Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 305-315
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.19373.2074

Bifurcation and chaos in a one mass discrete time vocal fold dynamical model

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(Communicated by Javad Damirchi)

Abstract

In this article, we are going to study the stability and bifurcation of a two-dimensional discrete time vocal fold model. The existence and local stability of the unique fixed point of the model is investigated. It is shown that a Neimark-Sacker bifurcation occurs and an invariant circle will appear. We give sufficient conditions for this system to be chaotic in the sense of Marotto. Numerically it is shown that our model has positive Lyapunov exponent and is sensitive dependence on initial conditions. Some numerical simulations are presented to illustrate our theoretical results.

Keywords: Vocal fold, Neimark-Sacker bifurcation, Limit cycle, Marotto's chaos, Lyapunov exponent, Sensitive dependence on initial conditions.
2010 MSC: Primary: 37C50; Secondary: 37C15.

1. Introduction

For any attempt to synthetic speech we need to know how sound is made. Physically, sound is generated whenever there is a disturbance of the equilibrium of density (or pressure) of a gas, liquid or solid (Fig. 1). Voice is the unique signal generated by the human vocal production system and is considered as the sounds originated from a flow of air from the lungs, which causes the vocal folds to vibrate, and that are subsequently modified by the vocal tract. In essence, the opposed membranes at the base of the human vocal tract are the source of voiced sounds. Air flow induced instability of this structure, which is known as the vocal folds. Indeed, voice is produced by movement of the two lateral opposing vocal folds located in the larynx (see textbooks in Otolaryngology, such as [17]) caused by the air flow through the trachea and generated by lung. In briefly, voice is the result of

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Received: December 2019 *Accepted:* June 2020

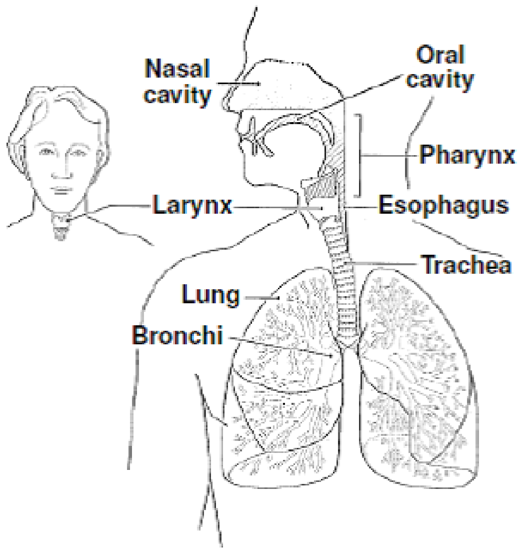


Figure 1: vocal folds are housed by the larynx, an organ placed in the neck of mammals, and they are composed of layered muscular and nonmuscular tissues (Titze, 1994, and references therein). The air space between the vocal folds is called glottis. The glottis is an important landmark in speech anatomy: widely used terms are, e.g., supraglottal (above the glottis) and subglottal (below the glottis)[22].

a balance between two forces: the force of the air leaving the lungs and the muscle strength of the larynx, where the vocal folds are located. For more detail see [10, 20, 21].
For a long time, Speech synthesis and Artificial intelligence researchers attempts to simulate the human voice production. Various mechanical and mathematical models are introduced for describing of the human organs which are connected with voice production and also of the process of phonation [4, 19, 21]. Most of these models are differential equations or compartmental models. But, sometimes discrete models give better and more accurate results. For example, in population models discrete models are better for small populations and for the discrete time prey-predator models the dynamics can produce a more efficient set of patterns than those observed in continuous-time models [12, 1, 19]. Signal processing of speech can be represented in discrete time as a digital filter [14, 11]. It is very important to be able to compare theoretical results with experimental records. In [11], the authors consider a discrete model of vocal fold that can be used to obtain the parameters from a recording of the vocal fold behaviour. This model enable us to identify the parameters of the model directly from discrete measurements of the vocal fold signal and syntheses the speech signals. As we can see in [19] the operation would be problematic for a general signal controller. In addition, the oversaturation control time in the continuous model is longer than in the discrete model, indicating that the discrete model is better than the continuous model. Also, the numerical simulations of continuous-time models are obtained by discretizing the models. The present work aims to describe the vibration properties of human vocal fold as a discrete-time dynamical model. Our study is focused on discretizing of a continuous model for vocal fold oscillations which introduced by Jorge C. Lucero [13]. Firstly, we briefly review the bi-dimensional oscillator, which is our case for discretization and has been proposed as a simple continuous time model for the vocal fold motion at phonation [13]. Then, based on forward-Euler method, we obtain our related discrete dynamical model for vocal fold. In the third section, the local stability is considered. In Section 4, we prove that our system undergoes the Neimark-Sacker bifurcation. In Section 5, we investigate the existence of chaos in the sense of Marroto's definition. By numerical simulation we estimate that the systems has positive

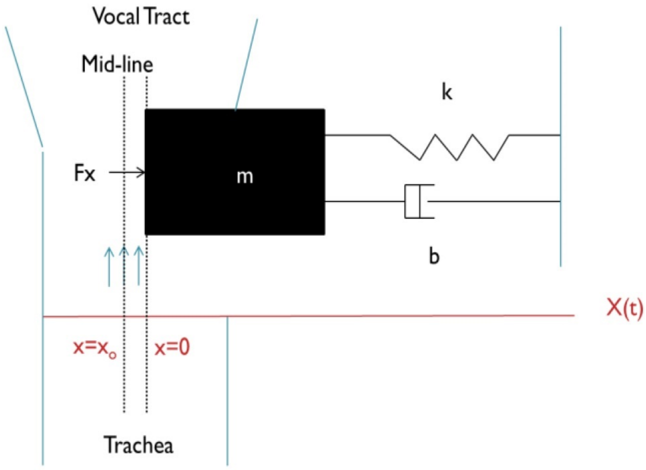


Figure 2: Mass-spring oscillator of vocal folds, F_x is the pressure of air flow from the lung[2, 6].

Lyapunov exponent and sensitive dependence on initial conditions, in Section 6. Finally, a brief conclusion is given in the last section.

2. Preliminaries

The fundamental equation of a self-sustained mass-spring oscillator results:

$$m\ddot{x} + r(1+\eta x^2)\dot{x} + kx = \frac{2P_L\tau\dot{x}}{x_0 + x + \dot{x}}, \qquad x_0 + x + \dot{x} > 0, \tag{2.1}$$

where m is the mass (the inertial property of the mechanical system), k is the spring constant which represents the effective stiffness (elasticity) of various tissue layers of the vocal cords, r is the damping, η is a phenomenological nonlinear coefficient, P_L is the lung pressure, τ is the (sufficiently small) time delay related to the period of the oscillation and x_0 is the half-width at the rest position (Fig 2). It's note that equation (2.1) includes a nonlinear damping term $r(1+\eta x^2)$ and introduced by Lucero in [13]. As applied to the vocal folds, this element represents the viscosity of the tissue, i.e., the energy absorber in the tissue [21].

Now, we can reduce the number of parameters by introducing dimensionless variable $u = \frac{x}{x_0}$, $s = t\sqrt{\frac{k}{m}}$ and also dimensionless parameters

$$\alpha = \frac{r}{\sqrt{mk}}, \quad \beta = \eta x_0^2, \quad \delta = \tau\sqrt{\frac{k}{m}}, \quad \gamma = \frac{2\tau P_L}{x_0\sqrt{mk}},$$

which implies

$$\ddot{u} + \alpha(1+\beta u^2)\dot{u} + u = \frac{\gamma\dot{u}}{1+u+\delta\dot{u}}, \qquad 1+u+\delta\dot{u} > 0, \tag{2.2}$$

where the dot means the derivatives with respect to s. Taking $v = \dot{u}$ converts (2.2) to the following equivalent bi-dimensional system:

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\alpha(1+\beta u^2)v - u + \frac{\gamma v}{1+u+\delta v}, \end{cases} \tag{2.3}$$

where, $1 + u + \delta v > 0$.

Fix $\alpha = 0.32,\ \beta = 100,\ \delta = 0.97$ and γ be the controller parameter. By varying γ a Hopf bifurcation

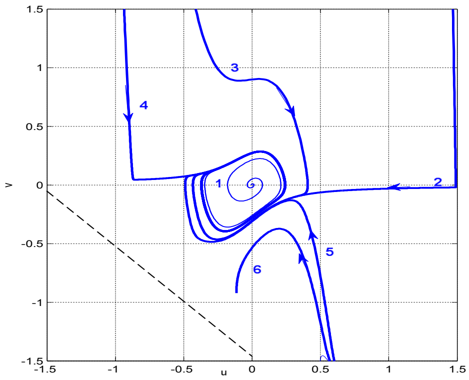


Figure 3: A phase portrait with six trajectories for Eq. (2.3), and $\alpha = 0.32,\ \beta = 100,\ \gamma = 0.78,\ \delta = 0.97$.

occurs and when this parameter crosses its bifurcation value and increases, a stable limit cycle is expelled from the origin, and grows in amplitude. Fig.2 shows a phase portrait with six trajectories for the above parameters. Trajectory 1 started at $(0.01, 0.1)$ and moves away from the origin and approaches the limit cycle, this means $(u, v) = (0, 0)$ is unstable. Trajectories 2, 3, 4 and 5 initiate at $(1.4, 1.5)$, $(-0.4, 1.5)$, $(-0.9, 1.5)$ and $(0.6, -1.5)$ respectively. These trajectories are within basin of attraction of the limit cycle. Trajectory 6 where initiates at $(0.55, -1.5)$ is outside the basin of attraction. This trajectory, as well as the whole region below trajectory 5, are outside the region of validity of the model as a representation of the vocal fold oscillator, since they do not correspond to licit vocal fold motion. More details on this equation can be found in [13].

Using the forward Euler method, we obtain the following discrete model

$$\begin{cases} u_{n+1} = u_n + hv_n, \\ v_{n+1} = v_n - h[\alpha(1+\beta u_n^2)v_n + u_n - \frac{\gamma v_n}{1+u_n+\delta v_n}], \end{cases} \tag{2.4}$$

where $h > 0$ is the step size.

3. Local Stability

In this section, we will study the existence and local stability of equilibria of system (2.4). For this purpose we need the following lemma:

Lemma 3.1. *[12] Assume $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants and let $F(1) > 0$. Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then the following statements hold.*
(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$;
(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ if and only if $F(-1) < 0$;
(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$;
(iv) $\lambda_1 = -1$ and $\lambda_2 \neq -1$ if and only if $F(-1) = 0$ and $B \neq 0, 2$;
(v) λ_1 and λ_2 are a pair of conjugate complex roots and, $|\lambda_1| = |\lambda_2| = 1$ if and only if $-2 < B < 2$ and $C = 1$;
(vi) $\lambda_1 = \lambda_2 = 1$ if and only if $F(-1) = 0$ and $B = 2$.

This model has one fixed point $(u, v) = (0, 0)$.

Proposition 3.2. *In system (2.4), the fixed point $(0, 0)$ is*

1. *a sink if $h < \alpha - \gamma$ and*

(a) $|\gamma - \alpha| < 2$, *or*

(b) $|\gamma - \alpha| > 2$ *and* $h < (\gamma - \alpha) - \sqrt{(\gamma-\alpha)^2 - 4}$, *or* $h > (\gamma - \alpha) + \sqrt{(\gamma-\alpha)^2 - 4}$.

2. *a source if* $h > \alpha - \gamma$ *and* $h < (\gamma - \alpha) - \sqrt{(\gamma-\alpha)^2 - 4}$, *or* $h > (\gamma - \alpha) + \sqrt{(\gamma-\alpha)^2 - 4}$.
3. *a saddle if* $(\gamma - \alpha) - \sqrt{(\gamma-\alpha)^2 - 4} < h < (\gamma - \alpha) + \sqrt{(\gamma-\alpha)^2 - 4}$.
4. *non-hyperbolic if*
   (a) $h = \alpha - \gamma$ *and* $h < 2$, *or*
   (b) $h = (\gamma - \alpha) \pm \sqrt{(\gamma-\alpha)^2 - 4}$ *and* $h \neq \frac{-2}{\gamma-\alpha}, \frac{-4}{\gamma-\alpha}$, *or*
   (c) $\alpha - \gamma = h = 2$.

Proof . We can apply the Lemma 3.1 to model (2.4). The Jacobian matrix associated with (2.4) in $(0, 0)$ is given by

$$J_{|(0,0)} = \begin{pmatrix} 1 & h \\ -h & 1 + (\gamma - \alpha)h \end{pmatrix}. \tag{3.1}$$

The characteristic polynomial of matrix (3.1) is obtain by

$$F(\lambda) = \lambda^2 + B\lambda + C = \lambda^2 - \big(2 + (\gamma - \alpha)h\big)\lambda + h^2 + (\gamma - \alpha)h + 1. \tag{3.2}$$

It is easy to see that $F(1) = h^2 > 0$. By simple calculus we obtain

$$F(-1) = h^2 + 2(\gamma - \alpha)h + 4. \tag{3.3}$$

Therefore, $F(-1) = 0$ iff $h = (\gamma - \alpha) \pm \sqrt{(\gamma-\alpha)^2 - 4}$. So, $F(-1) > 0$ if

1. $\Delta = 4(\gamma - \alpha) - 16 < 0$ or $|\gamma - \alpha| < 2$, or
2. $\Delta > 0$ or $|\gamma - \alpha| > 2$ and $(\gamma - \alpha) - \sqrt{(\gamma-\alpha)^2 - 4} < h < (\gamma - \alpha) + \sqrt{(\gamma-\alpha)^2 - 4}$.

We also have

$$C < 1 \Longleftrightarrow h(h + (\gamma - \alpha)) < 0 \quad \text{or} \quad h < \alpha - \gamma.$$

It's clear that $F(-1) > 0$ and $C > 1$ iff $h > \alpha - \gamma$ and $h < (\gamma - \alpha) - \sqrt{(\gamma-\alpha)^2 - 4}$, or $h > (\gamma - \alpha) + \sqrt{(\gamma-\alpha)^2 - 4}$. Moreover, $C = 1$ iff $h = \alpha - \gamma$ and also $B = 2$ iff $h = \frac{-4}{\gamma-\alpha}$. By checking the conditions in the Lemma 3.1 the proof is complete. □

4. Neimark-Sacker bifurcation

In this section, based on the argument given in the previous section and choosing h as the bifurcation parameter, we investigate the Neimark-Sacker bifurcation in system (2.4). The conditions in $4(a)$ of Proposition 3.2 can be written as the following set

$$NS = \{(\alpha, \gamma, h) \ \mid \ h = \alpha - \gamma, \quad h < 2\}.$$

We claim that when $h^* = \alpha - \gamma$ and $h^* < 2$ the system undergoes the Neimark-Sacker bifurcation. We can rewrite system (2.4) as

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & h \\ -h & 1 + (\gamma - \alpha)h \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -\gamma huv - \gamma\delta hv^2 + (1 - \alpha\beta)hu^2v + 2\gamma\delta huv^2 + \gamma\delta^2 hv^3 + O(4) \end{pmatrix}. \tag{4.1}$$

By changing the variable h to $\widetilde{h}+h^*$ we have that:

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ b_{11}uv + b_{02}v^2 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + O(4) \end{pmatrix}, \tag{4.2}$$

where, we have the coefficients as follows:

$$\begin{aligned} &a_{11} = 1, a_{12} = \widetilde{h}+h^*, a_{21} = -(\widetilde{h}+h^*), a_{22} = 1 + (\gamma - \alpha)(\widetilde{h}+h^*) \\ &b_{11} = -\gamma(\widetilde{h}+h^*), b_{02} = -\gamma\delta(\widetilde{h}+h^*), b_{03} = -\gamma\delta^2(\widetilde{h}+h^*), \\ &b_{21} = (1-\alpha\beta)(\widetilde{h}+h^*), b_{12} = 2\gamma\delta(\widetilde{h}+h^*). \end{aligned}$$

Since parameters belong to NS, then we have the complex eigenvalues $\lambda_{1,2} = \zeta \pm i\eta$ when $\zeta = \frac{2-h^*(\widetilde{h}+h^*)}{2}$ and $\eta = (\frac{\widetilde{h}+h^*}{2})\sqrt{4-(h^*)^2}$. We also have

$$|\lambda_{1,2}| = 1 + \widetilde{h}(\widetilde{h}+h^*), \quad \frac{d|\lambda|}{d\widetilde{h}}|_{\widetilde{h}=0} = h^* \neq 0. \tag{4.3}$$

It is easy to see that for $h^* < 2$

$$\lambda_{1,2}^k \neq, \qquad k = 1,2,3,4. \tag{4.4}$$

Because $\lambda_{1,2}|_{\widetilde{h}=0} = e^{\pm i\omega}$ where $\omega = \cos^{-1}(\frac{2-{h^*}^2}{2})$. Using the translation $\binom{u}{v} = J\binom{\widetilde{u}}{\widetilde{v}}$ where

$$J = \begin{pmatrix} a_{21} & 0 \\ \zeta - a_{11} & -\eta \end{pmatrix}, \tag{4.5}$$

or

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a_{21}\widetilde{u} \\ (\zeta - a_{11})\widetilde{u} - \eta\widetilde{v} \end{pmatrix},$$

we can write system (4.2) in the normal Jordan form as

$$\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \mapsto \begin{pmatrix} \zeta & -\eta \\ \eta & \zeta \end{pmatrix} \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} + \begin{pmatrix} 0 \\ g(\widetilde{u},\widetilde{v}) \end{pmatrix},$$

where

$$\begin{aligned} g(\widetilde{u},\widetilde{v}) = &-\eta b_{02}\widetilde{v}^2 - \frac{b_{02}(\zeta-1)^2 + b_{11}a_{12}(\zeta-1)}{\eta}\widetilde{u}^2 \\ &+ (2b_{02}(\zeta-1) + b_{11}a_{12})\widetilde{u}\widetilde{v} + b_{03}\eta^2\widetilde{v}^3 \\ &- (3b_{03}\eta(\zeta-1) + b_{12}a_{12}\eta)\widetilde{u}\widetilde{v}^2 \\ &+ (3b_{03}(\zeta-1)^2 + 2b_{12}a_{12}(\zeta-1) + b_{21}a_{12}^2)\widetilde{u}^2\widetilde{v} \\ &- \frac{b_{03}(\zeta-1)^3 + b_{12}a_{12}(\zeta-1)^2 + b_{21}a_{12}^2(\zeta-1)}{\eta}\widetilde{u}^3 + O(4). \end{aligned}$$

For investigate the existence of Neimark-Sacker bifurcation, we use the following statement [7]: Assuming that the bifurcating system (restricted to the center manifold) is in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos(c) & -\sin(c) \\ \sin(c) & \cos(c) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \tag{4.6}$$

with eigenvalues $\lambda, \overline{\lambda} = e^{\pm ic}$, one obtains

$$a = -Re\{\frac{(1-2\lambda)\overline{\lambda}^2}{1-\lambda}c_{11}c_{20}\} - \frac{1}{2}|c_{11}|^2 - \mid c_{02} \mid^2 + Re(\overline{\lambda}c_{21}),$$

where

$$\begin{aligned} c_{20} &= \frac{1}{8}\{(f_{xx} - f_{yy} + 2g_{xy}) + i(g_{xx} - g_{yy} - 2f_{xy})\}, \\ c_{11} &= \frac{1}{4}\{(f_{xx} - f_{yy}) + i(g_{xx} - g_{yy})\}, \\ c_{02} &= \frac{1}{8}\{(f_{xx} - f_{yy} - 2g_{xy}) + i(g_{xx} - g_{yy} + 2f_{xy})\} \end{aligned}$$

and

$$c_{21} = \frac{1}{16}\{(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + i(g_{xxx} + g_{xyy} - f_{xxy} - f_{yyy})\}.$$

Since $f(\widetilde{u}, \widetilde{v}) = 0$ after calculating, we get

$$\begin{aligned} c_{20} &= \frac{1}{8}\{2(2b_{02}(\zeta-1) + b_{11}a_{12})) + i(-2\frac{b_{02}(\zeta-1)^2 + b_{11}a_{12}(\zeta-1)}{\eta} + \eta b_{02})\}, \\ c_{11} &= \frac{1}{4}\{i(-2\frac{b_{02}(\zeta-1)^2 + b_{11}a_{12}(\zeta-1)}{\eta} - 2\eta b_{02})\}, \\ c_{02} &= \frac{1}{8}\{(-2(2b_{02}(\zeta-1) + b_{11}a_{12})) + i(-2\frac{b_{02}(\zeta-1)^2 + b_{11}a_{12}(\zeta-1)}{\eta} - 2\eta b_{02})\} \end{aligned}$$

and

$$\begin{aligned} c_{21} =& \frac{1}{16}\{(2(3b_{03}(\zeta-1)^2 + 2b_{12}a_{12}(\zeta-1) + b_{21}a_{12}^2) + 6b_{03}\eta^2) + \\ & i(-6\frac{b_{03}(\zeta-1)^3 + b_{12}a_{12}(\zeta-1)^2 + b_{21}a_{12}^2(\zeta-1)}{\eta} - 2(3b_{03}\eta(\zeta-1) + b_{12}a_{12}\eta))\}. \end{aligned}$$

From above analysis we write the theorem as follows:

Theorem 4.1. *If $(\alpha, \gamma, h) \in NS$ and the condition (4.3) hold and $a \neq 0$, then the system (2.4) passes through a Neimark- Sacker bifurcation at the origin. Moreover, if $a < 0$ (respectively $a > 0$), then an attracting (respectively repelling) invariant closed curve bifurcates from the fixed point for $h > h^*$ (respectively $h < h^*$).*

5. Chaos in Marotto's sense

In this section, we prove system (2.4) possesses chaotic behavior in the sense of Marotto's definition. We first present some definitions and theorem. For a differentiable function $f : R^n \to R^n$, let $Df(p)$ denote the Jacobian matrix of f evaluated at the point $p \in R^n$, and $|Df(p)|$ its determiniant.

Definition 5.1. *[15] Let f be differentiable in closed ball $B_r(z)$. The point $z \in R^n$ is an expanding fixed point of f in $B_r(z)$, if $f(z) = z$ and all eigenvalues of $Df(x)$ exceed 1 in norm for all $x \in B_r(z)$.*

Definition 5.2. *[15] Assume that z is an expanding fixed point of f in $B_r(z)$ for some $r > 0$. Then z is said to be a snap-back repeller of f if there exists a point $x_0 \in B_r(z)$ with $x_0 \neq z$; $f^k(x_0) = z$ and $|Df^k(x_0)| \neq 0$ for some positive integer k*

Theorem 5.3. *[15] If f possesses a snap-back repeller then the map f is chaotic.*

Now we theoretically give the condition such that system (2.4) have a snap-back repeller point. Consider the characteristic equation for System (2.4) at $P = (a, b)$ is

$$\begin{aligned} H(\lambda) &= \lambda^2 \\ &+ (h\alpha(1+\beta a^2) - \frac{h\gamma(1+a)}{(1+a+\delta b)^2} - 2)\lambda \\ &+ (-h\alpha(1+\beta a^2) + \frac{h\gamma(1+a)}{(1+a+\delta b)^2}) + h^2(2\alpha\beta ab + 1 + \frac{h\gamma b}{(1+a+\delta b)^2}) + 1, \end{aligned}$$

and put $A(a, b) = h\alpha(1+\beta a^2) - \frac{h\gamma(1+a)}{(1+a+\delta b)^2} - 2$ and $B(a, b) = (-h\alpha(1+\beta a^2) + \frac{h\gamma(1+a)}{(1+a+\delta b)^2}) + h^2(2\alpha\beta ab + 1 + \frac{h\gamma b}{(1+a+\delta b)^2}) + 1$.
If there exists a positive constant r , such that for all $(x, y) \in R^2$ satisfying $|x-a| < r, |y-b| < r$, the following conditions are satisfied:

$$\Delta = A^2(x, y) - 4B(x, y) < 0, \;\; B(x, y) > 1. \tag{5.1}$$

Then the characteristic Equation have a pair of conjugate complex eigenvalues satisfying $|\lambda_{1,2}| > 1$, and $P = (a, b)$ is an expanding fixed point of System (2.4) in $D(P) = \{(x, y) \in R^2; |x-a| < r, |y-b| < r\}$. Since $P_0 = (0, 0)$ is the only fixed point of System (2.4) and by $\Delta < 0$ and $B(x, y) > 1$ in (5.1), we have the following lemma that obtain necessary conditions of Marotto's chaos.

Lemma 5.4. *If $(\gamma-\alpha)^2 = \varepsilon < 4$ and $h + (\gamma-\alpha) = \sigma > 0$ then there exists $r = r(\varepsilon, \sigma)$ then there exists a neighborhood $D(P_0) = \{(x, y) \in R^2; |x-a| < r, |y-b| < r\}$, such that the norm of eigenvalues of point $(x, y) \in D(P_0)$ is greater than 1, and the fixed point P_0 is an expanding point in $D(P_0)$.*

According to the above argument, we need to prove that, in the neighborhood $B_r(z)$ of the fixed point z, a point z_0 can be found such that the kth iteration of z_0 under the map f comes back to z and $|Df^k(z_0)| \neq 0$. Now, we prove that there exists at least one snap-back repeller point after two iterations.
Indeed, we have

$$\begin{cases} x_0 = u + hv, \\ y_0 = v - h[\alpha(1+\beta u^2)v + u - \frac{\gamma v}{1+u+\delta v}], \end{cases} \tag{5.2}$$

and

$$\begin{cases} 0 = x_0 + hy_0, \\ 0 = y_0 - h[\alpha(1+\beta x_0^2)y_0 + x_0 - \frac{\gamma y_0}{1+x_0+\delta y_0}], \end{cases} \tag{5.3}$$

Now a f^2 map has been constructed to map the point $P_1 = (u, v)$ to the fixed point $z = (0, 0)$ after two iterations if there are nonzero solutions for equations (5.2) and (5.3). Here, we suppose $\delta = h$. By calculation, the nonzero solutions for (5.3) is $x_0 = \pm\sqrt{\frac{1}{\alpha\beta}(\gamma + h + \frac{1}{h} - \alpha)}$ and $y_0 = \frac{-x_0}{h}$. Now, we need to find a point $P_1 = (u, v)$ such that $f(u, v) = P_0$. Thus, we obtain equation:

$$h\alpha\beta u^3 - h\alpha\beta u^2 + u(h\alpha + \frac{\gamma}{h(1+x_0)}) - h\alpha x_0 + x_0 - \frac{\gamma x_0}{h(1+x_0)} = 0. \tag{5.4}$$

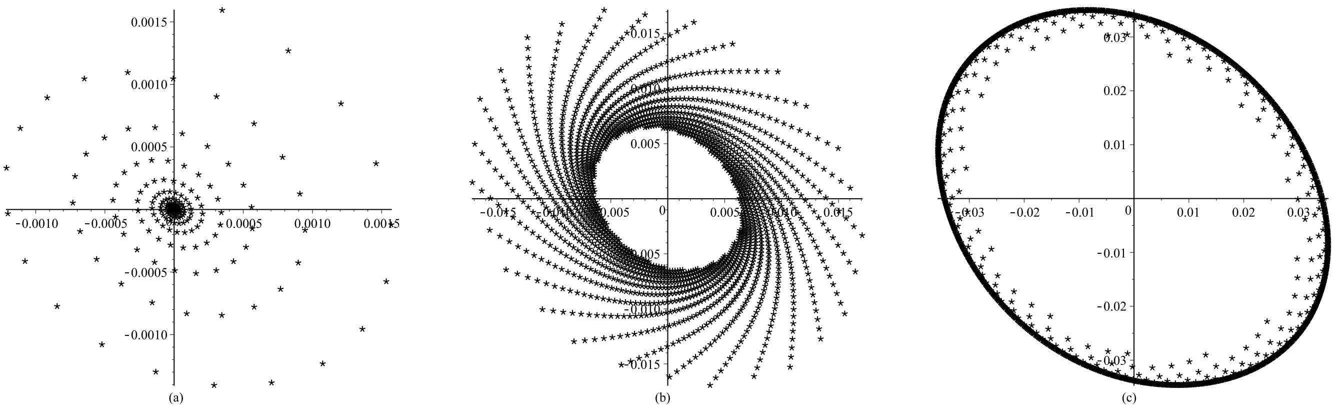


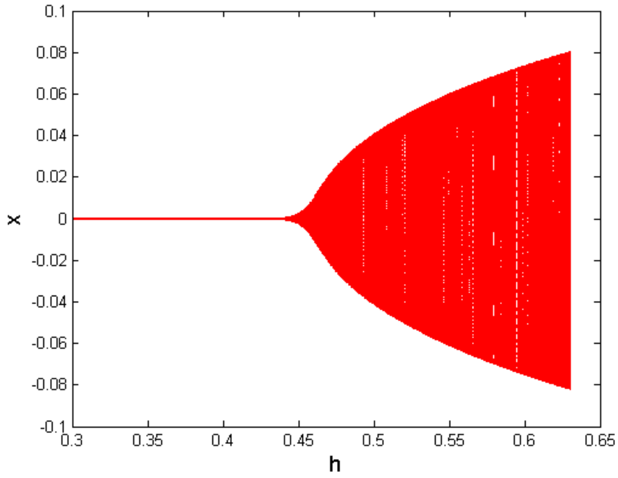
Figure 4: The Phase portrait of model (2.4) for $\alpha = 0.78, \beta = 100, \gamma = 0.32, \delta = 0.97$, and $(u_0, v_0) = (0.02, 0.002)$. (a) $h = 0.06$, in this case $h < \alpha - \gamma$, so the origin is a sink, (b) $h = 0.19$, in this case $h = \alpha - \gamma$, so the origin is a non-hyperbolic fixed point (c) $h = 0.6$, in this case $h > \alpha - \gamma$, so the origin is a source.

This is clear that equation (5.4) has at least one real root. Next we prove that there is a real root located in the neighborhood $B_r(z)$. It is notice that the constant item $-h\alpha x_0 + x_0 - \frac{\gamma x_0}{h(1+x_0)}$ in (5.4) may change from negative to positive. This implies that there is a real nonzero root in the small neighborhood of the origin or $B_r(z = 0)$. In the other hand, it is obvious that $|Df^2(P_1)| = |Df(z)||Df(P_1)| \neq 0$. Because $Df(x, y) = 0$ implies that $x^2 = \frac{\gamma-\alpha+h}{3\alpha\beta}$ and this is different from the solutions of equations (5.2) and (5.3). Thus, the following theorem is established.

Theorem 5.5. *Assume that $z = (0, 0)$ is an expanding fixed point in $B_r(z)$, if the solutions $(u, v), (x_0, y_0)$ of (5.2) and (5.3) satisfy $(u, v) \in B_r(z)$ and $|Df^2(P_1)| \neq 0$. Then z is a snap-back repeller of system (2.4), and hence system (2.4) is chaotic in the sense of Marotto.*

6. Numerical Simulation

In this section we show some numerical simulations for the system (2.4) to support our theoretical results. We choose $\alpha = 0.78, \beta = 100, \gamma = 0.32, \delta = 0.97$, and $(u_0, v_0) = (0.02, 0.002)$. For different values of parameter h we plot the phase portrait in Fig. 4. When $h < \alpha - \gamma$, the origin is a sink, for $h = \alpha - \gamma$ the origin becomes a non-hyperbolic and for $h > \alpha - \gamma$ it becomes an unstable fixed point (or source). Moreover, for the last case, a stable limit cycle has been arisen. Hence, Fig. 4 verify Proposition 3.2 and Theorem 4.1. In [13], Lucero considered real data for typical adult as $\alpha = 0.32, \beta = 100, \gamma = 0.78, \delta = 0.97$. For these real data the discrete model has always a limit cycle, because $\alpha - \gamma < 0$ and h is a positive parameters. Note that, the limit cycle in the discrete model is far from the singular line $1 + u + \delta v = 0$. Therefore, our results is reliable. We also numerically compute the bifurcation of system (2.4). The bifurcation diagram is shown in Fig. 5. In the previous section we analytically proved that system (2.4) has chaotic behaviour by Marotto's definition. Moreover, we interest to investigate chaotic behaviours of the vocal fold model in other sense. In what follows, we study some indication of chaos by numerical methods. The usual numerical test for the chaos is calculation of the maximum Lyapunov exponent. A positive Lyapunov exponent indicates chaos [18]. Therefore, we compute the maximum Lyapunov exponent of model (2.4) with respect to h. Fig. 6 shows that system (2.4) has positive Lyapunov exponent for $h \in [0.46, 0.6]$, so we expect the chaotic behaviour in this interval. In the other hand, from Devaney's point of view [5], one of the components of chaotic systems is "sensitive dependence to initial conditions". In the next subsection, we numerically obtain sensitive dependence to initial conditions for the model.



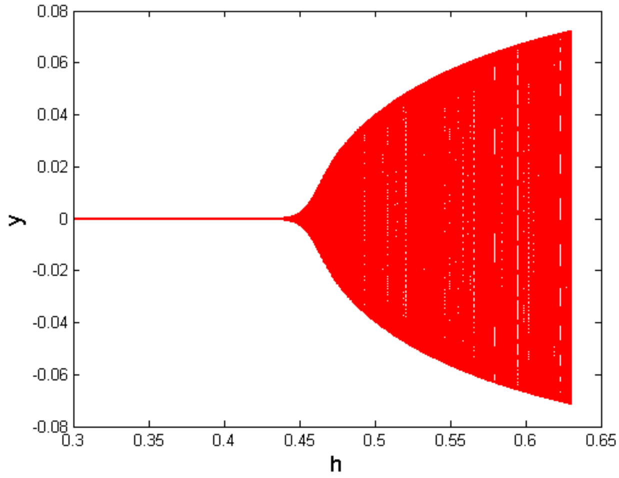


Figure 5: The bifurcation diagrams of the model for $\alpha = 0.78, \beta = 100, \gamma = 0.32, \delta = 0.97$, and $(u_0, v_0) = (0.02, 0.05)$.

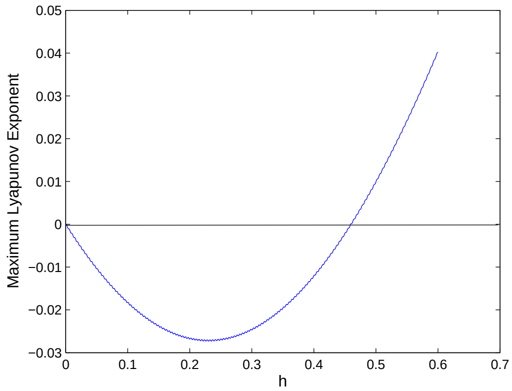


Figure 6: Maximum Lyapunov exponent of system (2.4) for $\alpha = 0.78,\ \beta = 100,\ \gamma = 0.32,\ \delta = 0.97$.

6.1. Sensitive dependence to initial conditions

To demonstrate the sensitivity to initial conditions of model (2.4), two orbits are considered. We fix the parameters $\alpha = 0.78$, $\beta = 100$, $\gamma = 0.32$, $\delta = 0.97$ and $h = 0.55$. Consider the initial points $(x_0, y_0) = (-0.19, -0.1)$ and $(\tilde{x}_0, \tilde{y}_0) = (x_0 + 0.01, y_0)$. At the beginning, the two time series are very close to each other; but after 20 iterations, the difference between them builds up rapidly. For example $(x_{20}, y_{20}) = (-0.070603, -0.002022)$ and $(\tilde{x}_{20}, \tilde{y}_{20}) = (-1.1714 \times 10^{3176}, 2.1974 \times 10^{6353})$. Fig. 6 demonstrates and confirms the above argument and shows a sensitive dependence on initial conditions for the model.

7. Conclusion and discussion

In this paper, we investigated the stability, bifurcation and chaos in of a two-dimensional discrete time vocal fold model. The discrete-time model exhibit more plentiful and complicated dynamical behaviours than a continuous-time model of the same type. Indeed, for small step size the discrete-time model is a good approximation of the continuous-time model, when the step size becomes large the discrete models shows more complicated behaviours. As discussed in Section 6, the discrete-time model have the chaotic behaviour for $h \simeq 4.5$ and more, (see figure 6). Many studies show that disordered voices from laryngeal pathologies such as vocal polyps and vocal nodules might exhibit chaotic behaviors [8]. So, consider the chaotic conditions in vocal fold dynamical model is very important. In this paper we obtained some conditions that the discrete time vocal fold model is chaotic in the sense of Marotto.

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