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# Weak subsequential continuity in fuzzy metric spaces and application

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**Abstract**

Compatibility of type (E) and weak subsequential continuity is utilized in a fuzzy metric space for the existence of a common fixed point. Illustrations and an application are stated to elucidate our outcomes.

*Keywords:* Compatibility of type (E), fuzzy metric space, dynamic programming functional equation, weak subsequential continuity.
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## 1. Introduction

The fuzzy metric space is one of the consequential generalizations of metric space due to its interesting applications in stability theory, applied science, mathematical programming, engineering science, modelling theory, medical science, control theory, image processing and communication. The idea was initiated by Zadeh [20], where he defined the fuzzy sets. Later Kramosil and Michalek [7] familiarised with the fuzzy metric space which is further improved by George and Veeramanti [6] using the continuous t-norms [5, 6, 7, 8, 10, 11, 13, 14, 17, 20]. Now we establish common fixed point exploiting compatibility of type (E) and weak subsequential continuity to demonstrate the usefulness of these notions for contractive, $\phi$-contractive as well as an integral type contractive condition in the fuzzy metric space. In the sequel illustrations and an application to solve a functional equations is also stated to elucidate our outcomes.

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## 2. Preliminaries

**Definition 2.1.** *[2] Let $\exists$ a sequence $\{x_n\} \in X$ satisfying $\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{H}x_n = z \in X$. A pair $(\mathcal{A}, \mathcal{H})$ over the standard metric space $(X, d)$ is*

1. *weakly subsequentially continuous iff $\lim_{n\to\infty} \mathcal{A}\mathcal{H}x_n = \mathcal{A}z$ or $\lim_{n\to\infty} \mathcal{H}\mathcal{A}x_n = \mathcal{H}z$.*
2. *$\mathcal{H}$-subsequentially continuous, $\lim_{n\to\infty} \mathcal{H}\mathcal{A}x_n = \mathcal{H}z$.*
3. *$\mathcal{A}$-subsequentially continuous, iff $\lim \mathcal{A}\mathcal{H}x_n = \mathcal{A}z$.*
4. *compatible of type (E) [15] if*
$$\lim_{n\to\infty} \mathcal{H}^2 x_n = \lim_{n\to\infty} \mathcal{H}\mathcal{A}x_n = \mathcal{A}z$$
*and*
$$\lim_{n\to\infty} \mathcal{A}^2 x_n = \lim_{n\to\infty} \mathcal{A}\mathcal{H}x_n = \mathcal{H}z.$$
5. *$\mathcal{A}$-compatible of type (E) [15] if*
$$\lim_{n\to\infty} \mathcal{A}^2 x_n = \lim_{n\to\infty} \mathcal{A}\mathcal{H}x_n = \mathcal{H}z.$$
6. *$\mathcal{H}$-compatible of type (E) [15] if*
$$\lim_{n\to\infty} \mathcal{H}^2 x_n = \lim_{n\to\infty} \mathcal{H}\mathcal{A}x_n = \mathcal{A}z.$$

Now we give an example of $\mathcal{A}$-subsequentially continuous and weakly subsequentially continuous mappings in the fuzzy metric space:

**Example 2.2.** *Let $X = [0, 2]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$ with continuous t-norm: $a * b = ab$, $t > 0$. We define $\mathcal{A}$ and $\mathcal{H}$ as:*

$$\mathcal{A}x = \begin{cases} 1 + x, & 0 \le x \le 1 \\ \frac{x+1}{2}, & 1 < x \le 2, \end{cases} \quad \mathcal{H}x = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

*Observe that $\mathcal{A}$ and $\mathcal{H}$ are not continuous at 1.*
*We consider a sequence $\{x_n\}$, where $x_n = \dfrac{1}{n}$, $n \ge 1$.*
*So,*
$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{H}x_n = 1$$
$$\lim_{n\to\infty} \mathcal{A}\mathcal{H}x_n = \lim_{n\to\infty} \mathcal{A}(1 - \frac{1}{n}) = 2 = \mathcal{A}1,$$
*i.e., $(\mathcal{A}, \mathcal{H})$ is A-subsequentially continuous as well as weakly subsequentially continuous.*

*$\{y_n\}$ is a sequence, where $y_n = 1 + \dfrac{1}{n}$, $n \ge 1$, then*
$$\lim_{n\to\infty} \mathcal{A}y_n = \lim_{n\to\infty} \mathcal{H}y_n = 1,$$
*but*
$$\lim_{n\to\infty} \mathcal{H}\mathcal{A}y_n = \lim_{n\to\infty} \mathcal{H}(1 + \frac{1}{2n}) = 1 \ne \mathcal{H}1.$$
*So, $(\mathcal{A}, \mathcal{H})$ is not reciprocally continuous.*

Observe that the weakly subsequentially continuous or $\mathcal{A}$-subsequentially continuous or $\mathcal{H}$-subsequentially continuous mappings are never reciprocally continuous maps [12] (see also Tomar and Karapinar [18]).

Now we give an example of compatible of type (E) in a fuzzy metric spaces:

**Example 2.3.** *Let* $X = [0,\infty)$ *and* $M(x,y,t) = \frac{t}{t+|x-y|}$ *with the t-norm* $a * b = ab$. *We define* $\mathcal{A}$ *and* $\mathcal{H}$ *as follows:*

$$\mathcal{A}x = \begin{cases} 2, & 0 \leq x \leq 2 \\ x+1, & x > 2, \end{cases} \quad \mathcal{H}x = \begin{cases} \frac{x+2}{2}, & 0 \leq x \leq 2 \\ 0, & x > 2. \end{cases}$$

*If the sequence* $\{x_n\} \in X$ *is defined by* $x_n = 2 - \dfrac{1}{n}$, *then*

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{H}x_n = 2,$$

$$\lim_{n\to\infty} \mathcal{A}^2 x_n = \lim_{n\to\infty} \mathcal{A}\mathcal{H}x_n = 2 = \mathcal{H}2,$$
$$\lim_{n\to\infty} \mathcal{H}^2 x_n = \lim_{n\to\infty} \mathcal{H}\mathcal{A}x_n = 2 = \mathcal{A}2.$$

*Hence,* $(\mathcal{A},\mathcal{H})$ *is compatible of type (E).*

Observe that compatibility of type (E) implies $\mathcal{A}$-compatibility as well as $\mathcal{H}$-compatibility of type (E) but the reverse is not correct.

**Lemma 2.4.** *[11] If in a fuzzy metric space* $(X, M, *)$, $\exists$ *a constant* $k \in (0,1)$ *satisfying*

$$M(x,y,kt) \geq M(x,y,t),$$

$t > 0$ *and fixed* $x, y \in X$, *then* $x = y$.

## 3. Main results

Now we utilize the idea of weak subsequential continuity and compatibility of type (E) in a fuzzy metric space.

**Theorem 3.1.** *Let* $(\mathcal{A},\mathcal{H})$ *and* $(\mathcal{B},\mathcal{K})$ *be compatible of type (E) as well as weakly subsequentially continuous pairs of a fuzzy metric space* $(X, M, *)$. *Then pairs* $(\mathcal{A},\mathcal{H})$ *and* $(\mathcal{B},\mathcal{K})$ *have a coincidence point. If:*

$$M(\mathcal{H}x,\mathcal{K}y,kt) \geq \min\left\{ \begin{array}{c} M(\mathcal{A}x,\mathcal{B}y,t), M(\mathcal{A}x,\mathcal{H}x,t), \\ M(\mathcal{B}y,\mathcal{K}y,t), M(\mathcal{A}x,\mathcal{K}y,t), M(\mathcal{B}y,\mathcal{H}x,t) \end{array} \right\} \tag{3.1}$$

$k \in (0,1)$, $x, y \in X$ *and* $t > 0$, *then* $\mathcal{A},\mathcal{B},\mathcal{H}$ *and* $\mathcal{K}$ *have a unique common fixed point in* $(X, M, *)$.

**Proof .** As $(\mathcal{A},\mathcal{H})$ is weakly subsequentially continuous, $\exists$ a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n\to\infty}\mathcal{A}x_n=\lim_{n\to\infty}\mathcal{H}x_n=z\in X$$

and

$$\lim_{n\to\infty}\mathcal{A}\mathcal{H}x_n=\mathcal{A}z$$

or

$$\lim_{n\to\infty}\mathcal{H}\mathcal{A}x_n=\mathcal{H}z.$$

Also $(\mathcal{A},\mathcal{H})$ is compatible of type (E), So

$$\lim_{n\to\infty}\mathcal{A}\mathcal{H}x_n=\lim_{n\to\infty}\mathcal{A}^2x_n=\mathcal{H}z$$

and

$$\lim_{n\to\infty}\mathcal{H}\mathcal{A}x_n=\lim_{n\to\infty}\mathcal{H}^2x_n=\mathcal{A}z.$$

Consequently, $\mathcal{A}z=\mathcal{H}z$. Similarly $(\mathcal{B},\mathcal{K})$ is weakly subsequentially continuous $\exists$ a sequence $\{y_n\}$ such that

$$\lim_{n\to\infty}\mathcal{B}y_n=\lim_{n\to\infty}\mathcal{K}y_n=w\in X$$

and

$$\lim_{n\to\infty}\mathcal{B}\mathcal{K}y_n=\mathcal{B}w.$$

or

$$\lim_{n\to\infty}\mathcal{K}\mathcal{B}y_n=\mathcal{K}w.$$

Also $(\mathcal{B},\mathcal{K})$ is compatible of type (E)

$$\lim_{n\to\infty}\mathcal{B}\mathcal{K}y_n=\lim_{n\to\infty}\mathcal{B}^2y_n=\mathcal{K}w$$

$$\lim_{n\to\infty}\mathcal{K}\mathcal{B}y_n=\lim_{n\to\infty}\mathcal{K}^2y_n=\mathcal{B}w.$$

So, $\mathcal{B}w=\mathcal{K}w$.
We assert $\mathcal{A}z=\mathcal{B}w$. Using $x=z$ and $y=w$ in (3.1):

$$M(\mathcal{H}z,\mathcal{K}w,kt)\geq\min\left\{\begin{array}{c}M(\mathcal{A}z,\mathcal{B}w,t),M(\mathcal{A}z,\mathcal{H}z,t),\\ M(\mathcal{B}w,\mathcal{K}w,t),M(\mathcal{A}z,\mathcal{K}w,t),M(\mathcal{B}w,\mathcal{H}z,t)\end{array}\right\}$$

$$M(\mathcal{A}z,\mathcal{B}w,kt)\geq\min\{M(\mathcal{A}z,\mathcal{B}w,t),1,1,M(\mathcal{A}z,\mathcal{B}w,t),M(\mathcal{A}z,\mathcal{B}w,t)\},$$

i.e.,

$$M(\mathcal{A}z,\mathcal{B}w,kt)\geq M(\mathcal{A}z,\mathcal{B}w,t).$$

From Lemma 2.4, $\mathcal{A}z=\mathcal{B}w$.
Now we prove $z=\mathcal{A}z$. Substituting $x=x_n$ and $y=w$ in (3.1):

$$M(\mathcal{H}x_n,\mathcal{K}w,kt)\geq\min\left\{\begin{array}{c}M(\mathcal{A}x_n,\mathcal{B}w,t),M(\mathcal{A}x_n,\mathcal{H}x_n,t),\\ M(\mathcal{B}w,\mathcal{K}w,t),M(\mathcal{A}x_n,\mathcal{K}w,t),M(\mathcal{B}w,\mathcal{H}x_n,t)\end{array}\right\}.$$

Letting $n\to\infty$:

$$M(z,\mathcal{K}w,kt)\geq\min\{M(z,\mathcal{B}w,t),1,1,M(z,\mathcal{K}w,t),M(\mathcal{B}w,z,t)\},$$

i.e.,

$$M(z, \mathcal{A}z, kt) \geq M(z, \mathcal{A}z, t).$$

From Lemma 2.4, $z = \mathcal{A}z = \mathcal{H}z$.
Substituting $x = x_n$ and $y = y_n$ in (3.1):

$$M(\mathcal{H}x_n, \mathcal{K}y_n, kt) \geq \min\left\{ \begin{array}{c} M(\mathcal{A}x_n, \mathcal{B}y_n, t), M(\mathcal{A}x_n, \mathcal{H}x_n, t), \\ M(\mathcal{B}y_n, \mathcal{K}yn, t), M(\mathcal{A}x_n, \mathcal{K}y_n, t), M(\mathcal{B}y_n, \mathcal{H}x_n, t) \end{array} \right\}.$$

Letting $n \to \infty$:

$$M(z, w, kt) \geq \min\{M(z, w, t), 1, 1, M(z, w, t), M(w, z, t)\},$$

i.e., $M(z, w, kt) \geq M(z, w, t)$, i.e., $z = w$ (Lemma 2.4).
Hence, $z$ is a common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{H}$ and $\mathcal{K}$.
Suppose $q$ is another common fixed point. Substituting $x = z$ and $y = q$ in (3.1):

$$M(\mathcal{H}z, \mathcal{K}q, kt) \geq \min\{M(\mathcal{A}z, \mathcal{B}q, t), M(\mathcal{A}z, \mathcal{H}q, t), M(\mathcal{B}q, \mathcal{K}q, t), M(\mathcal{A}z, \mathcal{K}q, t), M(\mathcal{B}q, \mathcal{H}z, t)\}$$

$$M(z, q, kt) \geq M(z, q, t).$$

Hence, $z = q$. Consequently, $z$ is unique. □

**Example 3.2.** *Let $X = [0, 2]$ and $M = \frac{t}{t+|x-y|}$ with t-norm defined by $a * b = \min\{a, b\}$, $x, y \in X$ and $t > 0$. Let*

$$\mathcal{A}x = \begin{cases} x, & 0 \leq x \leq 1 \\ \frac{1}{2}, & 1 < x \leq 2, \end{cases} \qquad \mathcal{B}x = \begin{cases} \frac{x+1}{2}, & 0 \leq x \leq 1 \\ 2, & 1 < x \leq 2, \end{cases}$$

$$\mathcal{H}x = \begin{cases} 1, & 0 \leq x \leq 1 \\ \frac{5}{4}, & 1 < x \leq 2, \end{cases} \qquad \mathcal{K}x = \begin{cases} 2 - x, & 0 \leq x \leq 1 \\ \frac{3}{4}, & 1 < x \leq 2. \end{cases}$$

*Consider a sequence $\{x_n\}$ defined as $x_n = 1 - \frac{1}{n}$, $n \geq 1$.*
*Clearly,*

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{H}x_n = 1.$$

*Also,*

$$\lim_{n\to\infty} \mathcal{A}\mathcal{H}x_n = \mathcal{A}1$$

$$\lim_{n\to\infty} \mathcal{H}\mathcal{H}x_n = \lim_{n\to\infty} \mathcal{H}\mathcal{A}x_n = \mathcal{A}1$$

*and*

$$\lim_{n\to\infty} \mathcal{A}\mathcal{A}x_n = \lim_{n\to\infty} \mathcal{A}\mathcal{H}x_n = \mathcal{H}1,$$

*Consider a sequence $\{y_n\}$ defined by $y_n = 1$, $n \geq 1$.*
*Clearly,*

$$\lim_{n\to\infty} \mathcal{B}y_n = \lim_{n\to\infty} \mathcal{K}y_n = 1$$

*and*

$$\lim_{n\to\infty} \mathcal{B}\mathcal{K}y_n = \mathcal{B}1,$$

$$\lim_{n\to\infty} \mathcal{K}\mathcal{K}y_n = \lim_{n\to\infty} \mathcal{K}\mathcal{B}y_n = \mathcal{B}1$$

*and*

$$\lim_{n\to\infty} \mathcal{B}\mathcal{B}y_n = \lim_{n\to\infty} \mathcal{B}\mathcal{K}y_n = \mathcal{K}1,$$

*i.e., $(\mathcal{A}, \mathcal{H})$ and $(\mathcal{B}, \mathcal{K})$ are compatible of type (E) as well as weakly subsequentially continuous. Now,*

1. *When* $x, y \in [0, 1]$*:*

$$M(\mathcal{H}x, \mathcal{K}y, t)) = \frac{t}{t + |y-1|} \geq \frac{t}{t + \frac{3}{2}|y-1|} = M(\mathcal{B}y, \mathcal{K}y, t),$$

*then for any* $k \in [\frac{2}{3}, 1)$*:*

$$M(\mathcal{H}x, \mathcal{K}y, kt)) \geq M(\mathcal{B}y, \mathcal{K}y, t).$$

2. *When* $x \in [0, 1]$ *and* $y \in (1, 2]$*:*

$$M(\mathcal{H}x, \mathcal{K}y, t) = \frac{t}{t + \frac{1}{4}} \geq \frac{t}{t+1} = M(\mathcal{B}y, \mathcal{H}x, t),$$

*then for any* $k \in [\frac{1}{4}, 1)$*:*

$$M(\mathcal{H}x, \mathcal{K}y, kt) \geq M(\mathcal{B}y, \mathcal{H}x, t).$$

3. *When* $x \in (1, 2]$ *and* $y \in [0, 1]$*:*

$$M(\mathcal{H}x, \mathcal{K}y, t) = \frac{t}{t + |\frac{3}{4} - y|} \geq \frac{t}{t + |\frac{3}{2} - y|} = M(\mathcal{A}x, \mathcal{K}y, t),$$

*then for any* $k \in [\frac{1}{2}, 1)$*:*

$$M(\mathcal{H}x, \mathcal{K}y, kt)) \geq M(\mathcal{A}x, \mathcal{K}y, t)).$$

4. *When* $x, y \in (1, 2]$*:*

$$M(\mathcal{H}x, \mathcal{K}y, t) = \frac{t}{t + \frac{1}{2}} \geq \frac{t}{t + \frac{3}{2}} = M(\mathcal{A}x, \mathcal{B}y, t),$$

*then for any* $k \in [\frac{1}{3}, 1)$*:*

$$M(\mathcal{H}x, \mathcal{K}y, kt)) \geq M(\mathcal{A}x, \mathcal{B}y, t).$$

*Hence, for any* $k \in [\frac{2}{3}, 1)$ *and for all* $x, y \in [0, \infty)$*, the inequality (3.1) holds.*
*Consequently, the hypotheses of Theorem 3.1 are verified and* $1$ *is the unique common fixed point for* $\mathcal{A}, \mathcal{B}, \mathcal{H}$ *and* $\mathcal{K}$*. It is interesting to observe that none of the mappings is continuous. Moreover, neither* $\mathcal{A}X \subseteq \mathcal{B}X$ *nor* $\mathcal{H}X \subseteq \mathcal{K}X$*.*

If $\mathcal{A} = \mathcal{B}$ and $\mathcal{H} = \mathcal{K}$:

**Corollary 3.3.** *Let* $(\mathcal{A}, \mathcal{H})$ *be compatible of type (E) and weakly subsequentially continuous pair of a fuzzy metric space* $(X, M, *)$*. Then the pair* $(\mathcal{A}, \mathcal{H})$ *has a coincidence point. If:*

$$M(\mathcal{H}x, \mathcal{H}y, kt) \geq \min \left\{ \begin{array}{c} M(\mathcal{A}x, \mathcal{A}y, t), M(\mathcal{A}x, \mathcal{H}x, t), \\ M(\mathcal{A}y, \mathcal{H}y, t), M(\mathcal{A}x, \mathcal{H}y, t), M(\mathcal{A}y, \mathcal{H}x, t) \end{array} \right\} \quad (3.2)$$

$x, y \in X,\ k \in (0, 1)$ *and* $t > 0$*, then* $\mathcal{A}$ *and* $\mathcal{H}$ *have a unique common fixed point.*

**Corollary 3.4.** *Corollary 3.3 is true even if the pair* $(\mathcal{A}, \mathcal{H})$ *is* $\mathcal{A}$*-compatible of type (E) and* $\mathcal{A}$*-subsequentially continuous.*

**Example 3.5.** *Let* $X = [0, \infty)$ *and* $M = \frac{t}{t+|x-y|}$ *with t-norm,* $a * b = \min\{a, b\}$, $x, y \in X$, $t > 0$. *Define:*

$$\mathcal{A}x = \begin{cases} 3x, & 0 \leq x \leq 1 \\ 2x-1, & x > 1, \end{cases} \qquad \mathcal{H}x = \begin{cases} \frac{x}{4}, & 0 \leq x \leq 1 \\ 0, & x > 1. \end{cases}$$

*Consider a sequence* $\{x_n\}$, *where* $x_n = \frac{1}{n}$, $n \geq 1$.
*Clearly,*

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{H}x_n = 0.$$

*Also we have:*

$$\lim_{n\to\infty} \mathcal{A}\mathcal{H}x_n = \mathcal{A}0$$

$$\lim_{n\to\infty} \mathcal{A}^2 x_n = \lim_{n\to\infty} \mathcal{A}\mathcal{H}x_n = \mathcal{H}0,$$

*i.e., the pair* $(\mathcal{A}, \mathcal{H})$ *is* $\mathcal{A}$*-compatible of type (E) as well as* $\mathcal{A}$*-subsequentially continuous. Now,*

1. *When* $x, y \in [0, 1]$*:*

$$M(\mathcal{H}x, \mathcal{H}y, t) = \frac{t}{t + \frac{1}{4}|x-y|} \geq \frac{t}{t + 3|x-y|} = M(\mathcal{A}x, \mathcal{A}y, t),$$

*then for any* $k \in [\frac{1}{12}, 1)$*:*

$$M(\mathcal{H}x, \mathcal{H}y, kt)) \geq M(\mathcal{A}x, \mathcal{A}y, t).$$

2. *When* $x \in [0, 1]$ *and* $y > 1$*:*

$$M(\mathcal{H}x, \mathcal{H}y, t) = \frac{t}{t + \frac{1}{4}x} \geq \frac{t}{t + \frac{11}{4}x} = M(\mathcal{A}x, \mathcal{H}x, t),$$

*then for any* $k \in [\frac{1}{11}, 1)$*:*

$$M(\mathcal{H}x, \mathcal{H}y, kt) \geq M(\mathcal{A}x, \mathcal{H}x, t).$$

3. *When* $x \in (1, \infty)$ *and* $y \in [0, 1]$*:*

$$M(\mathcal{H}x, \mathcal{H}y, t) = \frac{t}{t + \frac{1}{4}y} \geq \frac{t}{t + \frac{11}{4}y} = M(\mathcal{A}y, \mathcal{H}y, t),$$

*then for any* $k \in [\frac{1}{11}, 1)$*:*

$$M(\mathcal{H}x, \mathcal{H}y, kt) \geq M(\mathcal{A}y, \mathcal{H}x, t).$$

4. *When* $x, y \in (1, \infty)$*:*

$$M(\mathcal{H}x, \mathcal{H}y, t) = \frac{t}{t + 0} \geq \frac{t}{t + 2|x-y|} = M(\mathcal{A}x, \mathcal{A}y, t),$$

*then for any* $k \in (0, 1)$*:*

$$M(\mathcal{H}x, \mathcal{H}y, kt) \geq M(\mathcal{A}x, \mathcal{A}y, t).$$

Hence, for any $k \in [\frac{1}{11}, 1)$ the inequality **??** holds and as a result all the hypotheses of Corollary 3.4 are verified and 0 is the unique common fixed point for $\mathcal{A}$ and $\mathcal{H}$. It is interesting to observe that both $\mathcal{A}$ and $\mathcal{H}$ are discontinuous and neither $\mathcal{A}X \not\subseteq \mathcal{H}X$ nor $\mathcal{H}X \not\subseteq \mathcal{A}X$.
If $\mathcal{A} = \mathcal{B}$ in Theorem 3.1:

**Corollary 3.6.** *Let $(\mathcal{A},\mathcal{H})$ and $(\mathcal{A},\mathcal{K})$ be compatible of type (E) and weakly subsequentially continuous pairs of a fuzzy metric space $(X,M,*)$. Then $\mathcal{A},\mathcal{H}$ and $\mathcal{K}$ have a coincidence point. If*

$$M(\mathcal{H}x,\mathcal{K}y,kt)\geq\min\left\{\begin{array}{c}M(\mathcal{A}x,\mathcal{A}y,t),M(\mathcal{A}x,\mathcal{H}x,t),\\ M(\mathcal{A}y,\mathcal{K}y,t),M(\mathcal{A}x,\mathcal{K}y,t),M(\mathcal{A}y,\mathcal{H}x,t)\end{array}\right\} \tag{3.3}$$

*$x,y\in X$ and $t>0$, then $\mathcal{A},\mathcal{H}$ and $\mathcal{K}$ have a unique common fixed point.*

If $\mathcal{H}=\mathcal{K}$:

**Corollary 3.7.** *Let $(\mathcal{A},\mathcal{H})$ be compatible of type (E) and weakly subsequentially continuous pairs of a fuzzy metric space $(X,M,*)$. Then $\mathcal{A}$ and $\mathcal{H}$ have a coincidence point.*

$$M(\mathcal{H}x,\mathcal{K}y,kt)\geq\min\left\{\begin{array}{c}M(\mathcal{A}x,\mathcal{A}y,t),M(\mathcal{A}x,\mathcal{H}x,t),\\ M(\mathcal{A}y,\mathcal{H}y,t),M(\mathcal{A}x,\mathcal{H}y,t),M(\mathcal{A}y,\mathcal{H}x,t)t\end{array}\right\} \tag{3.4}$$

*$x,y\in X,k\in(0,1)$ and $t>0$, then $\mathcal{A}$ and $\mathcal{H}$ have a unique common fixed point.*

Now we utilize the weak subsequential continuity and compatibility of type (E) for $\phi$-contrative type mapping.

**Theorem 3.8.** *Theorem 3.1 remains true even if we replace 3.1 by*

$$M(\mathcal{H}x,\mathcal{K}y,kt)\geq\phi\Big(\min\left\{\begin{array}{c}M(\mathcal{A}x,\mathcal{B}y,t),M(\mathcal{A}x,\mathcal{H}x,t),\\ M(\mathcal{B}y,\mathcal{K}y,t),M(\mathcal{A}x,\mathcal{K}y,t),M(\mathcal{B}y,\mathcal{H}x,t)\end{array}\right\}\Big) \tag{3.5}$$

*$x,y\in X,k\in(0,1)$ and $t>0$, $\phi:[0,1]\to[0,1]$ is a lower semi continuous such that $\phi(t)>t$, $t\in(0,1)$ with $\phi(0)=0$ and $\phi(1)=1$.*

**Proof .** Theorem 3.8 follow the pattern of Theorem 3.1 as $t>0$, $\phi(t)>t$.□ Now we exploit integral type contractive condition in a fuzzy metric space.

**Theorem 3.9.** *Theorem 3.1 remains true even if we replace 3.1 by*

$$\int_0^{M(\mathcal{H}x,\mathcal{K}y,kt)}\varphi(t)dt\geq\int_0^{m(x,y)}\varphi(t)dt, \tag{3.6}$$

*where*

$$m(x,y)=\min\{M(\mathcal{A}x,\mathcal{B}y,t),M(\mathcal{A}x,\mathcal{H}x,t),M(\mathcal{B}y,\mathcal{K}y,t),M(\mathcal{A}x,\mathcal{K}y,t),M(\mathcal{B}y,\mathcal{H}x,t)\}$$

*and $x,y\in X,k\in(0,1)$ and $t>0$, $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue integrable and summable and for each $\varepsilon>0$, $\int_0^\varepsilon\varphi(t)dt>0$.*

**Proof .** Following Theorem 3.1, $\mathcal{A}z=\mathcal{B}z$ and $\mathcal{B}w=\mathcal{K}w$.
We prove $\mathcal{A}z=\mathcal{B}w$. Taking $x=z$ and $y=w$ in (3.6) we get:

$$\int_0^{M(\mathcal{H}z,\mathcal{K}w,kt)}\varphi(t)dt\geq\int_0^{\min\{M(\mathcal{A}z,\mathcal{B}w,t),M(\mathcal{A}z,\mathcal{H}z,t),M(\mathcal{B}w,\mathcal{K}w,t),M(\mathcal{A}z,\mathcal{K}w,t),M(\mathcal{B}w,\mathcal{H}z,t)\}}\varphi(t)dt.$$

Since, $\mathcal{A}z = \mathcal{H}z$ and $\mathcal{B}w = \mathcal{K}w$, we get:

$$\int_0^{M(\mathcal{A}z,\mathcal{B}w,kt)} \varphi(t)dt \geq \int_0^{\min\{M(\mathcal{A}z,\mathcal{B}w,t),1,1,M(\mathcal{A}z,\mathcal{B}w,t),M(\mathcal{A}z,\mathcal{B}w,t)\}} \varphi(t)dt$$
$$= \int_0^{M(\mathcal{A}z,\mathcal{B}w,t)} \varphi(t)dt,$$

i.e.,

$$M(\mathcal{A}z,\mathcal{B}w,kt) \geq M(\mathcal{A}z,\mathcal{B}w,t).$$

From Lemma 2.4, $\mathcal{A}z = \mathcal{B}w$.
Substituting $x = x_n$ and $y = w$ in (3.6):

$$\int_0^{M(\mathcal{H}x_n,\mathcal{K}w,kt)} \varphi(t)dt \geq \int_0^{m(x_n,w)} \varphi(t)dt.$$

Letting $n \to \infty$:

$$\int_0^{M(z,\mathcal{K}w,kt)} \varphi(t)dt \geq \int_0^{M(z,\mathcal{B}w,t)} \varphi(s)ds,$$

i.e.,

$$\int_0^{M(z,\mathcal{A}z,kt)} \varphi(t)dt \geq \int_0^{M(z,\mathcal{A}z,t)} \varphi(t)dt,$$

i.e.,

$$M(z,\mathcal{A}z,kt) \geq M(z,\mathcal{A}z,t),$$

i.e., $z = \mathcal{A}z = \mathcal{H}z$ (from Lemma 2.4).
Nextly, taking $x = x_n$ and $y = w$ in (3.6):

$$\int_0^{M(\mathcal{H}x_n,\mathcal{K}yn,kt)} \varphi(t)dt \geq \int_0^{m(x_n,y_n)} \varphi(t)dt.$$

Passing the limit, when $n \to \infty$:

$$\int_0^{M(z,w,kt)} \varphi(t)dt \geq \int_0^{m(z,w)} \varphi(t)dt = \int_0^{M(z,w,t)} \varphi(t)dt,$$

i.e.,

$$M(z,w,kt) \geq M(z,w,t).$$

Consequently, $z$ is a common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{H}$ and $\mathcal{K}$.
If $q$ is another fixed point, then using (3.6):

$$\int_0^{M(\mathcal{H}z,\mathcal{K}q,kt)} \varphi(t)dt \geq \int_0^{m(z,q)} \varphi(t)dt = \int_0^{M(z,q,t)} \varphi(t)dt.$$

Hence, $z = q$ and $z$ is unique. □

**Remark 3.10.** *All results are true even if we replace compatibility of type (E) and weak subsequential continuity by any one of the following:*

1. *$\mathcal{H}$ (or $\mathcal{A}$)-compatibility of type (E) and $\mathcal{H}$ (or $\mathcal{A}$)-subsequential continuity,*
2. *$\mathcal{A}$ (or $\mathcal{H}$)-compatibility of type (E) or compatibility of type (E) and subsequential continuity.*

## 4. Application

We utilize Corollary 3.4 to solve functional equations arising in dynamic programming as an application. It was first studied by Bellman [1] using famous Banach fixed point theorem. Let $W \subset X$ be state space and $D \subset Y$ be decision space. Let $B(W)$ be the set of bounded functions on $W$. Define:

$$M(h,k,t) = e^{-\frac{d(h,k)}{t}},$$

with t-norm $a * b = \min\{a,b\}$, $a,b \in [0,1]$, where $d(h,k) = \|h(\tau) - k(\tau)\|_\infty = \sup_{\tau \in W} |h-k|_\tau$. Then $(X, M, *)$ is a fuzzy metric space.

**Theorem 4.1.** *Let $\mathcal{H}$ and $\mathcal{A}$ be self mappings of $(X, M, *)$. If the following hypotheses hold:*

*(a) $H$ and $K$ are bounded,*

*(b) $\exists$ a $\delta \in (0,1)$ such that:*

$$|H(x,y,f(\tau(x,y))) - K(x,y,g(\tau(x,y)))| \leq \delta(h,k,t),$$

*where,*

$$\delta(h,k,t) = \min\{M(\mathcal{A}x, \mathcal{A}y, t), M(\mathcal{A}x, \mathcal{H}x, t), M(\mathcal{A}y, \mathcal{H}y, t), M(\mathcal{A}x, \mathcal{H}y, t), M(\mathcal{A}y, \mathcal{H}x, t)\},$$

*$x, y \in W$ and $h, k \in B(W)$,*

*(c) $\exists$ a sequence $\{h_n\} \in W$, satisfying*

$$\lim_{n\to\infty} \mathcal{A}h_n = \lim_{n\to\infty} \mathcal{H}h_n = h \in B(W),$$

$$\lim_{n\to\infty} \mathcal{A}\mathcal{H}h_n = \mathcal{A}h$$

*and*

$$\lim_{n\to\infty} \mathcal{A}^2 h_n = \lim_{n\to\infty} \mathcal{A}\mathcal{H}h_n = \mathcal{H}h.$$

*Then the system*

$$\begin{cases} \mathcal{H}f(t) = \sup_{x\in W}\{u(x,t) + H(x,y,f(\tau(x,y)))\} \\ \mathcal{A}g(t) = \sup_{x\in W}\{u(x,y) + K(x,y,g(\tau(x,y)))\}, \end{cases} \tag{4.1}$$

*has a unique bounded solution.*

**Proof .** The system has a unique solution iff $\mathcal{H}$ and $\mathcal{A}$ have a unique common fixed point.
For all $h, k \in B(W)$ and $\varepsilon > 0$, $\exists$ $y, z \in W$ such that

$$\mathcal{H}h < u(x,y) + H(x,y,h(\tau(x,y))) + \varepsilon \tag{4.2}$$

$$\mathcal{A}k < u(x,z) + K(x,z,k(\tau(x,z))) + \varepsilon \tag{4.3}$$

and since,

$$\mathcal{H}h \geq u(x,z) + H(x,z,h(\tau(x,z))), \tag{4.4}$$

$$\mathcal{A}k \geq u(x,y) + K(x,y,k(\tau(x,y))), \tag{4.5}$$

then from (4.2) and (4.5)

$$\begin{aligned}\mathcal{H}h - \mathcal{A}k &\leq H(x, y, h(\tau(x, y))) - K(x, y, k(\tau(x, y))) + \varepsilon \\ &\leq \delta(h, k) + \varepsilon. \end{aligned} \tag{4.6}$$

Also from (4.3) and (4.4)

$$\begin{aligned}\mathcal{H}h - \mathcal{A}k &> H(x, y, h(\tau(x, y))) - K(x, y, k(\tau(x, y))) - \varepsilon \\ &\geq -\delta d(h, k) - \varepsilon. \end{aligned} \tag{4.7}$$

Consequently, inequalities (4.5) and (4.7) implies that

$$\begin{aligned} d(\mathcal{H}h, \mathcal{A}k) = \sup |\mathcal{H}h - \mathcal{A}k| &\leq |H(x, y, h(\tau(x, y))) - K(x, y, k(\tau(x, y)))| + \varepsilon \\ &\leq \delta(h, k) + \varepsilon. \end{aligned}$$

Since, $\varepsilon > 0$ is arbitrary

$$d(\mathcal{H}h, \mathcal{A}k) \leq d(h, k). \tag{4.8}$$

So,

$$e^{-\frac{d(\mathcal{H}h, \mathcal{A}k)}{t}} \geq e^{-\frac{d(h,k)}{t}}.$$

$\Rightarrow$

$$M(\mathcal{H}x, \mathcal{H}y, kt) \geq \delta(h, k, t).$$

The condition $(c)$ implies that $(\mathcal{A}, \mathcal{H})$ is $\mathcal{A}$-compatible of type (E) as well as $\mathcal{A}$-subsequentially continuous as a result all the hypotheses of Corollary 3.4 are verified and consequently, $\mathcal{H}$ and $\mathcal{A}$ have a unique common fixed point. Hence, (4.1) has a unique solution.□

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