0,1],$$

*with the kernel function* $\ell(x,\vartheta)=-\vartheta$*, the nonlinear fuction* $\psi(\vartheta,u(\vartheta))=[u(\vartheta)]^3$*,* $g(x)=x+\frac{x^5}{5}$*; and its analytical solution is given by* $u(x)=x$*.*
*The transformed equation is*

$$u(x)-\int_0^1 \tilde{\ell}(x,\vartheta(x,\lambda))\psi(\vartheta(x,\lambda),u(\vartheta(x,\lambda)))\,d\lambda = g(x),\ \ x\in[0,1],$$

*where* $\tilde{\ell}(x,\vartheta(x,\lambda))=-x^2\lambda$ *and* $\psi(\vartheta(x,\lambda),u(\vartheta(x,\lambda)))=[u(x\lambda)]^3,\quad x,\lambda\in[0,1].$

From the tables, it is clear that the approximate solutions in iterated Galerkin method gives better convergence rates than the approximate solutions in Galerkin method and also the iterated multi-Galerkin method gives better convergence rates than Galerkin and iterated Galerkin methods. Note that the size of the system of nonlinear equations need to be solved, remains the same as in Galerkin method.

## References

[1] M. Ahues, A. Largillier and B. Limaye, *Spectral computations for bounded operators*, CRC Press, Boca Raton, 2001.

Table 3: Galerkin and iterated Galerkin methods

| $n$ | $\|u-u_n\|_\infty$ | $a$ | $\|u-\tilde{u}_n\|_\infty$ | $b$ |
|---|---|---|---|---|
| 2 | $2.6521169874556\times10^{-1}$ | - | $3.97699833561\times10^{-2}$ | - |
| 4 | $1.3356980012540\times10^{-1}$ | 0.99 | $1.24466899554\times10^{-2}$ | 1.93 |
| 8 | $5.916698447514\times10^{-2}$ | 1.01 | $2.66589320144\times10^{-3}$ | 2.04 |
| 16 | $2.902558964475\times10^{-2}$ | 1.00 | $8.12566001455\times10^{-4}$ | 1.81 |
| 32 | $1.4511496632140\times10^{-2}$ | 1.01 | $3.03665891224\times10^{-4}$ | 1.85 |
| 64 | $6.956894522541\times10^{-3}$ | 1.04 | $6.53369844224\times10^{-4}$ | 1.88 |
| 128 | $3.4222569335855\times10^{-3}$ | 1.03 | $1.54469822140\times10^{-5}$ | 2.02 |

Table 4: multi-Galerkin and iterated multi-Galerkin methods

| $n$ | $\|u-u_n^M\|_\infty$ | $c$ | $\|u-\tilde{u}_n^M\|_\infty$ | $d$ |
|---|---|---|---|---|
| 2 | $2.972556895\times10^{-2}$ | - | $1.272256981\times10^{-2}$ | - |
| 4 | $5.914589665\times10^{-3}$ | 2.12 | $2.425641244\times10^{-3}$ | 3.00 |
| 8 | $1.322569844\times10^{-3}$ | 2.02 | $2.965842487\times10^{-4}$ | 2.97 |
| 16 | $4.014556987\times10^{-4}$ | 1.99 | $4.356421458\times10^{-5}$ | 3.00 |
| 32 | $8.981259968\times10^{-5}$ | 2.03 | $5.965874244\times10^{-6}$ | 3.02 |
| 64 | $2.122569844\times10^{-5}$ | 2.02 | $7.472265347\times10^{-7}$ | 3.01 |


[2] H. Brunner, *Implicitly linear collocation methods for nonlinear Volterra equations*, Appl. Numer. Math. 9 (1992) 235–247.

[3] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, vol. 15. Cambridge University Press, Cambridge, 2004.

[4] H. Brunner and P.J. Houwen, *The Numerical Solution of Volterra Equations*, Elsevier Science Ltd, 1986.

[5] F. Chatelin, *Spectral Approximation of Linear Operators ci*, SIAM, 1983.

[6] Z. Chen, G. Long and G. Nelakanti, *The discrete multi-projection method for Fredholm integral equations of the second kind*, J. Integral Eq. Appl. 19(2) (2007) 143–162.

[7] G.N. Elnagar and M. Kazemi, *Chebyshev spectral solution of nonlinear Volterra-Hammerstein integral equations*, J. Comput. Appl. Math. 76 (1996) 147–158.

[8] F.Ghoreishi and M. Hadizadeh, *Numerical computation of the tau approximation for the Volterra-Hammerstein integral equations*, Numer. Algo. 52 (2009) 541–559.

[9] L. Grammont, R.P. Kulkarni and R.B. Vasconcelos, *Modified projection and the iterated modified projection methods for nonlinear integral equations*, J. Integral Eq. Appl. 25 (2013) 481–516.

[10] H. Kaneko and Y. Xu, *Degenerate kernel method for Hammerstein equations*, Math. Comput. 56 (1991) 141–148.

[11] R.P. Kulkarni, *A superconvergence result for solutions of compact operator equations*, Bull. Aust. Math. Soc. 68 (2003) 517–528.

[12] S. Kumar, *Superconvergence of a collocation-type method for Hammerstein equations*, IMA J. Numer. Anal. 7 (1987) 313–325.

[13] S. Kumar and I.H. Sloan, *A new collocation-type method for Hammerstein integral equations*, Math. Comput. 48 (1987) 585–593.

[14] G. Long, M.M. Sahani and G. Nelakanti, *Polynomially based multi-projection methods for Fredholm integral equations of the second kind*, Appl. Math. Comput. 215 (2009) 147–155.

[15] K. Maleknejad and P. Torabi, *Application of fixed point method for solving nonlinear Volterra-Hammerstein integral equation*, U.P.B. Sci. Bull. Series A: Appl. Math. Phys. 74 (2012) 45–56.

[16] M. Mandal and G. Nelakanti, *Superconvergence of Legendre spectral projection methods for Fredholm-Hammerstein integral equations*, J. Comput. Appl. Math. 319 (2017) 423–439.

[17] M. Mandal and G. Nelakanti, *Superconvergence results for linear second-kind Volterra integral equations*, J. Appl.

Math. Comput. 57 (2018) 247-–260.

[18] M. Mandal and G. Nelakanti, *Superconvergence results for Volterra-Urysohn integral equations of the second kind*, Int. Conf. Math. Comput. (2017) 358–379.

[19] M. Mandal and G. Nelakanti, *Legendre spectral Galerkin and multi-Galerkin methods for nonlinear Volterra integral equations of Hammerstein type*, J. Anal. 28 (2020) 323—349.

[20] M. Mandal and G. Nelakanti, *Superconvergence results of Legendre spectral projection methods for Volterra integral equations of second kind*, Comput. Appl. Math. 37 (2018) 4007–4022.

[21] G. M. Vainikko, *Galerkin's perturbation method and the general theory of approximate methods for non-linear equations*, USSR Comput. Math. Math. Phys. 7 (1967) 1–41 .

[22] M. Mandal and G. Nelakanti, *Superconvergence results for weakly singular Fredholm–Hammerstein integral equations*, Numerical Funct. Anal. Opt. 40(5) (2019) 548–570.