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# Inequalities for $tgs$-convex functions via some conformable fractional integrals

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**Abstract**

In this research article, we establish several Hermite-Hadamard type inequalities for $tgs$-convex functions via conformable fractional integrals and new fractional conformable integral operators.

*Keywords:* Hermite-Hadamard inequalities, $tgs$-convex functions, Conformable fractional integral, new fractional conformable integral operators.
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## 1. Introduction

In literature, convex functions and its generalization have become more importance due to its significant classical integral inequalities. The Hermite-Hadamard inequality [8, 9] for convex functions $\Upsilon : I \to \mathbb{R}$ on an interval $I$ of real line is defined as:

$$\Upsilon\left(\frac{b_1+b_2}{2}\right) \leq \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon(s)ds \leq \frac{\Upsilon(b_1)+\Upsilon(b_2)}{2}, \tag{1.1}$$

for all $b_1, b_2 \in I$ with $b_1 < b_2$. For more details see [2, 3, 7, 22, 11, 13, 14, 15, 18, 26].

Fractional calculus [12] has performeded major role in different scientific fields. In [23], Sarikaya et. al. showed some Hermite-Hadamard and Hermite-Hadamard type integral inequalities for fractional integrals. In [4, 5, 6, 16, 17, 19, 20, 21, 25], authors proved several Hermite-Hadamard type inequalities for variuos generalized fractional integrals.

Tunc et. al. [24] defined new class of functions called $tgs$-convex functions.

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**Definition 1.1 ([24]).** *A function $\Upsilon : I \subseteq \mathbb{R} \to \mathbb{R}$ is called tgs-convex, if it is nonnegative and satisfy the following inequality*

$$\Upsilon\left(\mu b_1 + (1-\mu)b_2\right) \leq \mu(1-\mu)[\Upsilon(b_1) + \Upsilon(b_2)], \tag{1.2}$$

*for all $b_1, b_2 \in I$ and $\mu \in [0,1]$.*

Abdeljawad [1] defined the conformable fractional integral as:

**Definition 1.2 ([1]).** *Let $\alpha \in (m, m+1]$ and $\gamma = \alpha - m$. Then the left and conformable fractional integrals starting at $b_1$ of order $\alpha > 0$ is defined by*

$$J_{\alpha}^{b_1} \Upsilon(s) = \frac{1}{m!}\int_{b_1}^{s} (s-t)^m (t-b_1)^{\gamma-1} \Upsilon(t)dt,$$

*and the right conformable fractional integrals is defined by*

$$J_{\alpha}^{b_2} \Upsilon(s) = \frac{1}{m!}\int_{s}^{b_2} (t-s)^m (b_2-t)^{\gamma-1} \Upsilon(t)dt.$$

Jarad et. al. [10] has defined the following new fractional integral operator.

**Definition 1.3 ([10]).** *Let $\gamma \in \mathbb{C}$, then the left and right sided fractional conformable integral operators of order $\alpha > 0$ are characterised as:*

$$ {}_{b_1}^{\gamma}\mathcal{J}^{\alpha} \Upsilon(s) = \frac{1}{\Gamma(\gamma)}\int_{b_1}^{s} \left(\frac{(s-b_1)^{\alpha} - (t-b_1)^{\alpha}}{\alpha}\right)^{\gamma-1} \frac{\Upsilon(t)}{(t-b_1)^{1-\alpha}}dt, \tag{1.3}$$

$$ {}^{\gamma}\mathcal{J}_{b_2}^{\alpha} \Upsilon(s) = \frac{1}{\Gamma(\gamma)}\int_{s}^{b_2} \left(\frac{(b_2-s)^{\alpha} - (b_2-t)^{\alpha}}{\alpha}\right)^{\gamma-1} \frac{\Upsilon(t)}{(b_2-t)^{1-\alpha}}dt. \tag{1.4}$$

The classical Beta and The incomplete Beta function is given as:

1. The Beta function:

   $\beta(b_1, b_2) = \int_0^1 t^{b_1-1}(1-t)^{b_2-1}dt$

2. The incomplete Beta function:

   $\beta_u(b_1, b_2) = \int_0^u t^{b_1-1}(1-t)^{b_2-1}dt, \quad u \in [0,1].$

Following relationship holds between classical Beta and incomplete Beta functions:

$$\beta(b_1, b_2) = \beta_u(b_1, b_2) + \beta_{1-u}(b_1, b_2).$$

Further,

$$\beta_u(b_1+1, b_2) = \frac{b_1\beta_u(b_1, b_2) - (\frac{1}{2})^{b_1+b_2}}{b_1+b_2},$$

and

$$\beta_u(b_1, b_2+1) = \frac{b_2\beta_u(b_1, b_2) - (\frac{1}{2})^{b_1+b_2}}{b_1+b_2}.$$

Our aim is to prove some Hermite-Hadamard type inequalities for $tgs$-convex functions via conformable as well as new conformable fractional integrals. In the coming section 2 we will prove integral inequalities for $tgs$-convex functions via conformable fractional integrals and then in the later section 3 we will prove integral inequalities for $tgs$-convex functions via new fractional conformable integral operators.

## 2. Inequalities via conformable fractional integrals

In this section, we show some integral properties for $tgs$-convex functions via conformable fractional integrals.

**Theorem 2.1.** *Let $\Upsilon : [b_1, b_2] \subset \mathbb{R} \to \mathbb{R}$ be a tgs-convex function such that $\Upsilon \in L_1[b_1, b_2]$, then*

$$\begin{aligned} &\frac{4\Gamma(\alpha-m)}{\Gamma(\alpha+1)} \Upsilon\left(\frac{b_1+b_2}{2}\right) \\ &\leq \frac{1}{(b_2-b_1)^\alpha}\left[J_\alpha^{b_1} \Upsilon\left(b_2\right)+J_\alpha^{b_2} \Upsilon\left(b_1\right)\right] \\ &\leq \frac{2(m+1)\Gamma(\alpha-m+1)}{\Gamma(\alpha+3)}(\Upsilon(b_1)+\Upsilon(b_2)). \end{aligned} \tag{2.1}$$

**Proof .** Using $tgs$-convexity of $\Upsilon$, we have

$$\Upsilon\left(\frac{x+y}{2}\right) \leq \frac{\Upsilon(x)+\Upsilon(y)}{4}. \tag{2.2}$$

Let $x = \mu b_1 + (1-\mu)b_2$ and $y = (1-\mu)b_1 + \mu b_2$, we get

$$4\,\Upsilon\left(\frac{b_1+b_2}{2}\right) \leq \Upsilon(\mu b_1+(1-\mu)b_2)+\Upsilon(\mu b_2+(1-\mu)b_1). \tag{2.3}$$

Multiplying (2.3) by $\frac{1}{m!}\mu^m(1-\mu)^{\alpha-m-1}$ with $\mu \in (0,1)$, $\alpha > 0$ and then integrating the resulting inequality with respect to $\mu$ over $[0,1]$, we find

$$\begin{aligned} &\frac{4}{m!} \Upsilon\left(\frac{b_1+b_2}{2}\right) \int_0^1 \mu^m(1-\mu)^{\alpha-m-1} d\mu \\ &\leq \frac{1}{m!}\int_0^1 \mu^m(1-\mu)^{\alpha-m-1}\, \Upsilon\left(\mu b_1+(1-\mu)b_2\right) d\mu \\ &\quad+\frac{1}{m!}\int_0^1 \mu^m(1-\mu)^{\alpha-m-1}\, \Upsilon\left(\mu b_2+(1-\mu)b_1\right) d\mu \\ &= I_1+I_2. \end{aligned} \tag{2.4}$$

By setting $t = \mu b_1 + (1-\mu)b_2$, we have

$$\begin{aligned} I_1 &= \frac{1}{m!}\int_0^1 \mu^m(1-\mu)^{\alpha-m-1}\, \Upsilon\left(\mu b_1+(1-\mu)b_2\right) d\mu \\ &= \frac{1}{m!}\int_{b_2}^{b_1} \left(\frac{t-b_2}{b_1-b_2}\right)^m\left(1-\frac{t-b_2}{b_1-b_2}\right)^{\alpha-m-1} \Upsilon\,(t)\frac{dt}{b_1-b_2} \\ &= \frac{1}{m!\,(b_2-b_1)^\alpha}\int_{b_1}^{b_2} (b_2-t)^m\,(t-b_1)^{\alpha-m-1}\, \Upsilon\,(t)dt \\ &= \frac{1}{(b_2-b_1)^\alpha} J_\alpha^{b_1}\, \Upsilon\,(b_2). \end{aligned} \tag{2.5}$$

Similarly, by setting $t = \mu b_2 + (1-\mu)b_1$, we have

$$\begin{aligned} I_2 &= \frac{1}{m!}\int_0^1 \mu^m (1-\mu)^{\alpha-m-1} \Upsilon\left(\mu b_2 + (1-\mu)b_1\right) d\mu \\ &= \frac{1}{m!}\int_{b_1}^{b_2} \left(\frac{t-b_1}{b_2-b_1}\right)^m \left(1-\frac{t-b_1}{b_2-b_1}\right)^{\alpha-m-1} \Upsilon(t)\frac{dt}{b_2-b_1} \\ &= \frac{1}{m!\,(b_2-b_1)^\alpha}\int_{b_1}^{b_2} (t-b_1)^m (b_2-t)^{\alpha-m-1} \Upsilon(t)dt \\ &= \frac{1}{(b_2-b_1)^\alpha} J_\alpha^{b_2} \Upsilon(b_1). \end{aligned} \tag{2.6}$$

Thus by using (2.5) and (2.6) in (2.4), we get the first inequality of (2.1).

Now consider,

$$\Upsilon(\mu b_1 + (1-\mu)b_2) \leq \mu(1-\mu)(\Upsilon(b_1) + \Upsilon(b_2)),$$

and

$$\Upsilon(\mu b_2 + (1-\mu)b_1) \leq \mu(1-\mu)(\Upsilon(b_2) + \Upsilon(b_1)).$$

By adding

$$\Upsilon(\mu b_1 + (1-\mu)b_2) + \Upsilon(\mu b_2 + (1-\mu)b_1) \leq 2\mu(1-\mu)(\Upsilon(b_1) + \Upsilon(b_2)). \tag{2.7}$$

Multiplying (2.7) by $\frac{1}{m!}\mu^m(1-\mu)^{\alpha-m-1}$ with $\mu \in (0,1)$, $\alpha > 0$ and then integrating the resulting inequality with respect to $\mu$ over $[0,1]$, we get

$$\begin{aligned} &\frac{1}{(b_2-b_1)^\alpha}\left[J_\alpha^{b_1} \Upsilon(b_2) + J_\alpha^{b_2} \Upsilon(b_1)\right] \\ &\leq \frac{2(m+1)\Gamma(\alpha-m+1)}{\Gamma(\alpha+3)}(\Upsilon(b_1) + \Upsilon(b_2)). \end{aligned} \tag{2.8}$$

Hence proof is completed. □

**Remark 2.2.** *In Theorem* 2.1*, if we take* $\alpha = n+1$*, then we obtain Theorem* 3.1 *in* [24].

**Lemma 2.3.** *Let* $\Upsilon : [b_1, b_2] \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* $(b_1, b_2)$ *with* $b_1 < b_2$ *such that* $\Upsilon' \in L_1[b_1, b_2]$*, then*

$$\begin{aligned} &\Delta_\Upsilon(b_1, b_2; \alpha; \beta; J) \\ &= \frac{b_2-b_1}{2}\int_0^1 \left(\beta_{1-t}(m+1, \alpha-m) - \beta_t(m+1, \alpha-m)\right) \Upsilon'\left(\mu b_1 + (1-\mu)b_2\right) d\mu, \end{aligned} \tag{2.9}$$

*where*

$$\begin{aligned} &\Delta_\Upsilon(b_1, b_2; \alpha; \beta; J) \\ &= \beta(m+1, \alpha-m)\left(\frac{\Upsilon(b_1) + \Upsilon(b_2)}{2}\right) - \frac{m!}{2(b_2-b_1)^\alpha}\left[J_\alpha^{b_1} \Upsilon(b_2) + J_\alpha^{b_2} \Upsilon(b_1)\right]. \end{aligned}$$

**Proof .** Consider,

$$\begin{aligned}
&\int_0^1 (\beta_{1-t}(m+1,\alpha-m) - \beta_t(m+1,\alpha-m)) \curlyvee' (\mu b_1 + (1-\mu)b_2) d\mu \\
&= \int_0^1 \beta_{1-t}(m+1,\alpha-m) \curlyvee' (\mu b_1 + (1-\mu)b_2) d\mu \\
&\quad - \int_0^1 \beta_t(m+1,\alpha-m) \curlyvee' (\mu b_1 + (1-\mu)b_2) d\mu \\
&= I_1 - I_2.
\end{aligned} \tag{2.10}$$

Then by integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1 \beta_{1-t}(m+1,\alpha-m) \curlyvee' (\mu b_1 + (1-\mu)b_2) d\mu \\
&= \int_0^1 \left( \int_0^{1-t} u^m (1-u)^{\alpha-m-1} du \right) \curlyvee' (\mu b_1 + (1-\mu)b_2) d\mu \\
&= \frac{1}{b_2-b_1} \beta(m+1,\alpha-m) \curlyvee (b_2) \\
&\quad - \frac{1}{b_2-b_1} \int_0^1 (1-t)^m t^{\alpha-m-1} \curlyvee (\mu b_1 + (1-\mu)b_2) d\mu \\
&= \frac{1}{b_2-b_1} \beta(m+1,\alpha-m) \curlyvee (a_2) \\
&\quad - \frac{1}{b_2-b_1} \int_{b_2}^{b_1} \left(1 - \frac{x-b_2}{b_1-b_2}\right)^m \left(\frac{x-b_2}{b_1-b_2}\right)^{\alpha-m-1} \frac{\curlyvee(x)}{b_1-b_2} dx \\
&= \frac{1}{b_2-b_1} \beta(m+1,\alpha-m) \curlyvee (b_2) - \frac{m!}{(b_2-b_1)^{\alpha+1}} J_{\alpha}^{b_2} \curlyvee (b_1).
\end{aligned} \tag{2.11}$$

Similarly, we have

$$\begin{aligned}
I_2 &= \int_0^1 \beta_t(m+1,\alpha-m) \curlyvee' (\mu b_1 + (1-\mu)b_2) d\mu \\
&= \int_0^1 \left( \int_0^{t} u^m (1-u)^{\alpha-m-1} du \right) \curlyvee' (\mu b_1 + (1-\mu)b_2) d\mu \\
&= -\frac{1}{b_2-b_1} \beta(m+1,\alpha-m) \curlyvee (b_1) \\
&\quad + \frac{1}{b_2-b_1} \int_0^1 t^m (1-t)^{\alpha-m-1} \curlyvee (\mu b_1 + (1-\mu)b_2) d\mu \\
&= -\frac{1}{b_2-b_1} \beta(m+1,\alpha-n) \curlyvee (b_1)
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{b_2-b_1} \int_{b_2}^{b_1} \left(\frac{x-b_2}{b_1-b_2}\right)^m \left(1 - \frac{x-b_2}{b_1-b_2}\right)^{\alpha-m-1} \frac{\curlyvee(x)}{b_1-b_2} dx \\
&= -\frac{1}{b_2-b_1} \beta(m+1,\alpha-m) \curlyvee (b_1) + \frac{m!}{(b_2-b_1)^{\alpha+1}} J_{\alpha}^{b_1} \curlyvee (b_2).
\end{aligned} \tag{2.12}$$

By substituting values of $I_1$ and $I_2$ in (2.10) and then multiplying by $\frac{a_2-a_1}{2}$ we get (2.9). $\square$

**Theorem 2.4.** *Let $\Upsilon : [b_1, b_2] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $(b_1, b_2)$ with $b_1 < b_2$ such that $\Upsilon' \in L_1[b_1, b_2]$. If $|\Upsilon'|^q$, with $q \geq 1$, is tgs-convex function, then the following inequality holds*

$$|\Delta_\Upsilon(b_1, b_2; \alpha; \beta; J)| \leq \frac{b_2 - b_1}{2}\lambda^{1-1/q}\left(\frac{|\Upsilon'(b_1)|^q + |\Upsilon'(b_2)|^q}{6}\right)^{1/q}, \tag{2.13}$$

*where*

$$\lambda = \beta(m+1, \alpha - m + 1) - \beta(m+1, \alpha - m) + \beta(m+2, \alpha - m).$$

**Proof .** Using Lemma 2.3, property of modulus, Power mean inequality and $tgs$-convexity of $|\Upsilon'|^q$, we have

$$\begin{aligned}
&|\Delta_\Upsilon(b_1, b_2; \alpha; \beta; J)| \\
&= \left|\frac{b_2 - b_1}{2}\int_0^1 (\beta_{1-t}(m+1, \alpha - m) - \beta_t(m+1, \alpha - m))\,\Upsilon'(\mu b_1 + (1-\mu)b_2)d\mu\right| \\
&\leq \frac{b_2 - b_1}{2}\left(\int_0^1 (\beta_{1-t}(m+1, \alpha - m) - \beta_t(m+1, \alpha - m))\,d\mu\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 |\Upsilon'(\mu b_1 + (1-\mu)b_2)|^q d\mu\right)^{\frac{1}{q}} \\
&\leq \frac{b_2 - b_1}{2}\lambda^{1-\frac{1}{q}}\left(\int_0^1 (\mu(1-\mu))(|\Upsilon'(b_1)|^q + |\Upsilon'(b_2)|^q)d\mu\right)^{\frac{1}{q}},
\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
\lambda &= \int_0^1 (\beta_{1-t}(m+1, \alpha - m) - \beta_t(m+1, \alpha - m))\,d\mu \\
&= \beta(m+1, \alpha - m + 1) - \beta(m+1, \alpha - m) + \beta(m+2, \alpha - m),
\end{aligned}$$

and

$$\int_0^1 \mu(1-\mu)d\mu = \frac{1}{6}.$$

Hence the proof is completed. □

**Theorem 2.5.** *Let $\Upsilon : [b_1, b_2] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $(b_1, b_2)$ with $b_1 < b_2$ such that $\Upsilon' \in L_1[b_1, b_2]$. If $|\Upsilon'|^q$, with $q, p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, is tgs-convex function, then the following inequality holds*

$$|\Delta_\Upsilon(b_1, b_2; \alpha; \beta; J)| \leq \frac{b_2 - b_1}{2}\nu^{1/p}\left(\frac{|\Upsilon'(b_1)|^q + |\Upsilon'(b_2)|^q}{6}\right)^{1/q}, \tag{2.15}$$

*where*

$$\nu = 2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} u^m(1-u)^{\alpha - m - 1}du\right)dt,$$

**Proof .** Using Lemma 2.3, property of modulus, Holder's inequality and $tgs$-convexity of $|\Upsilon'|^q$, we have

$$\begin{aligned}
&|\Delta_\Upsilon(b_1,b_2;\alpha;\beta;J)| \\
&= \left|\frac{b_2-b_1}{2}\int_0^1 (\beta_{1-t}(m+1,\alpha-m)-\beta_t(m+1,\alpha-m))\,\Upsilon'(\mu b_1+(1-\mu)b_2)d\mu\right| \\
&\le \frac{b_2-b_1}{2}\left(\int_0^1 |\beta_{1-t}(m+1,\alpha-m)-\beta_t(m+1,\alpha-m)|^p\,d\mu\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 |\Upsilon'(\mu b_1+(1-\mu)b_2)|^q d\mu\right)^{\frac{1}{q}} \\
&\le \frac{b_2-b_1}{2}\nu^{\frac{1}{p}}\left(\int_0^1 (\mu(1-\mu))(|\Upsilon'(b_1)|^q+|\Upsilon'(b_2)|^q)d\mu\right)^{\frac{1}{q}},
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
\nu &= \int_0^1 |\beta_{1-t}(m+1,\alpha-m)-\beta_t(m+1,\alpha-m)|^p\,dt \\
&= \int_0^{\frac{1}{2}} (\beta_{1-t}(m+1,\alpha-m)-\beta_t(m+1,\alpha-m))^p\,dt \\
&\quad + \int_{\frac{1}{2}}^1 (\beta_t(m+1,\alpha-m)-\beta_{1-t}(m+1,\alpha-m))^p\,dt \\
&= \int_0^{\frac{1}{2}}\left(\int_t^{1-t} u^m(1-u)^{\alpha-m-1}du\right)^p dt + \int_{\frac{1}{2}}^1\left(\int_{1-t}^{t} u^m(1-u)^{\alpha-m-1}du\right)^p dt \\
&= 2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} u^m(1-u)^{\alpha-m-1}du\right)^p dt,
\end{aligned}$$

and

$$\int_0^1 \mu(1-\mu)d\mu = \frac{1}{6}.$$

Hence the proof is completed. □

## 3. Inequalities via new fractional conformable integral operators

In this section, we show some integral properties for $tgs$-convex functions via new fractional conformable integral operators.

**Theorem 3.1.** *Let $\Upsilon : [b_1,b_2]\subset\mathbb{R}\to\mathbb{R}$ be a tgs-convex function such that $\Upsilon\in L_1[b_1,b_2]$, then*

$$\begin{aligned}
&\frac{4}{\gamma\alpha^\gamma}\Upsilon\left(\frac{b_1+b_2}{2}\right) \\
&\frac{\Gamma(\gamma)}{(b_2-b_1)^{\alpha\gamma}}\left[{}^{\gamma}_{b_1}\mathcal{J}^{\alpha}\,\Upsilon(a_2)+{}^{\gamma}\mathcal{J}^{\alpha}_{b_2}\,\Upsilon(b_1)\right] \\
&\le \frac{\beta(\frac{\alpha+1}{\alpha},\gamma)+\beta(\frac{\alpha+2}{\alpha},\gamma)}{\alpha}(\Upsilon(b_1)+\Upsilon(b_2)).
\end{aligned} \tag{3.1}$$

**Proof .** Multiplying (2.3) by $\left(\frac{1-\mu^\alpha}{\alpha}\right)^{\gamma-1}\mu^{\alpha-1}$ with $\mu\in(0,1)$, $\alpha>0$ and then integrating the resulting inequality with respect to $\mu$ over $[0,1]$, we find

$$\begin{aligned}&4\,\Upsilon\left(\frac{b_1+b_2}{2}\right)\int_0^1\left(\frac{1-\mu^\alpha}{\alpha}\right)^{\gamma-1}\mu^{\alpha-1}d\mu\\&\le\int_0^1\left(\frac{1-\mu^\alpha}{\alpha}\right)^{\gamma-1}\mu^{\alpha-1}\,\Upsilon\left(\mu b_1+(1-\mu)b_2\right)d\mu\\&\quad+\int_0^1\left(\frac{1-\mu^\alpha}{\alpha}\right)^{\gamma-1}\mu^{\alpha-1}\,\Upsilon\left(\mu b_2+(1-\mu)b_1\right)d\mu\\&=I_1+I_2.\end{aligned}\tag{3.2}$$

By setting $t=\mu b_1+(1-\mu)b_2$, we have

$$\begin{aligned}I_1&=\int_0^1\left(\frac{1-\mu^\alpha}{\alpha}\right)^{\gamma-1}\mu^{\alpha-1}\,\Upsilon\left(\mu b_1+(1-\mu)b_2\right)d\mu\\&=\int_{b_2}^{b_1}\left(\frac{1-\left(\frac{t-b_2}{b_1-b_2}\right)^\alpha}{\alpha}\right)^{\gamma-1}\left(\frac{t-b_2}{b_1-b_2}\right)^{\alpha-1}\Upsilon(t)\frac{dt}{b_1-b_2}\\&=\frac{1}{(b_2-b_1)^{\alpha\gamma}}\int_{b_1}^{b_2}\left(\frac{(b_2-b_1)^\alpha-(b_2-t)^\alpha}{\alpha}\right)^{\gamma-1}(b_2-t)^{\alpha-1}\,\Upsilon(t)dt\\&=\frac{\Gamma(\gamma)}{(b_2-b_1)^{\alpha\gamma}}\;{}^{\gamma}\mathcal{J}^{\alpha}_{b_2}\,\Upsilon(b_1).\end{aligned}\tag{3.3}$$

Similarly, by setting $t=\mu b_2+(1-\mu)b_1$, we have

$$\begin{aligned}I_2&=\int_0^1\left(\frac{1-\mu^\alpha}{\alpha}\right)^{\gamma-1}\mu^{\alpha-1}\,\Upsilon\left(\mu b_2+(1-\mu)b_1\right)d\mu\\&=\int_{b_2}^{b_1}\left(\frac{1-\left(\frac{t-b_1}{b_2-b_1}\right)^\alpha}{\alpha}\right)^{\gamma-1}\left(\frac{t-b_1}{b_2-b_1}\right)^{\alpha-1}\Upsilon(t)\frac{dt}{b_2-b_1}\\&=\frac{1}{(b_2-b_1)^{\alpha\gamma}}\int_{b_1}^{b_2}\left(\frac{(b_2-b_1)^\alpha-(t-b_1)^\alpha}{\alpha}\right)^{\gamma-1}(t-b_1)^{\alpha-1}\,\Upsilon(t)dt\\&=\frac{\Gamma(\gamma)}{(b_2-b_1)^{\alpha\gamma}}\;{}^{\gamma}_{b_1}\mathcal{J}^{\alpha}\,\Upsilon(b_2).\end{aligned}\tag{3.4}$$

Thus by using (3.3) and (3.4) in (3.2), we get the first inequality of (3.1). Now for the second inequality of (3.1) multiplying (2.7) by $\left(\frac{1-\mu^\alpha}{\alpha}\right)^{\gamma-1}\mu^{\alpha-1}$ with $\mu\in(0,1)$, $\alpha>0$ and then integrating the resulting inequality with respect to $\mu$ over $[0,1]$, we get

$$\begin{aligned}&\frac{\Gamma(\gamma)}{(b_2-b_1)^{\alpha\gamma}}\left[{}^{\gamma}_{b_1}\mathcal{J}^{\alpha}\,\Upsilon(b_2)+{}^{\gamma}\mathcal{J}^{\alpha}_{b_2}\,\Upsilon(b_1)\right]\\&\le\frac{\beta(\frac{\alpha+1}{\alpha},\gamma)+\beta(\frac{\alpha+2}{\alpha},\gamma)}{\alpha}(\Upsilon(b_1)+\Upsilon(b_2)).\end{aligned}\tag{3.5}$$

Hence proof is completed. □

**Lemma 3.2.** *Let $\Upsilon : [b_1, b_2] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $(b_1, b_2)$ with $b_1 < b_2$ such that $\Upsilon' \in L_1[b_1, b_2]$, then*

$$\begin{aligned}&\Delta_\Upsilon(b_1, b_2; \alpha; \gamma; \mathcal{J})\\&= \frac{(b_2-b_1)\alpha^\gamma}{2}\int_0^1 \left[\left(\frac{1-\mu^\alpha}{\alpha}\right)^\gamma - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\gamma\right] \Upsilon'\left(\mu b_1 + (1-\mu)b_2\right)d\mu,\end{aligned} \tag{3.6}$$

*where*

$$\begin{aligned}&\Delta_\Upsilon(b_1, b_2; \alpha; \gamma; \mathcal{J})\\&= \left(\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2}\right) - \frac{\alpha^\gamma\Gamma(\gamma+1)}{2(b_2-b_1)^{\alpha\gamma}}\left[{}^\gamma_{b_1}\mathcal{J}^\alpha \Upsilon(b_2) + {}^\gamma\mathcal{J}^\alpha_{b_2} \Upsilon(b_1)\right].\end{aligned}$$

**Proof .** Consider,

$$\begin{aligned}&\int_0^1 \left[\left(\frac{1-\mu^\alpha}{\alpha}\right)^\gamma - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\gamma\right] \Upsilon'\left(\mu b_1 + (1-\mu)b_2\right)d\mu\\&= \int_0^1 \left(\frac{1-\mu^\alpha}{\alpha}\right)^\gamma \Upsilon'\left(\mu b_1 + (1-\mu)b_2\right)d\mu\\&\quad - \int_0^1 \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\gamma \Upsilon'\left(\mu b_1 + (1-\mu)b_2\right)d\mu\\&= I_1 - I_2.\end{aligned} \tag{3.7}$$

Then by integration by parts, we have

$$\begin{aligned}I_1 &= \int_0^1 \left(\frac{1-\mu^\alpha}{\alpha}\right)^\gamma \Upsilon'\left(\mu b_1 + (1-\mu)b_2\right)d\mu\\&= \frac{1}{b_1-b_2}\left(\frac{1-\mu^\alpha}{\alpha}\right)^\gamma \Upsilon\left(\mu b_1 + (1-\mu)b_2\right)\Big|_0^1\\&\quad - \frac{1}{b_1-b_2}\int_0^1 \gamma\left(\frac{1-\mu^\alpha}{\alpha}\right)^{\gamma-1}\left(-\mu^{\alpha-1}\right)\Upsilon\left(\mu b_1 + (1-\mu)b_2\right)d\mu\\&= \frac{\Upsilon(b_2)}{(b_2-b_1)\alpha^\gamma} - \frac{\gamma}{b_2-b_1}\int_0^1 \left(\frac{1-\mu^\alpha}{\alpha}\right)^{\gamma-1}\mu^{\alpha-1}\,\Upsilon\left(\mu b_1 + (1-\mu)b_2\right)d\mu.\end{aligned} \tag{3.8}$$

Since by letting $t = \mu b_1 + (1-\mu)b_2$, we find

$$\int_0^1 \left(\frac{1-\mu^\alpha}{\alpha}\right)^{\gamma-1}\mu^{\alpha-1}\,\Upsilon\left(\mu b_1 + (1-\mu)b_2\right)d\mu = \frac{\Gamma(\gamma)}{(b_2-b_1)^{\alpha\gamma}}\,{}^\gamma\mathcal{J}^\alpha_{b_2}\Upsilon(b_1).$$

Thus by putting above value in (3.8), we get

$$I_1 = \frac{\Upsilon(b_2)}{(b_2-b_1)\alpha^\gamma} - \frac{\Gamma(\gamma+1)}{(b_2-b_1)^{\alpha\gamma+1}}\,{}^\gamma\mathcal{J}^\alpha_{b_2}\Upsilon(b_1).$$

Similarly, we have

$$\begin{aligned}
I_2 &= \int_0^1 \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\gamma \Upsilon'(\mu b_1 + (1-\mu)b_2)d\mu \\
&= \frac{1}{b_1-b_2}\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\gamma \Upsilon(\mu b_1 + (1-\mu)b_2)\Big|_0^1 \\
&\quad - \frac{1}{b_1-b_2}\int_0^1 \gamma\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^{\gamma-1}(1-\mu)^{\alpha-1}\Upsilon(\mu b_1 + (1-\mu)b_2)d\mu \\
&= \frac{-\Upsilon(b_1)}{(b_2-b_1)\alpha^\gamma} + \frac{\gamma}{b_2-b_1}\int_0^1 \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^{\gamma-1}(1-\mu)^{\alpha-1}\Upsilon(\mu b_1 + (1-\mu)b_2)d\mu.
\end{aligned} \tag{3.9}$$

Since by letting $t = \mu b_1 + (1-\mu)b_2$, we find

$$\int_0^1 \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^{\gamma-1}(1-\mu)^{\alpha-1}\Upsilon(\mu b_1 + (1-\mu)b_2)d\mu = \frac{\Gamma(\gamma)}{(b_2-b_1)^{\alpha\gamma}}\, {}^{\gamma}_{b_1}\mathcal{J}^\alpha\, \Upsilon(b_2).$$

Thus by putting above value in (3.9), we get

$$I_2 = \frac{-\Upsilon(b_2)}{(b_2-b_1)\alpha^\gamma} + \frac{\Gamma(\gamma+1)}{(b_2-b_1)^{\alpha\gamma+1}}\, {}^{\gamma}_{b_1}\mathcal{J}^\alpha\, \Upsilon(b_2).$$

By substituting values of $I_1$ and $I_2$ in (3.7) and then multiplying both sides by $\frac{(b_2-b_1)\alpha^\gamma}{2}$ we get (3.6). □

**Theorem 3.3.** *Let $\Upsilon : [b_1, b_2] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $(b_1, b_2)$ with $b_1 < b_2$ such that $\Upsilon' \in L_1[b_1, b_2]$. If $|\Upsilon'|^q$, with $q \geq 1$, is tgs-convex function, then the following inequality holds:*

$$|\Delta_\Upsilon(b_1, b_2; \alpha; \gamma; \mathcal{J})| \leq \frac{(b_2-b_1)\alpha^\gamma}{2}\tau^{1-1/q}\left(\frac{|\Upsilon'(b_1)|^q + |\Upsilon'(b_2)|^q}{6}\right)^{1/q}, \tag{3.10}$$

*where*

$$\tau = \frac{\beta\left(\frac{1}{\alpha}, \gamma+1\right)}{\alpha^{\gamma+1}} - \frac{\beta\left(\frac{1}{\alpha^2}, \gamma+1\right)}{\alpha^{\gamma+2}}.$$

**Proof .** Using Lemma 3.2, property of modulus, Power meaan inequality and $tgs$-convexity of $|\Upsilon'|^q$, we have

$$\begin{aligned}
&|\Delta_\Upsilon(b_1, b_2; \alpha; \gamma; \mathcal{J})| \\
&= \left|\frac{(b_2-b_1)\alpha^\gamma}{2}\int_0^1 \left[\left(\frac{1-\mu^\alpha}{\alpha}\right)^\gamma - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\gamma\right]\Upsilon'(\mu b_1 + (1-\mu)b_2)d\mu\right| \\
&\leq \frac{(b_2-b_1)\alpha^\gamma}{2}\left\{\int_0^1 \left[\left(\frac{1-\mu^\alpha}{\alpha}\right)^\gamma - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\gamma\right]d\mu\right\}^{1-1/q} \\
&\quad \times \left(\int_0^1 |\Upsilon'(\mu b_1 + (1-\mu)b_2)|^q d\mu\right)^{1/q} \\
&\leq \frac{(b_2-b_1)\alpha^\gamma}{2}\tau^{1-1/q}\left(\int_0^1 \mu(1-\mu)(|\Upsilon'(b_1)|^q + |\Upsilon'(b_2)|^q)d\mu\right)^{1/q},
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}\tau &= \int_0^1 \left[\left(\frac{1-\mu^\alpha}{\alpha}\right)^\gamma - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\gamma\right] d\mu \\ &= \int_0^1 \left(\frac{1-\mu^\alpha}{\alpha}\right)^\gamma d\mu - \int_0^1 \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\gamma d\mu \\ &= \frac{\beta\left(\frac{1}{\alpha}, \gamma+1\right)}{\alpha^{\gamma+1}} - \frac{\beta\left(\frac{1}{\alpha^2}, \gamma+1\right)}{\alpha^{\gamma+2}},\end{aligned} \tag{3.12}$$

and

$$\int_0^1 \mu(1-\mu)d\mu = \frac{1}{6}.$$

Thus by putting above values in (3.11), we get (3.10). □

## Conclusion

In section 2, from Theorem 2.1 we obtained the Hermite-Hadamard inequality for $tgs$-convex function via conformable fractional integrals. Then Lemma 2.3, we found an identity from which we proved Theorem 2.4 and 2.5, that is, Hermite-Hadamard type inequalities for $tgs$-convex function via conformable fractional integrals are obtained. In section 3, from Theorem 3.1 we obtained the Hermite-Hadamard inequality for $tgs$-convex function via new fractional conformable integral operators. Then from Lemma 3.2, we found an identity from which we proved Theorem 3.3, which is, Hermite-Hadamard type inequalities for $tgs$-convex function via new fractional conformable integral operators.

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## References

[1] T. Abdeljawad, *On confermable fractional calculus*, J. Comput. Appl. Math. 279 (2015) 57–66.

[2] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl. 335(2) (2007) 1294–1308.

[3] M. Avci, H. Kavurmaci and M. E. Ozdemir, *New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications*, Appl. Math. Comput. 217 (2011) 5171—5176.

[4] G. Farid, *A Treatment of the Hadamard inequality due to m-convexity via generalized fractional integral*, J. Fract. Calc. Appl. 9(1) (2018) 8–14.

[5] G. Farid, *Hadamard and Fejér-Hadamard inequalities for generalized fractional integral involving special functions*, Konuralp J. Math. 4(1) (2016) 108–113.

[6] G. Farid, A. Ur. Rehman and S. Mehmood, *Hadamard and Fejér-Hadamard type integral inequalities for harmonically convex functions via an extended generalized Mittag-Leffler function*, J. Math. Comput. Sci. 8(5) (2018) 630–643.

[7] Z. B. Fang and R. Shi, *On the $(p,h)$-convex function and some integral inequalities*, J. Inequal. Appl. 2014 (2014) p. 45.
[8] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl. (1893) 171–215.
[9] Ch. Hermite, *Sur deux limites d'une intégrale dénie*, Mathesis **3** (1883) 82.
[10] F. Jarad, E. Ugurlu, T. Abdeljawad and D. Baleanu, *On a new class of fractional operators*, Adv. Difference Equ. 2017 (2017).
[11] S. M. Kang, G. Farid, W. Nazeer and S. Mehmood, *$(h,m)$-convex functions and associated fractional Hadamard and Fejér-Hadamard inequalities via an extended generalized Mittag-Leffler function*, J. Inequal. Appl. 2019 (2019) p. 78.
[12] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential equations*, Elsevier Science B. V., Amsterdam, 2006.
[13] S. Mehmood, G. Farid, K. A. Khan and M. Yussouf, *New fractional Hadamard and Fejér-Hadamard inequalities associated with exponentially $(h,m)$-convex functions*, Eng. Appl. Sci. Lett. 3(2) (2020) 9–18.
[14] S. Mehmood, G. Farid, K. A. Khan and M. Yussouf, *New Hadamard and Fejér-Hadamard fractional inequalities for exponentially $m$-convex function*, Eng. Appl. Sci. Lett. 3(1) (2020) 45–55.
[15] N. Mehreen and M. Anwar, *Hermite-Hadamard type inequalities via exponentially $(p,h)$-convex functions*. IEEE Access, 8 (2020) 37589–37595
[16] N. Mehreen and M. Anwar, *Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for p-convex functions via conformable fractional integrals*, J. Inequal. Appl. 2020 (2020) p. 107.
[17] N. Mehreen and M. Anwar, *Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for p-convex functions via new fractional conformable integral operators*, J. Math. Compt. Sci. 19 (2019) 230–240.
[18] N. Mehreen and M. Anwar, *Hermite-Hadamard type inequalities via exponentially p-convex functions and exponentially $s$-convex functions in second sense with applications*, J. Inequal. Appl. 2019 (2019), p. 92.
[19] N. Mehreen and M. Anwar, *Integral inequalities for some convex functions via generalized fractional integrals*, J. Inequal. Appl. 2018 (2018) p. 208.
[20] N. Mehreen and M. Anwar, *Some inequalities via $\psi$-Riemann-Liouville fractional integrals*, AIMS Math. 4(5) (2019) 1403–1415.
[21] N. Mehreen and M. Anwar, *On some Hermite-Hadamard type inequalities for tgs-convex functions via generalized fractional integrals*, Adv. Differnce Equ. 2020 (2020) p. 6.
[22] I. Iscan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacet. J. Math. Stat. 43(6) (2014) 935–942.
[23] E. Set, M. Z. Sarikaya, A. Gözpnar, *Some Hermite-Hadamard type inequalities for convex functions via conformable fractional integrals and related inequalities*, preprint, 26 (2016) 221–229.
[24] M. Tunc, E. Gov, U. Şanal, *On tgs-convex function and their inequalities*, Facta Univ. Ser. Math. Inf. 30(5) (2015) 679–691.
[25] S. Ullah, G. Farid , K. A. Khan, A. Waheed and S. Mehmood, *Generalized fractional inequalities for quasi-convex functions*, Adv. Difference Equ. 2019 (2019) p. 15.
[26] K. S. Zhang and J. P. Wan, *p-convex functions and their properties*, Pure Appl. Math. 23(1) (2007) 130–133.