Int. J. Nonlinear Anal. Appl. 12 (202-) No. 2, 1719-1724
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5311

Study of the rings in which each element express as the sum of an idempotent and pure

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(Communicated by Madjid Eshaghi Gordji)

Abstract

This article aims to introduce the concept of $p-$clean rings as a generalization of some concepts such as clean rings and $r-$clean rings. As the first result, we prove that every clean ring is a $p-$clean ring and every $r-$clean ring is a $p-$clean ring. Furthermore, we give the relation between von Neumann local ring and $p-$clean ring. Finally, we investigate many properties of $p-$clean rings.

Keywords: ring, clean ring, r-clean ring, local ring and regular ring

1. Introduction

Goldman [7] studied the concept of unit element, where $\mathcal{R}$ be a ring, then an element u in $\mathcal{R}$ is called unit element if there exist v such that $u\ v = v\ u = 1_{\mathcal{R}}$. Let the set of unit elements in $\mathcal{R}$ denoted by $\mathrm{U}(\mathcal{R})$ that is $\mathrm{U}(\mathcal{R}) = \{\ u \in \mathcal{R}\ ;\ u\ v = v\ u = 1\ ,$ for some $v \in \mathcal{R}\ \}$. The concept of idempotent element was studied by de Melo Hernández [2], where an element e in $\mathcal{R}$ is called idempotent element if $e^2 = e$, let $\mathrm{Id}(\mathcal{R})$ be the set of idempotent elements in $\mathcal{R}$ that is $\mathrm{Id}(\mathcal{R}) = \{\ e \in \mathcal{R}\ ;\ e^2 = e\}$. The notion of regular element was first introdoced by von Neumann in 1936, where an element r in $\mathcal{R}$ is called regular element if there exist s in $\mathcal{R}$ such that $r = r\ s\ r$, a ring $\mathcal{R}$ is called regular ring if each element in $\mathcal{R}$ is regular. Many other authors interested in studying regular rings, for example see [10] and [9]. Let the set of regular element in $\mathcal{R}$ be denoted by $\mathrm{Reg}(\mathcal{R})$ that is $\mathrm{Reg}(\mathcal{R}) = \{\ r \in \mathcal{R}\ ;\ r = r\ s\ r$, for some $s \in \mathcal{R}\ \}$. The concept of clean ring first introduced by Nicholson in 1977 [8], where the ring $\mathcal{R}$ is called clean ring if for each $c \in \mathcal{R}$ there exist $e \in \mathrm{Id}(\mathcal{R})$ and $u \in \mathrm{U}(\mathcal{R})$ such that $c = e + u$. Many other authors interested in studying clean rings, for example see [5], [12]

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Received: March 2021 *Accepted:* June 2021

and [3]. Ashrafi and Nasibi [4] in 2013 introduced the concept of $r-$clean ring , where the ring $\mathcal{R}$ is called $r-$clean ring if for each $c \in \mathcal{R}$ there exist $e \in \mathrm{Id}(\mathcal{R})$ and $r \in \mathrm{Reg}(\mathcal{R})$ such that $c = e+r$. Many authors have studied $r-$ clean rings such as [1] and [13]. Let $\mathcal{R}$ be a ring, then an element p in $\mathcal{R}$ is called pure element if there exist q in $\mathcal{R}$ such that $p = pq$ [11] and the set of pure elements in $\mathcal{R}$ write Pu($\mathcal{R}$)=$\{ p \in \mathcal{R} : p=pq$, for some $q \in \mathcal{R} \}$. The concept of von Neumann local ring was studied by Anderson [2], where a ring $\mathcal{R}$ is called von Neumann local ring if for each $r \in \mathcal{R}$ we have either $r \in \mathrm{Reg}(\mathcal{R})$ or $1-r \in \mathrm{Reg}(\mathcal{R})$.

Definition 1.1. *An* element $c \in \mathcal{R}$ *is called* $p-$ clean *if there exist* $e \in \mathrm{Id}(\mathcal{R})$ *and* $p \in \mathrm{Pu}(\mathcal{R})$ *such that* $c = e + p$.

Definition 1.2. *Let* $\mathcal{R}$ *be a* ring*. Then* $\mathcal{R}$ *is called* $p-$ clean ring *if each element in* $\mathcal{R}$ *express as the sum of an idempotent and pure.*

Example 1.3. *The ring* $(Z_6 , +_6 , \cdot_6)$ *is a* $p-$ clean ring.

Example 1.4. *The ring* $(\mathbb{Z} , + , .)$ *is a* $p-$ clean ring.

Proposition 1.5. *Every* clean ring *is a* $p-$ clean ring.

Proof . Let $\mathcal{R}$ be a clean ring and $c \in \mathcal{R}$. Then $c = e + u$. Where $e \in \mathrm{Id}(\mathcal{R})$ and $u \in \mathrm{U}(\mathcal{R})$. To proof c is $p-$ clean , it remains only to prove that u is a pure element. Since $u \in \mathrm{U}(\mathcal{R})$, then there is $v \in \mathcal{R}$ such that $u\,v = v\,u = 1$, hence$1 \in \mathcal{R}$.

Now, $u = u \,.\, 1$, implies that u is a pure element. And hence $u \in \mathrm{Pu}(\mathcal{R})$, thus c is $p-$ clean.

Therefore $\mathcal{R}$ is a $p-$ clean ring.

The converse of above proposition is true. □

Example 1.6. *The ring* $(\mathbb{Z} , + , .)$ *is a* $p-$clean ring*. But not* clean *(because, not each element in* $\mathbb{Z}$ *is unite).*

Theorem 1.7. *Let* $\mathcal{R}$ *be a* ring *and* $\mathrm{Id}(\mathcal{R}) = \{0 , 1\}$*. Then* $\mathcal{R}$ *is* $p-$ clean ring *if and only if it is* clean ring*.*

Proof . Every clean ring is a $p-$clean ring by proposition 5. Conversely Let $\mathcal{R}$ be a $p-$clean ring and $c \in \mathcal{R}$. Then c is $p-$ clean, then there exists $e \in \mathrm{Id}(\mathcal{R})$ and $p \in \mathrm{Pu}(\mathcal{R})$ such that $c = e + p$. If $p = 0$, then $c = e = (1-e)+(2e-1)$. $(1-e)^2 = 1-2e+e^2 = 1-2e^2+e^2 = 1-e^2 = 1-e$ and hence $1-e \in \mathrm{Id}(\mathcal{R})$. Now, since $(2e-1)^2 = 4e^2 - 4e + 1 = 4e - 4e + 1 = 1$, then $(2e-1) \in \mathrm{U}(\mathcal{R})$ and hence c is clean element, thus $\mathcal{R}$ is clean ring. So, assume that $p \neq 0$. Since $p \in \mathrm{Pu}(\mathcal{R})$, then there is $d \in \mathcal{R}$ such that $p = pd$. Consider $d = qp$, then $p = pqp$. Now, $(pq)^2 = (pq)(pq) = (pqp)q = pq$ which is implies that $pq \in \mathrm{Id}(\mathcal{R})$, hence by hypothesis either $pq = 0$ or $pq = 1$. If $pq = 0$, then $p = 0$ which is contradiction, thus $pq = 1$. On the other hand, $(qp)^2 = (qp)(qp) = q(pqp) = qp$ which is implies that $qp \in \mathrm{Id}(\mathcal{R})$, hence by hypothesis either $qp = 0$ or $qp = 1$. If $qp = 0$, then $p = 0$ which is contradiction, thus $qp = 1$. Consequentially, $p \in \mathrm{U}(\mathcal{R})$. This implies that c is the sum of an idempotent and unit, and hence c is a clean element. Therefore, $\mathcal{R}$ is a clean ring. □

Theorem 1.8. *Let* $\mathcal{R}$ *be a* ring *and* $\mathrm{Id}(\mathcal{R}) = \{0 , 1\}$*. Then* $\mathcal{R}$ *is* $p-$ clean ring *if and only if it is a* von Neumann local ring*.*

Proof . Let $\mathcal{R}$ be a ring and $\mathrm{Id}(\mathcal{R}) = \{0\ ,\ 1\}$. Assume $\mathcal{R}$ is $p-$ clean ring, then by Theorem 7 we get $\mathcal{R}$ is clean ring. Let $c \in \mathcal{R}$ we have c is the sum of an idempotent and unit, that is there exists $e \in \mathrm{Id}(\mathcal{R})$ and $u \in \mathrm{U}(\mathcal{R})$ such that $c = e + u$, but the unit element is a regular element which is implies that $u \in \mathrm{Reg}(\mathcal{R})$. by hypothesis $e = 0$ or 1. If $e = 0$, then $c = e + u = u$, hence $c \in \mathrm{Reg}(\mathcal{R})$. If $e = 1$, then $c = 1 + u$ and $u = 1 - c$, hence $1 - c \in \mathrm{Reg}(\mathcal{R})$. Therefore $\mathcal{R}$ is a von Neumann local ring.

Conversely, let $\mathcal{R}$ be a von Neumann local ring and $c \in \mathcal{R}$. Then either $c \in \mathrm{Reg}(\mathcal{R})$ or $1 - c \in \mathrm{Reg}(\mathcal{R})$. if $c \in \mathrm{Reg}(\mathcal{R})$, put $c = 0 + c$, then to prove c is a $p-$clean, it remains only to prove that c is a pure element, since $c \in \mathrm{Reg}(\mathcal{R})$ then there is $d \in \mathcal{R}$ such that $c = cdc$. Consider $q = dc$, then $c = cq$, hence $c \in \mathrm{Pu}(\mathcal{R})$. If $1 - c \in \mathrm{Reg}(\mathcal{R})$, put $c = 1 + (1 - c)$. Since every regular element is pure, then $(1 - c) \in \mathrm{Pu}(\mathcal{R})$. Hence c is a $p-$ clean. Therefore $\mathcal{R}$ is a $p-$ clean ring. □

Proposition 1.9. *Every* $r-$ clean ring *is a* $p-$ clean ring .

Proof . Let $\mathcal{R}$ be a $r-$clean ring and let $c \in \mathcal{R}$. Then $c = e + r$. Where $e \in \mathrm{Id}(\mathcal{R})$ and $r \in \mathrm{Reg}(\mathcal{R})$. To proof c is $p-$clean element in $\mathcal{R}$, it is suffices we prove that r is pure element, Since $r \in \mathrm{Reg}(\mathcal{R})$, then there is $s \in \mathcal{R}$ such that $r = r\ s\ r$. Consider q = s r, then q $\in \mathcal{R}$. Hence r = r q, thus r is a pure element, consequentially c is $p-$ clean element. Therefore $\mathcal{R}$ is a $p-$ clean ring. The converse of above proposition is not true. □

Example 1.10. *The ring* $(\mathbb{Z}\ \ ,+\ \ ,\ \ .\ )$ *is a* $p-$ clean ring. *But not* $r-$ clean ring.

Proposition 1.11. *Every* pure element *of a* ring $\mathcal{R}$ *is a* $p-$ clean.

Proof . The proof follows from the definition of $p-$ clean element. □

Proposition 1.12. *Every* idempotent element *of a* ring $\mathcal{R}$ *is a* $p-$ clean.

Proof . The proof follows from the definition of $p-$ clean element. □

Proposition 1.13. *Let* $\mathcal{R}$ *be a* ring *and* $c \in \mathcal{R}$. *If* c *is* $p-$ clean, *then* $\forall n \in \mathbb{Z}^+$, c^n *is* $p-$ clean.

Proof . Let $\mathcal{R}$ be a ring and $c \in \mathcal{R}$. Assume that c is $p-$ clean, then there exists $e \in \mathrm{Id}(\mathcal{R})$ and p$\in \mathrm{Pu}(\mathcal{R})$ such that $c = e + p$. Now $c^n = (e+p)^n = e^n + p^n$, we must prove $e^n \in \mathrm{Id}(\mathcal{R})$ and $p^n \in \mathrm{Pu}(\mathcal{R})$. Since $e \in \mathrm{Id}(\mathcal{R})$ then $e^2 = e$. Now $(e^n)^2 = (e^2)^n = (e)^n$, hence $e^n \in \mathrm{Id}(\mathcal{R})$. Since p$\in \mathrm{Pu}(\mathcal{R})$, then there is q $\in \mathcal{R}$ such that p = pq. Hence $p^n = (pq)^n = p^n q^n$. Since q $\in \mathcal{R}$, then $q^n \in \mathcal{R}\ \forall n \in \mathbb{Z}^+$ thus $p^n \in \mathrm{Pu}(\mathcal{R})$ Therefore, c^n is a $p-$ clean ring. □

Proposition 1.14. *Let* $\mathcal{R}$ *be a* ring *and* $c \in \mathcal{R}$. *Then* c *is* $p-$clean *if and only if* $1 - c$ *is* $p-$clean.

Proof . Let $\mathcal{R}$ be a ring and $c \in \mathcal{R}$. Assume that c is $p-$ clean, then there exists $e \in \mathrm{Id}(\mathcal{R})$ and p$\in \mathrm{Pu}(\mathcal{R})$ such that $c = e + p$. Now $1 - c = 1 - (e + p) = (1 - e) + (-p)$, we must prove $1 - e \in \mathrm{Id}(\mathcal{R})$ and $(-\mathrm{p}) \in \mathrm{Pu}(\mathcal{R})$. Since $e \in \mathrm{Id}(\mathcal{R})$ then $e^2 = e$. Now $(1-e)^2 = 1 - 2e + e^2 = 1 - 2e^2 + e^2 = 1 - e^2 = 1 - e$, hence $1 - e \in \mathrm{Id}(\mathcal{R})$. Since p$\in \mathrm{Pu}(\mathcal{R})$, then there is q $\in \mathcal{R}$ such that p = pq. Hence $- -$ p $= -$pq $= (-$p$)$q, thus $(-\mathrm{p}) \in \mathrm{Pu}(\mathcal{R})$ Therefore, $(1 - c)$ is a $p-$ clean ring. Conversely: let $1 - c$ is $p-$clean, then $1 - c = e + p$ where $e \in \mathrm{Id}(\mathcal{R})$ and p$\in \mathrm{Pu}(\mathcal{R})$. Now, $c = 1 - (e + p) = (1 - e) + (-p)$, as a previous part we have $1 - e \in \mathrm{Id}(\mathcal{R})$ and$(-\mathrm{p}) \in \mathrm{Pu}(\mathcal{R})$ which implies that c is the sum of an idempotent and pure. Therefore, c is a $p-$ clean. □

Proposition 1.15. *Let* $\mathcal{R}$ *be a* ring *and* $c \in \mathcal{R}$. *If* c *is* $p-$clean, *then* $\forall n \in \mathbb{Z}^+$, $1 - c^n$ *is* $p-$clean.

Proof . The proof follows from proposition 13 and proposition 14 □

Proposition 1.16. *Let $\mathcal{R}$ be a* $\mathrm{p}-$clean ring *and $\mathcal{R}'$ be a* ring*. If f: $\mathcal{R} \to \mathcal{R}'$ is epimorphism, then $\mathcal{R}'$ is a $p-$*clean ring*.*

Proof . Let $c' \in \mathcal{R}'$. Since f: $\mathcal{R} \to \mathcal{R}'$ is epiomorphism, then $\in \mathcal{R}$ such that $\mathrm{c}' = f(\mathrm{c})$. But is a $p-$clean ring , then $c = e + p$ where $e \in \mathrm{Id}(\mathcal{R})$ and $\mathrm{p} \in \mathrm{Pu}(\mathcal{R})$ $\mathrm{c}' = f(\mathrm{c}) = f(\mathrm{e}+\mathrm{p}) = f(\mathrm{e}) + f(\mathrm{p})$ Now, we must prove $f(e) \in \mathrm{Id}(\mathcal{R})$ and $f(\mathrm{p}) \in \mathrm{Pu}(\mathcal{R}')$. Since $e \in \mathrm{Id}(\mathcal{R})$ then $e^2 = e$. Hence $f(e) = f(e^2) = [f(e)]^2$, thus $f(e) \in \mathrm{Id}(\mathcal{R})$, since $\mathrm{p} \in \mathrm{Pu}(\mathcal{R})$, then there is $\mathrm{q} \in \mathcal{R}$ such that $\mathrm{p} = \mathrm{pq}$. Hence $f(\mathrm{p}) = f(\mathrm{pq}) = f(\mathrm{p}).f(\mathrm{q})$, but $\mathrm{q} \in \mathcal{R}$, then $f(\mathrm{q}) \in \mathcal{R}'$, which implies that $f(\mathrm{p}) \in \mathrm{Pu}(\mathcal{R}')$, thus c' is the sum of an idempotent and pure. Therefore, $\mathcal{R}'$ is a $p-$clean ring. □

Proposition 1.17. *Let* I *be an* Ideal *of a $p-$*clean *ring $\mathcal{R}$. Then $\frac{\mathcal{R}}{\mathrm{I}}$ is a $p-$*clean *ring.*

Proof . Let $\mathrm{e} + \mathrm{I} \in \frac{\mathcal{R}}{\mathrm{I}}$. Then $c \in \mathcal{R}$, since $\mathcal{R}$ is $p-$clean ring, there exists $e \in \mathrm{Id}(\mathcal{R})$ and $\mathrm{p} \in \mathrm{Pu}(\mathcal{R})$ such that $c = e + p$. Hence $c + \mathrm{I} = e + p + \mathrm{I} = e + \mathrm{I} + p + \mathrm{I}$, to prove $c + \mathrm{I}$ is a $p-$clean element in $\frac{\mathcal{R}}{\mathrm{I}}$ we must prove that $e + p$ is an idempotent element in $\frac{\mathcal{R}}{\mathrm{I}}$ and $p + \mathrm{I}$ is a pure element in $\frac{\mathcal{R}}{\mathrm{I}}$. Since $e \in \mathrm{Id}(\mathcal{R})$, $e^2 = e$ and hence $e + \mathrm{I} = e^2 + \mathrm{I} = e.e + \mathrm{I} = (e + \mathrm{I})\ .\ (e + \mathrm{I}) = (e + \mathrm{I})^2$, thus $(e + \mathrm{I})$ is an idempotent element in $\frac{\mathcal{R}}{\mathrm{I}}$. Since $\mathrm{p} \in \mathrm{Pu}(\mathcal{R})$, there is $\mathrm{q} \in \mathcal{R}$ such that $\mathrm{p} = \mathrm{pq}$.
Now, $\mathrm{p+I} = \mathrm{pq} + \mathrm{I} = (\mathrm{p+I}) + (\mathrm{q+I})$, which implies that p+I is a pure element in $\frac{\mathcal{R}}{\mathrm{I}}$, thus $c + \mathrm{I}$ is the sum of an idempotent and pure. Therefore, $\frac{\mathcal{R}}{\mathrm{I}}$ is a $p-$clean ring. □

Proposition 1.18. *Let $\mathcal{R}_k$, $(k = 1, 2, .., n)$ be a $p-$*clean *ring. Then $\prod_{k=1}^{n} \mathcal{R}_k$ is a $p-$*clean *ring.*

Proof . Let $(c_1,\ c_2,\ \ldots, c_n) \in \prod_{k=1}^{n} \mathcal{R}_k$. Then $c_k \in \mathcal{R}_k$, $k = 1, 2, .., n$. Since $\mathcal{R}_k$ is $p-$clean ring, there exists $e_k \in \mathrm{Id}(\mathcal{R}_k)$ and $p_k \in \mathrm{Pu}(\mathcal{R}_k)$ such that $c_k = e_k + p_k$ $\forall k = 1, 2, .., n$. Hence

$$\begin{aligned}(c_1, c_2, \ldots, c_n) &= (e_1 + p_1, e_2 + p_2, \ldots, e_n + p_n)\\ &= (e_1, e_2, \ldots, e_n) + (p_1, p_2, \ldots, p_n).\end{aligned}$$

To prove $(c_1,\ c_2,\ \ldots, c_n)$ is a $p-$clean element in$\prod_{k=1}^{n} \mathcal{R}_k$, we must prove that $(e_1,\ e_2,\ \ldots, e_n)$ is an idempotent element in $\prod_{k=1}^{n} \mathcal{R}_k$ and $(p_1,\ p_2,\ \ldots, p_n)$ is a pure element in $\prod_{k=1}^{n} \mathcal{R}_k$. Since $e_k \in \mathrm{Id}(\mathcal{R}_k)$, ${e_k}^2 = e_k$, for all $k = 1, 2, .., n$, hence $(e_1,\ e_2,\ \ldots, e_n) = ({e_1}^2,\ {e_2}^2,\ \ldots, {e_n}^2)$ which is implies that $(e_1,\ e_2,\ \ldots, e_n) = (e_1,\ e_2,\ \ldots, e_n).\ (e_1,\ e_2,\ \ldots, e_n) = (e_1,\ e_2,\ \ldots, e_n)^2$ and thus $(e_1,\ e_2,\ \ldots, e_n)$ is an idempotent element in $\prod_{k=1}^{n} \mathcal{R}_k$. Since $p_k \in \mathrm{Pu}(\mathcal{R}_k)$ $\forall k = 1, 2, .., n.$, there is $q_k \in \mathcal{R}_k$ such that $p_k = p_k q_k$ $\forall k = 1, 2, .., n$. Hence,

$$\begin{aligned}(p_1,\ p_2,\ \ldots, p_n) &= (p_1 q_1,\ p_2 q_2,\ \ldots, p_n q_n)\\ &= (p_1, p_2, \ldots, p_n)\,.\,(q_1, q_2, \ldots, q_n)\,.\end{aligned}$$

Since $q_k \in \mathcal{R}_k$ $\forall k = 1, 2, .., n$, then $(q_1,\ q_2,\ \ldots, q_n) \in \prod_{k=1}^{n} \mathcal{R}_k$, which implies that $(p_1,\ p_2,\ \ldots, p_n)$ is a pure element in $\prod_{k=1}^{n} \mathcal{R}_k$, thus $(c_1,\ c_2,\ \ldots, c_n)$ is the sum of an idempotent and pure. Therefore, $\prod_{k=1}^{n} \mathcal{R}_k$ is a $p-$clean ring. □

Proposition 1.19. *Let $\mathcal{R}$ be an abelian* ring *and $c \in \mathcal{R}$ is $p-$*clean*. If $e \in \mathrm{Id}(\mathcal{R})$ and $(-c)$ is a $p-$*clean *in $\mathcal{R}$, then $(c + e)$ is a $p-$*clean*.*

Proof . Let $\mathcal{R}$ be an abelian ring and $c \in \mathcal{R}$ is $p-$clean. Then $1-c$ is $p-$clean by Proposition 14 Similarly as in Proposition 14, we can proof that $(-c)$ is a $p-$clean in $\mathcal{R}$ iff $(1+c)$ is a $p-$clean element.

Now, let $c = f + p$ where $f^2 = f$ and $p = pq$ $1 + c = g + w$ where $g^2 = g$ and $w = wz$.

$$\begin{aligned} c + e &= c + ce - ce + e = (c+1)\,e + c(1-e) \\ &= (g+w)\,e + (f+p)(1-e) = ge + we + f\,(1-e) + p(1-e) \\ &= ge + f\,(1-e) + we + p(1-e). \end{aligned}$$

We note that

$$\begin{aligned} (ge + f\,(1-e))^2 &= (ge + f\,(1-e)).(ge + f\,(1-e)) \\ &= (ge)^2 + gef\,(1-e) + f\,(1-e)\,ge + (f\,(1-e))^2 \\ &= ge + gef - ge^2f + fge - fe^2g + f(1-e) \\ &= ge + gef - gef + fge - feg + f(1-e) = ge + f(1-e). \end{aligned}$$

Also, $we + p(1-e)$ is a $p-$clean element in $\mathcal{R}$ (because $we + p\,(1-e)$ is a unit element)

$$\begin{aligned} (we + p(1-e)).(w^{-1}we + p^{-1}p(1-e)) &= we + wep^{-1}p + p\,(1-e)\,w^{-1}we + p(1-e) \\ &= we + wep^{-1}p - wep^{-1}pe + pw^{-1}we - pew^{-1}we + p\,(1-e) \\ &= we + p\,(1-e)\,. \end{aligned}$$

Since, $we + p\,(1-e)$ is a $p-$clean element in $\mathcal{R}$, $(c+e)$ is a $p-$clean.
□

2. Conclusion

Our aims in this work are to study the concepts of clean ring and $r-$clean ring which are stronger than of the concept $p-$clean ring. Furthermore, we give the relation between von Neumann local ring and $p-$clean ring and we prove that the finite direct product of $p-$clean rings is also $p-$clean ring. Finally, we studies many properties of $p-$clean rings.

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