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Approximation of Fourier series in terms of functions in L_p Spaces for $0 < p < 1$

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Abstract

Many results introduced for the absolutely convergence of Fourier series in terms of absolutely continuous functions. Here we study the convergence of Fourier series in terms of p-integrable functions series.

Keywords: modulus of continuity, Dyadic intervals, Haar system, Rademacher system

1. Introduction

A famous problem in trigonometric series is the following Theorem

Theorem 1.1. *A necessary and sufficient condition for the absolute convergence of the Fourier trigonometric series of any function $f(t) \in H^\omega$ of bounded variation is that*

$$\sum_{n=1}^{\infty} \frac{\sqrt{\omega(\frac{1}{n})}}{n} < \infty$$

Where H^ω be the class of continuous $2\pi-$ periodic functions whose modulus of continuity $\omega(f,\delta)$ satisfies the condition $\omega(f,\delta) = O\omega(\delta)$.

The sufficient condition of the above Theorem was studied in [8] and [3]. The necessary condition of the above theorem was established in [3] ,[4], [6], [2] and [5]. In our work we strength the result in [5]. We use arbitrarly bounded complete orthonormal system instead of trigonometric systed in $L_p(0,1)$ spaces for $0 < p < 1$. We prove the following Theorem.

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Theorem 1.2. *Suppose that* $\{f_n(t)\}$ *be an orthonormal system complete in* $L_p(0,1)$ *and such that*

$$|f_n(t)| \leq M, t \in [0,1], \quad n = 1, 2, \ldots$$

Where M *is positive constant.*
Then, for any modulus of continuity $\omega(\delta)$ *satisfying the condition*

$$\sum_{n=1}^{\infty} \frac{\sqrt{\omega(\frac{1}{n})}}{n} = \infty$$

There exists an absolutely continuous function $\Phi(t), \Phi(0) = \Phi(1)$, *such that*

$$\omega(\delta, \Phi) = O\{\omega(\delta)\}, \qquad \sum_{l=1}^{\infty} |a_l(\Phi)| = \infty,$$

where $a_l(\Phi) = \displaystyle\int_0^1 \Phi(t).f_l(t)dt.$

Definition 1.3. *The function* $\omega(\delta)$ *is called the modulus of continuity of* f *where*

$$\omega(\delta) = \omega(\delta, f)_p = \sup_{0<|h|<\delta_1} \| f(x+h) - f(x) \|_p .$$

and $f(x)$ *be defined in a closed interval.*

Definition 1.4. *Let* Ω_1 *be the set of* (n,k) *such that* $1 \leq k \leq 2^n, \quad n = 0, 1, \ldots, \Omega, \quad \Omega = \Omega_1 \cup (0,0)$, *denoted by*

$$\Delta_0^0 = (0,1), \qquad \Delta_0^{-0} = [0,1],$$
$$\Delta_n^k = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right), \quad \Delta_n^{-k} = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right), (n,k) \in \Omega_1$$

Such that the intervals are called dyadic intervals. Clearly, if two dyadic intervals intersection, then one of them contains the other. The inclusion $\Delta_n^k \supset \Delta_p^q$ is equivalent to conditions

$$p \geq n, 2^{p-n}(k-1) < q \leq 2^{p-n}k.$$

Put $\chi_0^0 \equiv 1$. *if* $(n,k) \in \Omega_1$, then

$$\chi_n^k = \left\{ \begin{array}{cc} 0, & t \in \bar{\Delta}_n^k \\ 1, & t \in \Delta_{n+1}^{2k-1} \\ -1, & t \in \Delta_{n+1}^{2k} \end{array} \right\}$$

The value of $\chi_n^k(t)$ in a discontinuity point t is defined as

$$\chi_n^k(t) = \lim_{\varepsilon \to 0} \frac{1}{2}\left(\chi_n^k(t-\varepsilon) + \Delta_n^k(t+\varepsilon)\right).$$

If $k = 1$ or $k = 2^n$, then the value $\chi_n^k(t)$ in 0 and 1 is defined so that $\chi_n^k(t)$ is continuous in 0 and 1.

The set of function $\chi_n^k(t), (nmk) \in \Omega_1$ is called the Haar system and denoted by (H.s).

Definition 1.5. *(Rademacher system) [1] Is an incomplete orthogonal system of functions on the unite interval of the following form*

$$\left\{t \to r_n(t) = sgn(\sin 2^{n+1}\pi t), \quad t \in [0,1], \quad n \in N\right\}$$

The Rademacher system is stochastically, and is closely related to the Walsh system.

2. The Auxiliary Results

In this section we give some auxiliary lemma that we need in our research

Lemma 2.1. *(Yano's Inequality) [7] The Yano's inequality given of the form*

$$meas\left\{x \in \Delta : (f(x))^{\frac{1}{p}} > 2\right\} \leq \frac{A}{2}\int_{\Delta} (f(x))^{\frac{1}{p}}\ dx.$$

Lemma 2.2. *(Parseval identity) [7]* $f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}), \quad o < x < L$. *Then*

$$\frac{2}{L}\int_0^L [f(x)]^2\, dx = \sum_{n=1}^{\infty} b_n^2.$$

3. The Proof of the Main Result

Here we shall prove our main result:
Theorem 1.2
Suppose that $f_n(t)$ be an orthonormal system complete in $L_p(0,1)$ and such that

$$|f_n(t)| \leq M, \quad t \in [0,1], \quad n = 1, 2, \ldots, \tag{3.1}$$

where M is positive constant.
Then, for any modulus of continuity $\omega(\delta)$ satisfying the condition

$$\sum_{n=1}^{\infty} \frac{\sqrt{\omega(\frac{1}{n})}}{n} = \infty \tag{3.2}$$

such that

$$\omega(\delta, \Phi) = O\{\omega(\delta)\}, \qquad \sum_{l=1}^{\infty} |a_l(\Phi)| = \infty, \tag{3.3}$$

where $a_l(\Phi) = \int_0^1 \Phi(t).f_l(t)dt$.
Proof . For each $m = 1, 2, \ldots$ suppose that N_m be the largest n such that $\omega(f, \frac{1}{2^n})_p \geq \frac{1}{2^m}$
Assume that $N_0 = 0$, since $\omega(f, \frac{1}{2^{N_{m+1}}})_p > \frac{1}{2}\omega(f, 2^{N_m})_p > \frac{1}{2^{m+1}}$
we have$N_m + 1 + N_m + 1$, and the sequence $\{N_m\}$ increasing, such that

$$\frac{1}{2^{m+1}} \leq \omega(f, \frac{1}{2^n})_p \leq \frac{1}{2^m} \tag{3.4}$$

where $N_m < n < N_{m+1}, \quad m = 0, 1, \ldots$ Let d_m be the largest positive integer such that

$$N_{m-d_{m+1}} + d_{m+1} = N_m \tag{3.5}$$

Let $\{L_n\}$ and $\{R_n\}$ be two sequences of indices such that $L_1 = 1,\ L_{n+1} > R_n, \quad n = 1, 2, \ldots$

$$R_n - L_{n+1} = 2^{m+1} \ \ for \ \ N_m < n \leq N_{m+1}, \quad m = 0, 1, \ldots \tag{3.6}$$

We will define the family of function $F_x(t)$ by formula

$$F_x(t) = \varphi_2(t) + \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \sum_{n=N_m+1}^{N_{m+1}} \sum_{k=L_n}^{R_n} \sum_{s\in G_{nk}} \mu_s(x)\varphi_s(t), \tag{3.7}$$

where $\{\varphi_s(t)\}_{s=0}^{\infty}$ is the Schauder system.
The set G_{nk} and the functions $\mu_s(\chi)$ satisfy the conditions

- $G_{nk} = \emptyset, \quad for \;\; n \in \{N_m\},$

$$\sum_{k=L_n}^{R_n} G_{nk} = \left\{2^n+1,\ldots,2^{n+1}\right\}, \quad n \notin N_m \tag{3.8}$$

- $G_{nk} \cap G_{nl} = \emptyset, \quad for \;\; k \neq l, \;\; and\, for \;\; s \in G_{nk},$

$$\mu_s(x) = \frac{1}{2^{\frac{k}{2}}} \sum_{p\in Q_s} \chi_p(x) \tag{3.9}$$

Where $\chi_p(x)_{p=1}^{\infty}$ is the Haar system.
$Q_s \subseteq \left\{2^k+1,\ldots,2^{k+1}\right\}, \;\; and \;\; Q_s \cap Q_r = \emptyset \;\; for \;\; s \neq r, \; if \; s,r \in G_{nk}$
For there more, one has the identity

$$\sum_{s\in G_{nk}} \mu_s(x) = rk(x) \tag{3.10}$$

Where $r_k(x)_{k=0}^{\infty}$ is the Rademacher system. It follows from (3.9) that the function of the $\mu_s(x)$ can take only three values, namely, 0 and ±1. The function $\mu_s(x)$ are defined by induction.
We may assume that $N_1 < 1$. For $2^1 < s \leq 2^2$, the set

$$\mu_3(x) = r_1(x), \quad \mu_4(x) = r_2(x) \tag{3.11}$$

The functions $\mu_3(x), \mu_4(x)$ satisfy relation (3.8)–(3.10) for
$m = 0, n = 1, R_1 = 2, L_1 = 1, G_11 = 3, G_12 = 4, Q_3 = 3, 4, Q_4 = 5, 6, 7, 8$. We shall limit ourselves to determining those functions $\mu_s(x)$ for which the functions $Q_s(t)$ have support in $[0, 1/2]$. Fore functions $Q_s(t)$ with support in $[1/2, 1]$.
Assume that $+1 < n < N_{m+1}$ and $\mu_s(x)$ are defined for $2^{n-1} < s \leq 2^n$, so that relation (3.8)–(3.10) satisfied.
Suppose that $s \in G_{n-1,k}$, where $L_{n-1} \leq k \leq R_{n-1}$. There exists two Schauder function $\emptyset_p(t)$, and $\emptyset_{p+1}(t), 2^n < p < 2^{n+1}$ with support contained in the support $\emptyset_s(t)$. Set

$$\mu_p(x) = \mu_s^+(x).r_v(x), \qquad \mu_{p+1}(x) = \mu_s^-(x).r_v(x) \tag{3.12}$$

where $v = L_{n+k} - L_{n-1}, \qquad \mu_s^+(x) = \max\{0, \mu_s(x)\}, \qquad \mu_s^-(x) = \max\{0, -\mu_s(x)\}$
Assume that
$n = N_m + 1, n \notin N_m$ and the functions $\mu_s(x)$ are defined and don't vanish for $2^{n-d_m-1} < s \leq 2^{n-d_m}$.
Let $s \in G_{n-d_m-1,k}$ where $L_{n-d_m-1} \leq k \leq R_{n-d_m-1}$. There are 2^{d_m+1} Schauder function $\phi_{p+q}(t)$ where

$1 \leq q \leq 2^d_{m+1}, 2^n \leq p \leq 2^{n+1} - 2^d_{m+1}$, with support in the support of the function $\emptyset_s(t)$.
Set

$$\begin{aligned} \mu_{p+q}(x) &= \mu_s^+(x).r_v(x), \quad and \\ \mu_{p+q}(x) &= \mu_s^-(x).r_v(x), \\ If\ 2^{d_m}, q \leq 2^{d_{m+1}}, \quad & where \quad v = L_n + 2^{d_m}(k - L_n - d_m - 1) + q - 2^{d_m} - 1 \end{aligned} \tag{3.13}$$

Now, (1.3)–(1.5) determine completely the system $\mu_s(x)$. We mention some properties of the functions $F_x(t)$ which follow from (3.7)–(1.5). For every $x \in [0, 1]$ the series (3.7) converges uniformly in t to a continuous singular function which is monotone on $[0, 1/2]$ and $[1/2, 1]$.
If $\frac{1}{2^{N_{m+1}}} < \delta \leq \frac{1}{2^{N_m}}$, then $\omega(\delta, F_x) \leq \frac{1}{2^{m-1}} \leq 4\omega\left(\frac{1}{2^{N_{m+1}}}\right), \quad for\ all\ x \in [0, 1]$

$$\omega(\delta, F_x) \leq 4\omega(\delta) \tag{3.14}$$

Denote

$$C_{sl} = \int_0^1 \varphi_s(t) f_l(t) dt \tag{3.15}$$

We have

$$a_l(F_x) \equiv \int_0^1 F_x(t) f_l(t) dt = C_{2,l} + \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \sum_{n=N_m+1}^{N_{m+1}} \sum_{k=L_n}^{R_n} \sum_{s \in G_{nk}} \mu_s(x).C_{st} \tag{3.16}$$

Since by (3.9) the supports of the functions $\mu_s(x)$ for $s \in G_{nk}$ are disjoint, $\sum_{s \in G_{nk}} \mu_s(x).C_{st}$ can be viewed as a single function. In the view of this not we set

$$\sum_{n=N_m+1}^{N_{m+1}} \sum_{k=L_n}^{R_n} \sum_{s \in G_{nk}} \mu_s(x).C_{st} = \sum_{k=1}^{2^{m+1}} \delta_{nkl}(x) \tag{3.17}$$

We shall prove that for every $l = 1, 2, \ldots,$ the following inequality holds:

$$\begin{aligned} \int_0^1 \sum_{m=0}^{\infty} \frac{1}{2^{2m/2}} \sum_{n=N_m+1}^{N_{m+1}} 2^n. \left(\sum_{k=1}^{2^{m+1}} \delta^p_{nkl}(x) dx \right)^{\frac{1}{p}} &\leq B. \left(\int_0^1 \left| C_{2,t} + \sum_{m=0}^{\infty} \frac{1}{2^{2m/2}} \sum_{n=N_m+1}^{N_{m+1}} \sum_{k=1}^{2^{m+1}} \delta_{nkl} \right| dx \right)^{\frac{1}{p}} \\ &= B. \parallel a_l(F_x) \parallel_p \end{aligned} \tag{3.18}$$

where B is some positive constant. The uniform boundedness of the system $\{f_l(t)\}$.
For $2^n < s \leq 2^{n+1}$ the inequality

$$| C_{st} | = \left| \int_0^1 \varphi_s(t).f_l(t) dt \right| \leq M. \int_0^1 \varphi_s(t) dt \leq \frac{M}{2^n}, \quad where \quad l = 1, 2, \ldots.$$

By (3.9) and (3.17), the latter estimate implies

$$| \delta_{nkl} | \leq \frac{M}{2^n} \tag{3.19}$$

For all $x \in [0, 1], k = 1, 2, \ldots, 2^{m+1}$ *and* $l = 1, 2, \ldots.$
For the sake of brevity, in the proof of (3.18), we shall write δ_{nk} instead of δ_{nkl}. Let Δ_{nq} be a dyadic intervals of length 2^{-L_n},

$$\Delta_{nq} = \left(\frac{q}{2^{L_n}}, \frac{q+1}{2^{L_n}} \right) \tag{3.20}$$

Suppose that

$$F_y = \left\{ x \in \Delta_{nq} : \left(\sum_{k=1}^{2^{m+1}} \delta_{nk}^p(x) \right)^{\frac{1}{p}} > y \right\}. \tag{3.21}$$

Applying Yano's inequality to the function $\sum_{k=1}^{2^{m+1}} \delta_{nk}(x)$ on the interval Δ_{nq}, we obtain

$$meas \left\{ x \in \Delta_{nq} : \left(\sum_{k=1}^{2^{m+1}} \delta_{nk}^p(x) \right)^{\frac{1}{p}} > z \right\} \leq \frac{A}{z} \int_{\Delta_{nk}} \left| \sum_{k=1}^{2^{m+1}} \delta_{nk}(x) \right|^{\frac{1}{p}} dx \tag{3.22}$$

where A is a positive constant. So, by (3.21) and (3.22),

$$meas\ F_y \leq \frac{A}{\sqrt{y}} \sum_{k=1}^{2^{m+1}} (|\delta_{nk}(x)|^p)^{\frac{1}{p}} \tag{3.23}$$

According to (3.19)

$$\sum_{k=1}^{2^{m+1}} \delta_{nk}^p(x) \leq \frac{M^2.2^{m+1}}{2^{2n}} \tag{3.24}$$

From the relations (3.21), (3.23) and (3.24), we have the estimate

$$\begin{aligned} \int_{\Delta_{nk}} \left(\sum_{k=1}^{2^{m+1}} \delta_{nk}^p(x) \right)^{\frac{1}{p}} dx &= \int_0^{\infty} meas\ F_y\ dy = \int_0^{M^2 2^{m+1-2n}} meas\ F_y\ dy \\ &\leq A. \sum_{k=1}^{2^{m+1}} (|\delta_{nk}(x)|^p)^{\frac{1}{p}} \int_0^{M^2 2^{m+1-2n}} \frac{dy}{\sqrt{y}} \\ &= \frac{A.M.\sqrt{2^{m+1}}}{2^{n-1}}. \sum_{k=1}^{2^{m+1}} (|\delta_{nk}(x)|^p)^{\frac{1}{p}} \end{aligned} \tag{3.25}$$

Let $\mathfrak{m}_{nq}$ be the family of all sets $E_{nq} \in \Delta_{nq}$ such that each E_{nq} is the union of disjoint dyadic intervals I_{nqj} of length satisfying the inequalities

$$\frac{1}{2^{R_n}} \leq |I_{nqj}| \leq \frac{1}{2^{L_n}}.$$

The $\sum_{k=1}^{2^{m+1}} \delta_{nkl}(x)$ as a polynomial in Haar system. Then let

$$P_{E_{nq}} \left\{ \sum_{k=1}^{2^{m+1}} (\delta_{nkl}(x))^{\frac{1}{p}} \right\} \tag{3.26}$$

be the sum of those summands in this polynomial whose supports are contained in the set E_{nq}. Similarly, the symbol

$$P_{E_{nq}} \left\{ \sum_{k=1}^{2^{m+1}} (\delta_{nkl}^p(x))^{\frac{1}{p}} \right\} \tag{3.27}$$

Means that from each function $\delta^p_{nk}(x)$ we take only the summands with support in E_{nq}. Inequality (3.25) still holds if $\sum_{k=1}^{2^{m+1}} (\delta_{nk}(x))^{\frac{1}{p}}$ *and* $\sum_{k=1}^{2^{m+1}} (\delta^p_{nk}(x))^{\frac{1}{p}}$ are replaced by their projections

$$P_{E_{nq}}\left\{\sum_{k=1}^{2^{m+1}} (\delta_{nk}(x))^{\frac{1}{p}}\right\} \quad and \quad P_{E_{nq}}\left\{\sum_{k=1}^{2^{m+1}} (\delta^p_{nk}(x))^{\frac{1}{p}}\right\}, \text{on the set } E_{nq} \in \mathfrak{m}_{nq}$$

namely, we have the following inequality

$$\int_{\Delta_{nk}} P_{E_{nq}}\left\{\sum_{k=1}^{2^{m+1}} (\delta^p_{nk}(x))^{\frac{1}{p}}\right\} \leq \frac{A.M.\sqrt{2^{m+1}}}{2^{n-1}}.\int_{\Delta_{nk}} \frac{A.M.\sqrt{2^{m+1}}}{2^{n-1}} \mid P_{E_{nq}}\left\{\sum_{k=1}^{2^{m+1}} (\delta_{nk}(x))\right\} \mid^{\frac{1}{p}} \tag{3.28}$$

For any set $E_{nq} \in \mathfrak{m}_{nq}$ there exists a unique positive integer $S \equiv s(E_{nq})$ such that

$$\begin{aligned} \frac{A.M.\sqrt{2^{m+1}}}{2^{n-1}}\left(\int_{\Delta_{nk}} \mid P_{E_{nq}}\left\{\sum_{k=1}^{2^{m+1}} (\delta_{nk}(x))\right\} \mid^p dx\right)^{\frac{1}{p}} &< \left(\int_{\Delta_{nk}} \mid P_{E_{nq}}\left\{\sum_{k=1}^{2^{m+1}} (\delta^p_{nk}(x))\right\} \mid dx\right)^{\frac{1}{p}} \\ \leq \frac{A.M.\sqrt{2^{m+1}}}{2^{n-1}}\left(\int_{\Delta_{nk}} \left|P_{E_{nq}}\right|\left\{\sum_{k=1}^{2^{m+1}} \delta_{nk}(x)\right\}\right|^p dx\right)^{\frac{1}{p}} & \end{aligned} \tag{3.29}$$

Let

$$\lambda_1 = min\ s(E_{nq}), \quad E_{nq} \in \mathfrak{m}_{nq} \tag{3.30}$$

Let us call the set E_{nq} maximal if there is $E_{nq} \notin \mathfrak{m}_{nq}$ for which $E_{nq} \subset F_{nq}$ and $s(E_{nq}) = s(F_{nq})$. Let $E(\lambda_1, n, q)$ be a maximal set for which

$$s\left(E(\lambda_1, n, q)\right) = \lambda_1 \tag{3.31}$$

Assume that $s\left(E(\lambda_1, n, q)\right) = \lambda_1 \neq \Delta_{nq}$. Consider the functions

$$\sum^{(\lambda_1)} \delta_{nk}(x) = \left(\sum_{k=1}^{2^{m+1}} \delta_{nk}(x)\right)^{\frac{1}{p}} - P_{E(\lambda_1,n,q)}\left(\sum_{k=1}^{2^{m+1}} \delta_{nk}(x)\right)^{\frac{1}{p}} \tag{3.32}$$

$$\sum^{(\lambda_1)} \delta^p_{nk}(x) = \left(\sum_{k=1}^{2^{m+1}} \delta^p_{nk}(x)\right)^{\frac{1}{p}} - P_{E(\lambda_1,n,q)}\left(\sum_{k=1}^{2^{m+1}} \delta^p_{nk}(x)\right)^{\frac{1}{p}} \tag{3.33}$$

We can apply the same argument as we applied to $\left(\sum_{k=1}^{2^{m+1}} \delta_{nk}(x)\right)^{\frac{1}{p}}$ and $\left(\sum_{k=1}^{2^{m+1}} \delta^p_{nk}(x)\right)^{\frac{1}{p}}$. From (3.29), (3.32) and (3.33) for any set $E_{nq} \in \mathfrak{m}_{nq}$ there exists a unique integers $\gamma \equiv \gamma(E_{nq})$ for which,

$$\begin{aligned} \frac{A.M.\sqrt{2^{m+1}}}{2^{n-1}}\left(\int_{\Delta_{nq}} \left|P_{E_{nq}}\left\{\sum^{\lambda_1} (\delta_{nk}(x))\right\}\right|^p dx\right)^{\frac{1}{p}} &< \left(\int_{\Delta_{nq}} \left|P_{E_{nq}}\left\{\sum^{(\lambda_1)} (\delta^p_{nk}(x))\right\}\right| dx\right)^{\frac{1}{p}} \\ \leq \frac{A.M.\sqrt{2^{m+1}}}{2^{n-1}}\left(\int_{\Delta_{nq}} \left|P_{E_{nq}}\left\{\sum^{\lambda_1} \delta_{nk}(x)\right\}\right|^p dx\right)^{\frac{1}{p}} & \end{aligned} \tag{3.34}$$

Let $E(\lambda_2, n, q)$ be the maximal sub set for which

$$\lambda_2 \equiv \min \gamma(E_{nq}) = \gamma(E(\lambda_2, n, q)), \quad E(\lambda_2, n, q) \in \mathfrak{m}_{nq} \tag{3.35}$$

From the definition of this set we have

$$E(\lambda_1, n, q) \subset E(\lambda_2, n, q). \tag{3.36}$$

$$\text{We prove the inequality} \quad \lambda_1 < \lambda_2. \tag{3.37}$$

Suppose that $\lambda_1 \leq \lambda_2$. By definition of $E(\lambda_1, n, q)$ and $E(\lambda_2, n, q)$ we have

$$\begin{aligned}
\left(\int_{\Delta_{nq}} P_{E(\lambda_2,n,q)} \left\{ \sum_{k=1}^{2^{m+1}} (\delta_{nk}^p(x)) \right\} dx \right)^{\frac{1}{p}} &= \left(\int_{\Delta_{nq}} P_{E(\lambda_1,n,q)} \left\{ \sum_{k=1}^{2^{m+1}} (\delta_{nk}^p(x)) \right\} dx \right)^{\frac{1}{p}} \\
&= \left(\int_{\Delta_{nq}} P_{E(\lambda_2,n,q)} \left\{ \sum^{\lambda_2} (\delta_{nk}^p(x)) \right\} dx \right)^{\frac{1}{p}} \\
&> \frac{A.M.\sqrt{2^{m+1}}}{2^{n-1}} \left(\int_{\Delta_{nq}} \left| P_{E(\lambda_1,2,n,q)} \left\{ \sum_{k=1}^{2^{m+1}} \delta_{nk}(x) \right\} \right|^p dx \right)^{\frac{1}{p}} \\
&+ \frac{A.M.\sqrt{2^{m+1}}}{2^{n-1}} \left(\int_{\Delta_{nq}} \left| P_{E(\lambda_2,n,q)} \left\{ \sum^{\lambda_2} \delta_{nkl}(x) \right\} \right|^p dx \right)^{\frac{1}{p}}
\end{aligned} \tag{3.38}$$

yet (3.38) contrasts the maximality of $E(\lambda_1, n, q)$, and this proves (3.37). If $E(\lambda_2, n, q) \neq \Delta_{nq}$, then we define the number λ_3 and the set $E(\lambda_3, n, q)$ in an analogous fashion. We continue this process until for some v we obtain $E(\lambda_v, n, q) = \Delta_{nq}$. As a result we have the family of positive integers $\{\lambda_i\}$, the set $E(\lambda_i, n, q)$ and the functions $\sum^{\lambda_i} \delta_{nk}(x)$ and $\sum^{\lambda_i} \delta_{nk}^p(x)$ the following properties

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_0 \leq \lambda_2 \leq \dots \lambda_v \leq, \tag{3.39}$$

$$\emptyset = E(0, n, q) \subset E(\lambda_1, n, q) \subset E(\lambda_2, n, q) \subset \dots \subset E(\lambda_v, n, q) = \Delta_{nq} \tag{3.40}$$

$$\sum^{\lambda_i} \delta_{nk}(x) = \left(\sum_{k=1}^{2^{m+1}} \delta_{nk}(x) \right)^{\frac{1}{p}} - P_{E(\lambda_i,n,q)} \left\{ \sum_{k=1}^{2^{m+1}} \delta_{nk}(x) \right\}^{\frac{1}{p}} \tag{3.41}$$

$$\sum^{\lambda_i} \delta_{nk}^p(x) = \left(\sum_{k=1}^{2^{m+1}} \delta_{nk}(x)^p(x) \right)^{\frac{1}{p}} - P_{E(\lambda_i,n,q)} \left\{ \sum_{k=1}^{2^{m+1}} \delta_{nk}^p(x)(x) \right\}^{\frac{1}{p}} \tag{3.42}$$

where $i = 0, 1, \dots, v$

$$\sum_{i=1}^{v} P_{E(\lambda_i,n,q)} \left\{ \sum^{(\lambda_i - 1)} \delta_{nk}(x) \right\}^{\frac{1}{p}} = \left(\sum_{k=1}^{2^{m+1}} \delta_{nk}(x) \right)^{\frac{1}{p}} \tag{3.43}$$

$$\sum_{i=1}^{v} P_{E(\lambda_i,n,q)} \left\{ \sum^{(\lambda_i - 1)} \delta_{nk}^p(x) \right\}^{\frac{1}{p}} = \left(\sum_{k=1}^{2^{m+1}} \delta_{nk}^p(x) \right)^{\frac{1}{p}} \tag{3.44}$$

In addition, we have

$$\frac{A.M.\sqrt{2^{m+1}}}{2^{n+\lambda_i}}\left(\int_{\Delta_{nq}}\left|P_{E(\lambda_i,n,q)}\left\{\sum^{(\lambda_i-1)}\delta_{nk}(x)\right\}\right|^p dx\right)^{\frac{1}{p}}$$
$$<\left(\int_{\Delta_{nq}}P_{E(\lambda_i,n,q)}\left\{\sum^{(\lambda_i-1)}\delta^p_{nk}(x)\right\}dx\right)^{\frac{1}{p}}$$
$$\leq\frac{A.M.\sqrt{2^{m+1}}}{2^{n+\lambda_i-1}}\left(\int_{\Delta_{nq}}\left|P_{E(\lambda_i,n,q)}\left\{\sum^{(\lambda_i-1)}\delta_{nk}(x)\right\}\right|^p dx\right)^{\frac{1}{p}} \tag{3.45}$$

And, for an arbitrary set $F_{nq}\in\mathfrak{m}_{nq}$

$$\left(\int_{\Delta_{nq}}P_{F_{nq}}\left\{\sum^{(\lambda_i-1)}\delta^p_{nk}(x)\right\}dx\right)^{\frac{1}{p}}\leq\frac{A.M.\sqrt{2^{m+1}}}{2^{n+\lambda_i}}\left(\int_{\Delta_{nq}}\left|P_{F_{nq}}\left\{\sum^{(\lambda_i-1)}\delta_{nk}(x)\right\}\right|^p dx\right)^{\frac{1}{p}} \tag{3.46}$$

Where $i=1,2,\ldots,v$. According to (3.39), to each interval Δ_{nq} there corresponds a certain in finite system of pairwise distinct positive integers $\lambda_i\equiv\lambda_i(n,q)$, where $i=1,\ldots,v(n,q)$. Let $C_{np},\ p=0,1,\ldots,$ be the set of those indices $q, 1\leq q\leq 2^{L_n}$, for which $\lambda_i(n,q)=p$ for some i.
Set

$$S^p_{nq}(x) = P_{E(\lambda_i,n,q)}\left\{\sum^{(\lambda_i-1)}\delta_{nk}(x)\right\}^{\frac{1}{p}} \tag{3.47}$$
$$T^p_{nq}(x) = P_{E(\lambda_i,n,q)}\left\{\sum^{(\lambda_i-1)}\delta^p_{nk}(x)\right\}^{\frac{1}{p}} \tag{3.48}$$

where $p=\lambda_i(n,q)$. We have

$$\left(\int_0^1\sum_{m=0}^{\infty}\frac{1}{2^{3m/2}}\sum_{n=N_m+1}^{N_{m+1}}2^n\sum_{k=1}^{2^{m+1}}\delta^p_{nk}(x)\,dx\right)^{\frac{1}{p}}=\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{2^{3m/2}}\sum_{n=N_m+1}^{N_{m+1}}2^n\sum_{q\in C_{np}}\parallel T^p_{nq}\parallel_{L_p[\Delta_{nq}]} \tag{3.49}$$

By Lemma 2-1 the measure of the set

$$G_y=\left\{T^p_{nq}(x)>y\right\}, \tag{3.50}$$

In the same fashion as in the proof of (3.23), we obtain

$$meas\ G_y = \frac{A}{\sqrt{y}}\parallel S^{(p)}_{nq}\parallel_{L_p[\Delta_{nq}]} \tag{3.51}$$
$$Let\qquad H(p,n,q) = \left\{T^{(p)}_{nq}(x)>\frac{M^2.2^m}{2^{2(n+p)+3}}\right\} \tag{3.52}$$

The set $H(p,n,q)$ belongs to $\mathfrak{m}_{nq}$. We shall show that

$$\int_{\Delta_{nq}}P_{H(p,n,q)}\left\{T^{(p)}_{nq}\right\}dx\leq\frac{1}{2}\parallel T^{(p)}_{nq}\parallel_{L_p[\Delta_{nq}]} \tag{3.53}$$

Indeed, by using (3.51) and the first inequality in (3.45), we obtain

$$\int_{\Delta_{nq}} P_{H(p,n,q)}\left\{T_{nq}^{(p)}\right\} dx \;\geq\; \frac{1}{2} \parallel T_{nq}^{(p)} \parallel_{L_p[\Delta_{nq}]} - \int_0^{M^2, 2^{m-2(n+p)-3}} meas\ G_y\ dy$$

$$\geq \parallel T_{nq}^{(p)} \parallel_{L_p[\Delta_{nq}]} - A. \parallel S_{nq}^{(p)} \parallel_{L_p[\Delta_{nq}]} . \int_0^{M^2, 2^{m-2(n+p)-3}} \frac{dy}{\sqrt{y}}$$

$$\geq \parallel T_{nq}^{(p)} \parallel_{L_p[\Delta_{nq}]} - \frac{A.M\sqrt{2^{m-1}}}{2^{n+p}}. \parallel S_{nq}^{(p)} \parallel_{L_p[\Delta_{nq}]} \;\geq\; \frac{1}{2} \parallel T_{nq}^{(p)} \parallel_{L_p[\Delta_{nq}]} .b$$

This prove (3.53).

Inequality (3.53) implies

$$\parallel T_{nq}^{(p)} \parallel_{L_p[H(p,n,q)]} \;\geq\; \frac{1}{2} \parallel T_{nq}^{(p)} \parallel_{L_p[\Delta_{nq}]} \tag{3.54}$$

Hence

$$\left(\int_0^1 \sum_{m=0}^{\infty} \frac{1}{2^{3m/2}} \sum_{n=N_m+1}^{N_{m+1}} 2^n . \sum_{k=1}^{2^{m+1}} \delta_{nk}^p dx\right)^{\frac{1}{p}} \leq c(p) \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{2^{3m/2}} \sum_{n=N_m+1}^{N_{m+1}} 2^n \sum_{q\in C_{np}} \parallel T_{nq}^{(p)} \parallel_{L_p[H(p,n,q)]} \tag{3.55}$$

Let use note that the length of intervals make up the set $H(p,r,s)$ for $r < n$ is a multiple of the length of the interval Δ_{nq}. For $n > r, \Delta_{nq}$ either belongs to the set $H(p,r,s)$ or does not intersect this set. Let D_{np} be the set of those indices $q \in C_{np}$ any of them

$$\Delta_{nq} \cap \sum_{r=1}^{n-1} \sum_{q\in C_{rp}}^{n} H(p,r,s) = \emptyset, \tag{3.56}$$

where for $j \equiv m(r)$

$$\begin{aligned} &\textstyle\sum_{m=0}^{\infty} \frac{1}{2^3 m/2} \sum_{n=N_m+1}^{N_{m+1}} 2^n \sum_{q\in C_{np}} \parallel T_{nq}^{(p)} \parallel_{L_p[H(p,n,q)]} \\ \leq &\textstyle\sum_{m=0}^{\infty} \frac{1}{2^3 m/2} \sum_{n=N_m+1}^{N_{m+1}} \sum_{q\in D_{np}} \sum_{r=n}^{\infty} 2^{r-j} \sum_{s\in C_{rp}} \parallel T_{rs}^{(p)} \parallel_{L_p[H(p,r,s)\cap H(p,n,q)]} \\ \leq &\textstyle\sum_{m=0}^{\infty} \frac{1}{2^3 m/2} \sum_{n=N_m+1}^{N_{m+1}} \sum_{q\in D_{np}} \sum_{r=n}^{\infty} 2^{r-j} \sum_{s\in C_{rp}} \parallel T_{rs}^{(p)} \parallel_{L_p[\Delta_{nq}\cap H(p,n,q)]} \end{aligned} \tag{3.57}$$

We benefit the second inequality in (3.45), we can obtain the upper bound for the mean value of the function $T_{rs}^{(p)}(x)$ on Δ_{rs}. In fact,

$$\begin{aligned} \parallel T_{rs}^{(p)} \parallel_{L_p[\Delta_{rs}]} \leq \frac{A.M.\sqrt{2^{j+1}}}{2^{r+p-1}}. \parallel S_{rs}^{(p)} \parallel_{L_p[\Delta_{rs}]} \;&\leq\; \frac{A.M.\sqrt{2^{j+1}}}{2^{r+p-1}} |\Delta_{rs}|^{\frac{1}{p}}. \parallel S_{rs}^{(p)} \parallel_{L_p[\Delta_{rs}]} \\ &= \frac{A.M.\sqrt{2^{j+1}}}{2^{r+p-1}} |\Delta_{rs}|^{\frac{1}{p}}. \parallel T_{rs}^{(p)} \parallel_{L_p[\Delta_{rs}]} . \end{aligned} \tag{3.58}$$

These returns

$$\frac{1}{|\Delta_{rs}|} \parallel T_{rs}^{(p)} \parallel_{L_p[\Delta_{rs}]} \leq \frac{A^2 M^2 2^{j+3}}{2^{(r+p)}}. \tag{3.59}$$

Now, we estimate the expression

$$\sum_{r=n}^{\infty} 2^{r-j} \sum_{s\in C_{rp}} \parallel T_{rs}^{(p)} \parallel_{L_p[\Delta_{rs}\cap H(p,n,q)]} = 2^{n-m} \parallel T_{nq}^{(p)} \parallel_{L_p[H(p,n,q)]} + \sum_{r=n+1}^{\infty} 2^{r-j} \sum_{s\in C_{rp}} \parallel T_{rs}^{(p)} \parallel_{L_p[\Delta_{rs}\cap H(p,n,q)]} \tag{3.60}$$

From (3.59), we obtain

$$\begin{aligned} \sum_{r=n+1}^{\infty} 2^{r-j} \sum_{s\in C_{rp}} \parallel T_{rs}^{(p)} \parallel_{L_p[\Delta_{rs}\cap H(p,n,q)]} &\leq \frac{A^2.M^2.2^3}{2^{2p}} \sum_{r=n+1}^{\infty} \frac{1}{2^r} \sum_{s\in C_{rp}} |\Delta_{rs} \cap H(p,n,q)| \\ &\leq \frac{A^2.M^2.2^3}{2^{n+2p}}\, meas\ H(p,n,q). \end{aligned} \tag{3.61}$$

By definition of the set $H(p, n, q)$ we get

$$2^n \parallel T_{nq}^{(p)} \parallel_{L_p[H(p,n,q)]} \geq \frac{M^2 2^m}{2^{n+2p+3}}\, meas\ H(p,n,q). \tag{3.62}$$

Thus

$$\sum_{r=n+1}^{\infty} 2^{r-j} \sum_{s?C_{rp}} \parallel T_{rs}^{(p)} \parallel_{L_p[\Delta_{rs}\cap H(p,n,q)]} \leq A^2.2^{n+6-m} \parallel T_{nq}^{(p)} \parallel_{L_p[H(p,n,q)]} \tag{3.63}$$

By (3.60) and (3.63)

$$\sum_{r=n}^{\infty} 2^{r-j} \sum_{s\in C_{rp}} \parallel T_{rs}^{(p)} \parallel_{L_p[\Delta_{rs}\cap H(p,n,q)]} \leq A^2.2^{n+6-m} \parallel T_{nq}^{(p)} \parallel_{L_p[H(p,n,q)]} \tag{3.64}$$

Taking into consideration

$$\parallel T_{nq}^{(p)} \parallel_{L_p[H(p,n,q)]} \leq 2. \int_{\Delta_{nq}} P_{H(p,n,q)} \left\{ T_{nq}^{(p)}(x) \right\} dx = 2 \left(\int_{\Delta_{nq}} P_{H(p,n,q)} \left\{ \sum^{(\lambda_i-1)} \delta_{nk}^p(x) \right\} dx \right)^{\frac{1}{p}} \tag{3.65}$$

Where $\lambda_i = p$,and using (3.64), we obtain

$$\parallel T_{nq}^{(p)} \parallel_{L_p[H(p,n,q)]} \leq \frac{4A.M.\sqrt{2^{M+1}}}{2^{N+P}} \left(\int_{\Delta_{nq}} \left| P_{H(p,n,q)} \left\{ \sum^{(\lambda_i-1)} \delta_{nk}(x) \right\} \right|^p dx \right)^{\frac{1}{p}} \tag{3.66}$$

By definition of $\sum^{(\lambda_i-1)} \delta_{nk}(x)$ we have

$$P_{H(p,n,q)} \left\{ \left| \sum^{\lambda_i-1} \delta_{nk}(x) \right|^p \right\}^{\frac{1}{p}} = P_{H(p,n,q)} \left\{ \sum_{k=1}^{2^{m+1}} \delta_{nk}(x) \right\}^{\frac{1}{p}} - P_{E(\lambda_i-1,n,q)\cap H_{C(p,n,q)}} \left\{ \sum_{k=1}^{2^{m+1}} \delta_{nk}(x) \right\}^{\frac{1}{p}} \tag{3.67}$$

It follows from (3.19) that the series

$$C_{2,l} + \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \sum_{n=N_m+1}^{N_{m+1}} \sum_{k=1}^{2^{m+1}} \delta_{nkl}(x)$$

For every set F_{nq} fromM_{nq}, $N_m < n < N_{m+1}$, one has the inequality

$$\frac{1}{2^{m+1}}\left(\int_{\Delta_{nq}}\left|P_{F_{nq}}\left\{\sum_{k=1}^{2^{m+1}}\delta_{nkl}(x)\right\}^{p}dx\right|\right)^{\frac{1}{p}}\leq 2\left(\int_{F_{nq}}\left|C_{2,l}+\sum_{j=0}^{\infty}\frac{1}{2^{j+1}}\sum_{\alpha=N_j+1}^{N_{j+1}}\sum_{\beta=1}^{2^{m+1}}\delta_{\alpha\beta l}(x)\right|^{p}dx\right)^{\frac{1}{p}} \tag{3.68}$$

Relation (3.57), (3.64), (3.66), (3.67) and (3.68) imply

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{1}{2^{\frac{3m}{2}}}\sum_{n=N_m+1}^{N_{m+1}}2^n\sum_{q\in C_{np}}\parallel T^{(p)}_{nql}\parallel_{L_p[H(p,n,q)]} &\leq A^2.2^7\sum_{m=0}^{\infty}\frac{1}{2^{\frac{3m}{2}}}\sum_{n=N_m+1}^{N_{m+1}}2^n\sum_{q?D_{np}}\parallel T^{(p)}_{nql}\parallel_{L_p[H(p,n,q)]}\\ &\leq\frac{A^3.M.2^{10}}{2^p}\sum_{m=0}^{\infty}\frac{1}{2^m}\sum_{n=N_m+1}^{N_{m+1}}2^n\sum_{q?D_{np}}(\int_{\Delta_{nq}}\left|P_{H(p,n,q)}\left\{\sum^{(\lambda_i-1)}\delta_{nkl}(x)\right\}^{p}dx\right|)^{\frac{1}{p}}\\ &\leq\frac{A^3.M.2^{13}}{2^p}\sum_{m=0}^{\infty}\sum_{n=N_m+1}^{N_{m+1}}\sum_{q?D_{np}}\parallel G_l\parallel_{L_p[H(p,n,q)]}.\end{aligned} \tag{3.69}$$

Where

$$G_l(x)=C_{2,l}+\sum_{j=0}^{\infty}\frac{1}{2^{j+1}}\sum_{\alpha=N_j+1}^{N_{j+1}}(\sum_{\beta=1}^{2^{m+1}}\delta_{\alpha\beta l}(x))^{\frac{1}{p}}$$

By definition of the sets D_{np} we get

$$\left\{\sum_{q?D_{np}}^{n}H(p,n,q)\right\}\cap\left\{\sum_{s\in D_{rp}}^{n}H(p,r,s)\right\}=\Phi \tag{3.70}$$

For $n\neq r$. Thus, by (3.69) and (3.70),

$$\sum_{m=0}^{\infty}\frac{1}{2^{\frac{3m}{2}}}\sum_{n=N_m+1}^{N_{m+1}}2^n\sum_{q?C_{np}}\parallel T^{(p)}_{nql}\parallel_{L_p[H(p,n,q)]}\leq\frac{A^3.M.2^{13}}{2^p}\left(\int_0^1\left|C_{2,l}+\sum_{m=0}^{\infty}\frac{1}{2^{m+1}}\sum_{n=N_m+1}^{N_{m+1}}\sum_{k=1}^{2^{m+1}}\delta_{nkl}(x)\right|^{p}dx\right)^{\frac{1}{p}} \tag{3.71}$$

Adding inequalities (3.71) over p, we obtain

$$\begin{aligned}\left(\int_0^1\sum_{m=0}^{\infty}\frac{1}{2^{\frac{3m}{2}}}\sum_{n=N_m+1}^{N_{m+1}}2^n\sum_{k=1}^{2^{m+1}}\delta^p_{nkl}(x)dx\right)^{\frac{1}{p}} &\leq A^3.M.2^{15}\left(\int_0^1\left|C_{2,l}+\sum_{m=0}^{\infty}\frac{1}{2^{m+1}}\sum_{n=N_m+1}^{N_{m+1}}\sum_{k=1}^{2^{m+1}}\delta_{nkl}(x)\right|^{p}dx\right)^{\frac{1}{p}}\\ &\leq A^3.M.2^{15}\parallel\delta_{nkl}(x)\parallel_{L_P[0,1]}\end{aligned} \tag{3.72}$$

By (3.72) this proves (3.18).

Next by summing inequalities (3.18) over l, we obtain

$$\begin{aligned}|F_x|_{L_p[0,1]} &\geq \frac{1}{B}\sum_{L=0}^{\infty}\left(\int_0^1\sum_{m=0}^{\infty}\frac{1}{2^{\frac{3m}{2}}}\sum_{n=N_m+1}^{N_{m+1}}2^n\sum_{k=1}^{2^{m+1}}\delta^p_{nkl}(x)dx\right)^{\frac{1}{P}}\\ &= \frac{1}{B}\sum_{m=0}^{\infty}\frac{1}{2^{\frac{3m}{2}}}\sum_{n=N_m+1}^{N_{m+1}}2^n\sum_{k=L_n}^{R_n}\sum_{s\in G_{nk}}\sum_{l=1}^{\infty}C^2_{sl}\left(\int_0^1 M^p_S(x)dx\right)^{\frac{1}{P}}\\ &\leq \frac{1}{B}\sum_{m=0}^{\infty}\frac{1}{2^{\frac{3m}{2}}}\sum_{n=N_m+1}^{N_{m+1}}2^n\sum_{k=L_n}^{R_n}\sum_{s\in G_{nk}}\sum_{l=1}^{\infty}C^2_{sl}\parallel M_s\parallel_{L_p[0,1]}\end{aligned} \tag{3.73}$$

The system $\{F_l(t)\}$ being complete, by Lemma 2.2 implies that $for s \in G_{nk}$

$$\sum_{l=1}^{\infty} C_{sl}^2 = \int_0^1 \Phi_s^2(t)dt = \frac{1}{3.2^n} \tag{3.74}$$

Thus, by (3), (3.10) and (3.74), for $N_m < n < N_{m+1}$ we have

$$\begin{aligned}\sum_{k=L_n}^{R_n} \sum_{s\in G_{nk}} \sum_{l=1}^{\infty} C_{sl}^2 . \left(\int_0^1 \mu_s^p(x)dx\right)^{\frac{1}{p}} &= \frac{1}{3.2^n} \sum_{k=L_n}^{R_n} \sum_{s\in G_{nk}} \left(\int_0^1 \mu_s^p(x)dx\right)^{\frac{1}{p}} \\ &= \frac{1}{3.2^n} \sum_{k=L_n}^{R_n} \left(\int_0^1 r_k^p(x)dx\right)^{\frac{1}{p}} = \frac{2^{m+1}}{3.2^n} \end{aligned} \tag{3.75}$$

By the relations between (3.73) and (3.74) impels that

$$\| F_x \|_{L_p[0,1]} \geq \frac{2}{3B} \sum_{m=0}^{\infty} \frac{1}{2^{\frac{3m}{2}}} \sum_{n=N_m+1}^{N_{m+1}} 1 \tag{3.76}$$

$$\geq \frac{2}{3B} \sum_{n=1}^{\infty} \left(\sup_{|h|\leq\delta} \| f(x+2^{-n}) - f(x) \|_p \right)^{\frac{1}{2}} - \sum_{n\in\{N_m\}}^{n} \left(\sup_{|h|\leq\delta} \| f(x+2^{-n}) - f(x) \|_p \right)^{\frac{1}{2}} \tag{3.77}$$

Since

$$\sum_{n=1}^{\infty} \left(\sup_{|h|\leq\delta} \| f(x+2^{-n}) - f(x) \|_p \right)^{\frac{1}{2}} = \infty \tag{3.78}$$

And

$$\sum_{n\in\{N_m\}}^{n} \left(\sup_{|h|\leq\delta} \| f(x+2^{-n}) - f(x) \|_p \right)^{\frac{1}{2}} < \infty \tag{3.79}$$

We obtain

$$\| F_x \|_{L_p[0,1]} = \infty \tag{3.80}$$

The follows from (3.14) and (3.79), there exists a continuous function of bounded variation $\Phi(t)$ satisfying conditions (3). In fact, consider the Banach space of all continuous functions of bounded variation $g(t)$ for which

$g(0) = g(1) = 0$ and $\omega(\delta, g) = O\{\omega(\delta)\}$, The norm in B is given by

$$\| g \|_B = \vee_0^1 + \inf\{k : \omega(\delta, g) \leq k\omega(\delta)\}.$$

The functions $H_n(g) = \sum_{l=1}^{n} |a_l(g)|$,

where $a_l(g) = \int_0^1 g(t)f_l(t)dt$, are convex, and, a according to (3.14), the functions $F_x(t)$ satisfy for all $x \in [0,1]$, the inequality

$$|F_x|_B \leq 6.$$

The existence of $\Phi(t)$
In order to construct the absolutely continuous function $\Phi(t)$ we proceed as follows.
Let use

$$F_x^p(t) = \varphi_2(t) + \sum_{m=0}^{p} \frac{1}{2^{m+1}} \sum_{n=N_m+1}^{N_{m+1}} \sum_{k=L_n}^{R_n} \sum_{s\in G_n} \mu_s(x)\varphi_s(t). \tag{3.81}$$

By the same argument as in the proof (3.79), we obtain

$$\lim_{p\to\infty} \int_0^1 \sum_{l=1}^{\infty} |a_l(F_x^{(p)})|dx = \infty \;\; i.e \;\; \lim_{p\to\infty} \parallel F_x^{(p)} \parallel_{L_p[0,1]} = \infty \tag{3.82}$$

Assume that for all $x \in [0,1]$ *and* $p = 1, 2, \ldots,$

$$\sum_{l=1}^{\infty} |a_l(F_x^{(p)}| < \infty. \tag{3.83}$$

If (3.82) is not satisfied for some x_0 and p_0, then we take for $\Phi(t)$ the function $F_{x_0}^{(p_0)}(t)$, because the function$F_x^{(p)}(t)$ are absolutely continuous and satisfy (3.14).
Now, we shall define by induction three sequences of indices $\{s_j\}, \{k_j\}$, and $\{p_j\}$, the sequence of numbers $\{\varepsilon_j\}$ and the sequence of points $\{x_j\}$.

$\varepsilon_1 = 1$ *and* $s_1 = 1$. Using (3.81) we find indices k_1, p_1 and a point x_1 such that

$$\sum_{l=1}^{k_1} |a_l F_{x_1}^{(p_1)}| \geq 1$$

Suppose that the indices s_i,k_i and p_i and the numbers x_i and ε_i have already been defined for $1 \leq i \leq j$. According to (3.82), there is an index $s_{j+1} > k_j$ such that

$$\sum_{l=1}^{j} \sum_{l=s_{j+1}}^{\infty} |a_l(F_{x_i}^{(p_i)})| \leq \frac{1}{2^j}. \tag{3.84}$$

$$\text{The number} \quad \varepsilon_{j+1} \quad \text{is chosen so that } \varepsilon_{j+1}.k_j \leq \frac{1}{2^j} \tag{3.85}$$

Finally, in view of (3.81), there exist indices k_{j+1} and p_{j+1} and a point x_{j+1} such that

$$\sum_{l=s_{j+1}}^{k_{j+1}} |a_l(F_{x_{j+1}}^{(p_{j+1})})| \geq \frac{1}{\varepsilon_{j+1}} \tag{3.86}$$

The set

$$\Phi(t) = \sum_{l=1}^{\infty} \varepsilon_i F_{x_i}^{p_i}(t). \tag{3.87}$$

Since the function $F_x^{(p)}(t), p = 1, 2, \ldots, x \in [0,1]$, are absolutely continuous in t and satisfy the relation $\int_0^1 \left| \frac{d}{dt} F_x^{(p)}(t) \right| dt = \vee_0^1 F_x^{(p)} = 2$, the lebesgue theorem yields

$$\Phi(t) = \sum_{i=1}^{\infty} \varepsilon_i \int_0^t \frac{dF_{x_i}^{(p_i)}(u)}{du} du = \int_0^t \left\{ \sum_{i=1}^{\infty} \varepsilon_i \frac{dF_{x_i}^{(p_i)}(u)}{du} \right\} du.$$

This shows that $\Phi(t)$ is absolutely continuous. Moreover, in view of (3.14) and (3.84) one has

$$\omega(\delta,\Phi)\leq\sum_{i=1}^{\infty}\varepsilon_i\omega(\delta,F_{x_i}^{(p_i)})\leq 4\omega(\delta)\sum_{\varepsilon=1}^{\infty}\varepsilon_i\leq 8\omega(\delta) \tag{3.88}$$

Furthermore

$$\begin{aligned}\sum_{l=1}^{\infty}|a_l(\Phi)| &\geq \sum_{j=1}^{\infty}\sum_{l=s_j}^{k_j}|a_l(\Phi)| \\ &\geq \sum_{i=1}^{\infty}\sum_{l=s_j}^{k_j}\left\{\varepsilon_i|a_l(F_{x_i}^{(p_i)})|-\sum_{i=1}^{i-1}\varepsilon_i|a_l(F_{x_i}^{(p_i)})|-\sum_{i=j+1}^{\infty}\varepsilon_i|a_l(F_{x_i}^{(p_i)})|\right\}\end{aligned} \tag{3.89}$$

Taking into a account the inequalities

$$|a_l(F_{x_i}^{(p_i)})|\leq\| F_{x_i}^{(p_i)}\|_p\leq\| F_{x_i}^{(p_i)}\|_c=1,$$

We obtain

$$\sum_{l=s_j}^{k_j}\sum_{i=j+1}^{\infty}\varepsilon_i|a_l(F_{x_i}^{(p_i)})| \leq \sum_{l=s_j}^{k_j}\sum_{l=j+1}^{\infty}\varepsilon_i \leq 2\varepsilon_{j+1}.k_j \leq \frac{1}{2^{j-1}}. \tag{3.90}$$

Inequalities (3.83), (3.85),(3.88),(3.89) imply

$$\sum_{l=1}^{\infty}|a_l(\Phi)|=\infty. \tag{3.91}$$

In view of (3.87) and (3.90), the absolutely continuous function $\Phi(t)$ satisfies the relations (3.3). This establishes the theorem. □

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