Int. J. Nonlinear Anal. Appl.
Volume 12, Special Issue, Winter and Spring 2021 , 425-446
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5332

On split equality variation inclusion problems in Banach spaces without operator norms

[1] Lateef O. Jolaoso, [2]Ferdinard U. Ogbuisi, [3,*]Oluwatosin T. Mewomo

[1,2,3] *School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa.*
[2] *DSI-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), Johannesburg, South Africa.*

(Communicated by Madjid Eshaghi Gordji)

Abstract

The purpose of this paper is to study the approximation of solutions of split equality variational inclusion problem in uniformly convex Banach spaces which are also uniformly smooth. We introduce an iterative algorithm in which the stepsize does not require prior knowledge of operator norms. This is very important in practice because norm of operators that are often involved in applications are rarely known explicitly. We prove a strong convergence theorem for the approximation of solutions of split equality variational inclusion problem in p-uniformly convex Banach spaces which are also uniformly smooth. Further, we give some applications and a numerical example of our main theorem to show how the sequence values affect the number of iterations. Our results improve, complement and extend many recent results in literature.

Keywords: Split equality problem, variational inclusion, Bregman distance, fixed point problem, operator norm, Banach spaces.
2010 MSC: 47H06, 47H09, 49J53, 65K10

1. Introduction

Let E be a Banach space with dual E^*. For $p > 1$, the generalized duality mapping $J_p^E : E \to 2^{E^*}$ is defined by

$$J_p^E(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^p, ||x^*|| = ||x||^{p-1}\},$$

*Corresponding author
Email address: [1]216074984@stu.ukzn.ac.za, [2]215082189@stu.ukzn.ac.za, [3]mewomoo@ukzn.ac.za ([1] Lateef O. Jolaoso, [2]Ferdinard U. Ogbuisi, [3,*]Oluwatosin T. Mewomo)

Received: December 2017 *Accepted:* February 2021

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. For $p = 2$, J_p^E reduces to the normalized duality mapping, $J_2^E \equiv J^E$. It is well known that if E is uniformly smooth, then J_p^E is single-valued and uniformly continuous on bounded subsets of E. Moreover, if E is reflexive and strictly convex with a strictly convex dual, then $(J_p^E)^{-1} = J_q^{E^*}$ is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E and thus $J_p^E J_q^{E^*} = I_{E^*}$ and $J_q^{E^*} J_p^E = I_E$, where I_E and I_{E^*} are the identity operators on E and E^* respectively, see [31, 42, 51] for more details. We note that in a real Hilbert space, the duality mappings reduce to the identity mapping.

A mapping $A : E \to 2^{E^*}$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x - y, u - v\rangle \geq 0$, for all $x, y \in D(A)$ and $u \in Ax, v \in Ay$. A monotone mapping A is said to be maximal if its graph $G(A) := \{(x, u) : u \in Ax\}$ is not contained in the graph of any other monotone mapping. It is know that a monotone mapping A is maximal if and only if for $(x, u) \in E \times E^*$, $\langle x - y, u - v\rangle \geq 0$ for every $(y, v) \in G(A)$ implies that $u \in Ax$. Also if A is maximal monotone, then the zero of A, $A^{-1}(0) := \{x \in E : 0 \in Ax\}$ is closed and convex (see [50]). The problem of finding a point $x^* \in E$ which satisfies

$$0 \in Ax^*, \tag{1.1}$$

where A is a maximal monotone operator is known as the Variational Inclusion Problem (VIP). Several iterative methods have been proposed for finding solutions of (1.1) and related optimization problems in Hilbert, Banach, Hadamard and p-uniformly convex metric spaces, (see [1, 6, 8, 9, 27, 17, 36, 37, 38] and the references therein). A well-known method for solving the equation (1.1) in Hilbert space H is the proximal point algorithm introduced by Martinet [32]: For given $x_1 \in H$,

$$x_{n+1} = J_{\lambda_n}^A x_n, \quad n \geq 1, \tag{1.2}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $J_\lambda^A = (I + \lambda A)^{-1}$ for $\lambda > 0$. In 1976, Rockafellar [48] proved that if $\liminf_{n\to\infty} \lambda_n > 0$ and $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ defined by (1.2) converges weakly to an element of $A^{-1}(0)$.
Let H_1 and H_2 be two real Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two set-valued maximal monotone mappings and $A : H_1 \to H_2$ be a bounded linear operator, A^* be the adjoint of A. The Split Variational Inclusion Problems (SVIP) is formulated as:

$$\text{find } x^* \in H_1, \text{ such that } 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*). \tag{1.3}$$

This problem was introduced by Moudafi [34] in 2011 and have been studied extensively (see for instance [13, 34, 41]). Recently, Bryne et.al. [13] proposed the following iterative method to solve the problem (1.3): For given $x_0 \in H_1$ and $\lambda > 0$, the iterative sequence $\{x_n\}$ is generated as follows;

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \tag{1.4}$$

and obtained a weak and a strong convergence theorem for solving problem (3.3). Inspired by the work of Bryne et.al. [13], Kazimi and Rizvi [30] proposed the following algorithm for approximating of solution of SVIP which is also a fixed point of a nonexpansive self-mapping S: For a given $x_0 \in H$, let the sequence $\{u_n\}$ and $\{x_n\}$ be generated by;

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n - \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \ n \geq 0, \end{cases} \tag{1.5}$$

and proved that both $\{u_n\}$ and $\{x_n\}$ converge weakly to $z \in F(S) \cap \Gamma$, where Γ is the solution set of SVIP (1.3).

Recently, Guo et.al. [21] considered the Split Equality Variational Inclusion Problem SEVIP in Hilbert spaces defined as; find $x^* \in H_1$ and $y^* \in H_2$ such that

$$\begin{cases} 0 \in B_1(x^*) \text{ and } 0 \in B_2(y^*), \\ Ax^* = By^*, \end{cases} \tag{1.6}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators. Recently, motivated by the work of Moudafi [34], Bryne et.al. [13], Kazmi and Rizvi [30] among others, Guo et.al. [21] proved a strong convergence theorem for approximating a solution SEVIP which is also a solution of certain variational inequality problem in Hilbert space.

Recently, effort have been made to approximate the solution of split feasibility problem and split variational inclusion problem in Hilbert and Banach spaces. For recent results on split feasibility problems, some of its generalization, and related optimization problems, (see [2, 3, 18, 23, 24, 26, 28, 29, 40, 55] and some of the references therein).

The resolvent operator $R_{\lambda T}$ associated with T for $\lambda > 0$ is given as

$$R_{\lambda T}(x) := \{z \in E : J_p^E x \in J_p^E z + \lambda T(z)\}.$$

Equivalently, $R_{\lambda T} := (J_p^E + \lambda T)^{-1} J_p^E$. $R_{\lambda T}$ is single valued and also $T^{-1}0 = F(R_{\lambda T})$. It is well known that $R_{\lambda T}$ is relative nonexpansive, that is

$$0 \leq \langle R_{\lambda T}(x) - R_{\lambda T}(y), J_p^E(x) - J_p^E(R_{\lambda T}(x)) - (J_p^E y - R_{\lambda T}(y))\rangle, \tag{1.7}$$

for all $x, y \in E$.

In this paper, motivated by the works of Bryne [13], Kazmi and Rizvi [30], Guo et.al. [21], Cruz and Shehu [12], we proposed a simultaneous iterative algorithm for approximating solution of split equality variational inclusion problem (1.6) in p-uniformly convex Banach spaces which are also uniformly smooth. Even in finite dimensions, computing the norm of bounded linear operator is a difficult task as shown by the following theorem of Hendrickx and Olshevsky [22].

Theorem 1.1. *[22]: Fix a rational $p \in [1, \infty)$ with $p \neq 1, 2$. Unless $P = NP$, there is no algorithm which given input ϵ and a matrix M with entries in $\{-1, 0, 1\}$, computes $||M||_p$ to relative accuracy ϵ, in time which is polynomial in ϵ^{-1} and the dimensions of the matrix.*

Thus, we introduce an iterative algorithm with a self adaptive stepsize and prove a strong convergence theorem for approximating solution of split equality variational inclusion problem in p-uniformly convex Banach spaces which are also uniformly smooth such that the ardours task of computing operator norms is avoided.

2. Preliminaries

In this section, we recall some definitions and known results which will be use in the sequel. We adopt the notations $x_n \rightharpoonup x$ and $x_n \to x$ to mean that x_n converges weakly to x and x_n converges strongly to x respectively.

Let E be a real Banach space and $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $dim(E) \geq 2$, the modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$, defined by

$$\delta_E := \inf \Big\{1 - ||\frac{x + y}{2}|| : ||x|| = ||y|| = 1; \ \epsilon = ||x - y||\Big\}.$$

E is said to be uniformly smooth if and only if $\delta_E(\epsilon) > 0$, for all $\epsilon \in (0, 2]$ and p-uniformly convex if there exists a $C_p > 0$, such that $\delta_E(\epsilon) \geq C_p\epsilon^p$ for any $\epsilon \in (0, 2]$.
The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) := \sup\left\{\frac{1}{2}(||x+y|| + ||x-y||) - 1 : ||x|| \leq 1, ||y|| \leq t\right\}.$$

A Banach space E is said to be uniformly smooth if and only if

$$\lim_{t\to\infty} \frac{\rho_E(t)}{t} = 0,$$

and q-$uniformly$ $smooth$ if there exists a $C_q > 0$ such that $\rho_E(t) \leq C_q t^q$ for any $t > 0$.
It is well known that E is p-uniformly convex and uniformly smooth if and only if its dual space E^* is q-uniformly smooth and uniformly convex. For more information on geometric of Banach spaces, see [4, 50, 51, 52].

Lemma 2.1. *(Xu [57]): Let $x, y \in E$ and $q > 1$. If a Banach space E is q-uniformly smooth, then there is a $C_q > 0$ so that*

$$||x - y||^q \leq ||x||^q - q\langle y, J_q^E(x)\rangle + C_q||y||^q.$$

Definition 2.2. *[54] Let $f : E \to \mathbb{R}$ be a convex and Gáteaux differentiable function. The Bregman distance with respect to f is defined by*

$$D_f(x, y) := f(y) - f(x) - \langle f'(x), y - x\rangle, \tag{2.1}$$

$\forall\ x, y \in E$. It is worth noting that the duality mapping J_p^E is actually the derivative of the function $f_p(x) = \frac{1}{p}||x||^p$ for $2 \leq p < \infty$. If $f = f_p$, then the Bregman distance with respect to f_p now becomes

$$\begin{aligned} D_p(x, y) &= \frac{1}{q}||x||^p - \langle J_p^E x, y\rangle + \frac{1}{p}||y||^p \\ &= \frac{1}{p}(||y||^p - ||x||^p) + \langle J_p^E x, x - y\rangle \\ &= \frac{1}{q}(||x||^p - ||y||^p) - \langle J_p^E x - J_p^E y, y\rangle. \end{aligned} \tag{2.2}$$

Infact, the Bregman distance is not symmetric and so it is not a metric but it posses the following important properties: for all $w, x, y, \in E$,

$$D_p(x, y) = D_p(x, w) + D_p(w, y) + \langle w - y, J_p^E x - J_p^E y\rangle, \tag{2.3}$$

and

$$D_p(x, y) + D_p(y, x) = \langle x - y, J_p^E x - J_p^E y\rangle. \tag{2.4}$$

We note that for the p-uniformly convex space, the metric and Bregman distance has the following relation (see [45])

$$\tau||x - y||^p \leq D_p(x, y) \leq \langle x - y, J_p^E x - J_p^E y\rangle, \tag{2.5}$$

where $\tau > 0$ is some fixed number.

Let C be a nonempty closed and convex subset of E. The metric projection

$$P_C x := \underset{y\in C}{argmin}\ ||x-y||,$$

for all $x \in E$ is the unique minimizer of the norm distance which can be characterized by a variational inequality:

$$\langle J_p^E(x - P_C x), z - P_C x\rangle \leq 0,\ \forall\ z \in C. \tag{2.6}$$

Similar to the metric projection, we define the Bregman projection as

$$\Pi_C x := \underset{y\in C}{argmin}\ D_p(x, y),$$

for all $x \in E$, which is the unique minimizer of the Bregman distance. The Bregman projection is also characterized by the variational inequality:

$$\langle J_p^E(x) - J_p^E(\Pi_C x), z - \Pi_C x\rangle \leq 0.\ \forall\ z \in C, \tag{2.7}$$

which implies that

$$D_p(\Pi_C x, z) \leq D_p(x, z) - D_p(x, \Pi_C x), \tag{2.8}$$

for all $z \in C$.

Following [7, 15], we make use of the function $V_p : E^* \times E \to [0,\infty)$, defined by

$$V_p(x, y) := \frac{1}{q}||x||^q - \langle x, y\rangle + \frac{1}{p}||y||^p,\ \forall\ x \in E^*,\ y \in E. \tag{2.9}$$

Then V_p is nonnegative and $V_p(x, y) = D_p(J_p^{E^*}(x), y)$ for all $x \in E^*$ and $y \in E$. Moreover, by the subdifferential inequality

$$\langle f'(x), y - x\rangle \leq f(y) - f(x),$$

with $f(x) = \frac{1}{q}||x||^q$ and $x \in E^*$, then $f'(x) = J_q^{E^*}$. Then we have

$$\langle J_q^{E^*}(x), y\rangle \leq \frac{1}{q}||x + y||^q - \frac{1}{q}||x||^q, \tag{2.10}$$

and from (2.10), we obtain (see [46])

$$V_p(\bar{x} + \bar{y}, x) \geq V_p(\bar{x}, x) + \langle \bar{y}, J_p^{E^*}(\bar{x}) - x\rangle, \tag{2.11}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$. In addition, V_p is convex in the first variable. Thus, for all $z \in E$,

$$D_p(J_q^{E^*} \sum_{i=1}^{N} t_i J_p^E(x_i), w) \leq \sum_{i=1}^{N} t_i D_p(x_i, w), \tag{2.12}$$

where $\{x_i\} \subset E$ and $\{t_i\} \subset (0, 1)$ with $\sum_{i=1}^{N} = 1$.

Lemma 2.3. *[43] Let E be a reflexive strictly convex and smooth Banach space, let $T : E \to 2^{E^*}$ be a maximal monotone operator with $T^{-1}0 \neq \emptyset$. Then for any $x \in E$, $u \in T^{-1}(0)$ and $\lambda > 0$, we have*

$$D_p(x, R_{\lambda T} x) + D_p(R_{\lambda T} x, u) \leq D_p(x, u).$$

Lemma 2.4. [53, 54] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + t_n\delta_n \quad \forall n \geq 0,$$

where $\{t_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that:

i. $\sum_{n=o}^{\infty} t_n = \infty$,

ii. $\limsup_{n\to\infty} \delta_n \leq 0$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Main result

Theorem 3.1. *Let E_1, E_2 and E_3 be p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. Let $T_1 : E_1 \to 2^{E_1^*}$ and $T_2 \to 2^{E_2^*}$ be maximal monotone operators such that $\Gamma := \{(\bar{x}, \bar{y}) \in T_1^{-1}(0) \times T_2^{-1}(0); A\bar{x} = B\bar{y}\}$ is nonempty. For fixed $u \in E_1$ and $v \in E_2$, choose an initial guess $x_1 \in E_1$ and $y_1 \in E_2$ arbitrarily and let $\{\alpha_n\} \subset [0, 1]$. Assume that the nth iterate $(x_n, y_n) \in E_1 \times E_2$ has been constructed; then we calculate the $(n+1)$th iterate (x_{n+1}, y_{n+1}) via the formula*

$$\begin{cases} u_n = R_{\lambda T_1} J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), \\ x_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(u_n)), \\ v_n = R_{\lambda T_2} J_q^{E_2^*}(J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)), \\ y_{n+1} = J_p^{E_2^*}(\alpha_n J_p^{E_2}(v) + (1-\alpha_n) J_p^{E_2}(v_n)), \end{cases} \tag{3.1}$$

where $\lambda > 0$, A^* and B^* are the adjoints of A and B respectively and the stepsize t_n is choosen in such a way that

$$t_n \in \Big(\epsilon, \Big(\frac{q||Ax_n - By_n||^p}{C_q||A^* J_p^{E_3}(Ax_n - By_n)||^q + Q_q||B^* J_p^{E_3}(Ax_n - By_n)||^q} - \epsilon\Big)^{\frac{1}{q-1}}\Big), \ n \in \Omega, \tag{3.2}$$

for small enough ϵ, otherwise $t_n = t$(t being any nonegative value), where the set of indices $\Omega = \{n : Ax_n - By_n \neq 0\}$. Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequence $\{(x_n, y_n)\}$ strongly converges to $(\bar{x}, \bar{y}) = (\Pi_{\Gamma_1} u, \Pi_{\Gamma_2} v)$, where $\Gamma_i = \{z \in E_i : 0 \in T_i(z)\}$ for $i = 1, 2$, and Π_{Γ_1} and Π_{Γ_2} are the Bregman projections onto Γ_1 and Γ_2 respectively.

Proof . We divide the proof into three steps:
STEP 1: We show that the step size (3.2) is well define. Observe that for any $(x, y) \in \Gamma$, we have

$$\langle A^* J_p^{E_3}(Ax_n - By_n), x_n - x\rangle = \langle J_p^{E_3}(Ax_n - By_n), Ax_n - Ax\rangle, \tag{3.3}$$

and

$$\langle B^* J_p^{E_3}(Ax_n - By_n), y - y_n\rangle = \langle J_p^{E_3}(Ax_n - By_n), By - By_n\rangle. \tag{3.4}$$

By adding (3.3) and (3.4) and taking into account the fact $Ax = By$, we have

$$\begin{aligned} ||Ax_n - By_n||^p &= \langle A^* J_p^{E_3}(Ax_n - By_n), x_n - x\rangle + \langle B^* J_p^{E_3}(Ax_n - By_n), y - y_n\rangle \\ &\leq ||A^* J_p^{E_3}(Ax_n - By_n)|| ||Ax_n - x|| + ||B^* J_p^{E_3}(Ax_n - By_n)|| ||y - y_n||. \end{aligned} \tag{3.5}$$

Therefore, for $n \in \Omega$, that is, $||Ax_n - By_n|| > 0$, we have $||A^* J_p^{E_3}(Ax_n - By_n)|| \neq 0$
or $||B^*(Ax_n - By_n)|| \neq 0$. Thus t_n is well defined.

STEP 2: We show that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Now let $(x^*, y^*) \in \Gamma$, then from (3.1), we have that

$$\begin{aligned} D_p(u_n, x^*) &= D_p(R_{\lambda T_1} J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), x^*) \\ &\leq D_p(J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), x^*) \qquad (3.6) \\ &= \frac{1}{q}||J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)||^q - \langle J_p^{E_1}(x_n), x^* \rangle \\ &+ t_n \langle A^* J_p^{E_3}(Ax_n - By_n), x^* \rangle + \frac{1}{p}||x^*||^p \\ &\leq \frac{1}{q}||J_p^{E_1}(x_n)||^q - t_n \langle J_p^{E_3}(Ax_n - By_n), Ax_n \rangle \\ &+ \frac{C_q}{q} t_n^q ||A^* J_p^{E_3}(Ax_n - By_n)||^q - \langle J_p^{E_1}(x_n), x^* \rangle \\ &+ t_n \langle J_p^{E_3}(Ax_n - By_n), Ax^* \rangle + \frac{1}{p}||x^*||^p \\ &= \frac{1}{q}||x_n||^p - \langle J_p^{E_1}(x_n), x^* \rangle + \frac{1}{p}||x^*||^p - t_n \langle J_p^{E_3}(Ax_n - By_n), Ax_n - Ax^* \rangle \\ &+ \frac{C_q}{q} t_n^q ||A^* J_p^{E_3}(Ax_n - By_n)||^q \\ &= D_p(x_n, x^*) - t_n \langle J_p^{E_3}(Ax_n - By_n), Ax_n - Ax^* \rangle \\ &+ \frac{C_q t_n^q}{q} ||A^* J_p^{E_3}(Ax_n - By_n)||^q. \qquad (3.7) \end{aligned}$$

Following similar process as above, we obtain

$$\begin{aligned} D_p(v_n, y^*) &\leq D_p(J_q^{E_2^*}(J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)), y^*) \qquad (3.8) \\ &\leq D_p(y_n, y^*) - t_n \langle J_p^{E_3}(Ax_n - By_n), By^* - By_n \rangle \\ &+ \frac{Q_q t_n^q}{q} ||B^* J_p^{E_3}(Ax_n - By_n)||^q. \qquad (3.9) \end{aligned}$$

Adding (3.7) and (3.9), noting that $Ax^* = By^*$, we have

$$\begin{aligned} D_p(u_n, x^*) + D_p(v_n, y^*) &\leq D_p(x_n, x^*) + D_p(y_n, y^*) \\ &- t_n \Big[||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q}(C_q ||A^* J_p^{E_3}(Ax_n - By_n)||^q \\ &+ Q_q ||B^* J_p^{E_3}(Ax_n - By_n)||^q) \Big]. \qquad (3.10) \end{aligned}$$

Thus

$$D_p(u_n, x^*) + D_p(v_n, y^*) \leq D_p(x_n, x^*) + D_p(y_n, y^*). \qquad (3.11)$$

Also from (3.1), we have

$$\begin{aligned} D_p(x_{n+1}, x^*) &= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1 - \alpha_n) J_p^{E_1}(u_n)), x^*) \\ &\leq \alpha_n D_p(u, x^*) + (1 - \alpha_n) D_p(u_n, x^*). \qquad (3.12) \end{aligned}$$

Similarly, we have

$$D_p(y_{n+1}, y^*) \leq \alpha_n D_p(v, y^*) + (1 - \alpha_n) D_p(v_n, y^*). \qquad (3.13)$$

Hence

$$\begin{aligned} D_p(x_{n+1},x^*)+D_p(y_{n+1},y^*) &\leq \alpha_n(D_p(u,x^*)+D_p(v,y^*)) \\ &\quad +(1-\alpha_n)(D_p(u_n,x^*)+D_p(v_n,y^*)) \\ &\leq \alpha_n(D_p(u,x^*)+D_p(v,y^*)) \\ &\quad +(1-\alpha_n)(D_p(x_n,x^*)+D_p(y_n,y^*)) \\ &\leq \max\{D_p(u,x^*)+(D_p(v,y^*),D_p(x_n,x^*)+D_p(y_n,y^*)\} \\ &\vdots \\ &\leq \max\{D_p(u,x^*)+(D_p(v,y^*),D_p(x_1,x^*)+D_p(y_1,y^*)\}. \end{aligned} \tag{3.14}$$

Thus $\{D_p(x_{n+1},x^*)+D_p(y_{n+1},y^*)\}$ is bounded. Consequently, $\{D_p(x_n,x^*)\}$ and $\{D_p(y_n,y^*)\}$ are bounded. It therefore, follows that $\{x_n\},\{y_n\},\{u_n\}$ and $\{v_n\}$ are bounded.

STEP 3: Next, we prove that $\{x_n\}$ converges strongly to $\bar{x}=\Pi_{\Gamma_1}u$ and $\{y_n\}$ converges strongly to $\bar{y}=\Pi_{\Gamma_2}v$. From (3.1), we have that

$$\begin{aligned} D_p(x_{n+1},x^*) &= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(u)+(1-\alpha_n)(u_n)),x^*) \\ &= V_p(\alpha_n J_p^{E_1}(u)+(1-\alpha_n)(u_n),x^*) \\ &= V_p(\alpha_n J_p^{E_1}(u)+(1-\alpha_n)(u_n)-\alpha_n(J_p^{E_1}(u)-J_p^{E_1}(x^*)),x^*) \\ &\quad +\langle \alpha_n(J_p^{E_1}(u)-J_p^{E_1}(x^*)),J_q^{E_1^*}(\alpha_n J_p^{E_1}(u)+(1-\alpha_n)(u_n))-x^*\rangle \\ &= V_p(\alpha_n J_p^{E_1}(x^*)+(1-\alpha_n)(u_n),x^*)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle \\ &= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(x^*)+(1-\alpha_n)(u_n)),x^*)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle \\ &\leq \alpha_n D_p(x^*,x^*)+(1-\alpha_n)D_p(u_n,x^*)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle \\ &= (1-\alpha_n)D_p(u_n,x^*)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle. \end{aligned} \tag{3.15}$$

Similarly, we have

$$D_p(y_{n+1},y^*)\leq(1-\alpha_n)D_p(v_n,y^*)+\alpha_n\langle J_p^{E_2}(u)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle. \tag{3.16}$$

Therefore, from (3.11) we have

$$\begin{aligned} D_p(x_{n+1},x^*)+D_p(y_{n+1},y^*) &\leq (1-\alpha_n)(D_p(u_n,x^*)+D_p(v_n,y^*)) \\ &\quad +\alpha_n(\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle \\ &\quad +\langle J_p^{E_2}(v)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle) \\ &\leq (1-\alpha_n)(D_p(x_n,x^*)+D_p(y_n,y^*)) \\ &\quad +\alpha_n(\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle \\ &\quad +\langle J_p^{E_2}(v)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle). \end{aligned} \tag{3.17}$$

Now, we set $\Theta_n(x^*,y^*):=D_p(x_n,x^*)+D_p(y_n,y^*)$, and divide the remaining part of the proof into two cases.

Case A: Suppose there exists $n_0\in\mathbb{N}$ such that $\{\Theta_n(x^*,y^*)\}$ is monotonically non-increasing for all $n\geq n_0$. Then $\{\Theta_n(x^*,y^*)\}$ converges as $n\to\infty$ and so

$$\Theta_n(x^*,y^*)-\Theta_{n+1}(x^*,y^*)\to 0, \quad n\to\infty.$$

Let $M_n := C_q||A^*J_p^{E_3}(Ax_n - By_n)||^q + Q_q||B^*J_p^{E_3}(Ax_n - By_n)||^q$, then from (3.10), we have

$$\begin{aligned} t_n\Big[||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q}M_n\Big] &\leq D_p(x_n, x^*) + D_p(y_n, y^*) \\ &\quad -(D_p(u_n, x^*) + (D_p(v_n, y^*)), \end{aligned} \tag{3.18}$$

and therefore,

$$\begin{aligned} t_n\Big[||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q}M_n\Big] &\leq D_p(x_n, x^*) + D_p(y_n, y^*) - (D_p(u_n, x^*) + (D_p(v_n, y^*)) \\ &= \Theta_n(x^*, y^*) - \Theta_{n+1}(x^*, y^*) + \Theta_{n+1}(x^*, y^*) \\ &\quad -(D_p(u_n, x^*) + (D_p(v_n, y^*)). \end{aligned} \tag{3.19}$$

Moreover, it follows from (3.17) and (3.19) and the fact that $\alpha_n \to 0$ as $n \to \infty$ that

$$\begin{aligned} t_n\Big[||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q}M_n\Big] &\leq \Theta_n(x^*, y^*) - \Theta_{n+1}(x^*, y^*) \\ &\quad +(1-\alpha_n)(D_p(u_n, x^*) + D_p(v_n, y^*)) \\ &\quad +\alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle \\ &\quad +\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle) \\ &\quad -(D_p(u_n, x^*) + (D_p(v_n, y^*)) \to 0, \quad n \to \infty. \end{aligned} \tag{3.20}$$

Again, by the condition on the stepsize t_n, we have that

$$t_n^{q-1} < \frac{q||Ax_n - By_n||^p}{M_n} - \epsilon,$$

which implies that

$$t_n^{q-1}M_n < q||Ax_n - By_n||^p - \epsilon M_n,$$

and thus

$$\frac{\epsilon M_n}{q} < ||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q}M_n \to 0, \quad n \to \infty.$$

Therefore,

$$C_q||A^*J_p^{E_3}(Ax_n - By_n)||^q + Q_q||B^*J_p^{E_3}(Ax_n - By_n)||^q \to 0, \quad n \to \infty.$$

It follows that

$$\lim_{n\to\infty} ||A^*J_p^{E^3}(Ax_n - By_n)||^q = 0 \ \text{ and } \ \lim_{n\to\infty} ||B^*J_p^{E^3}(Ax_n - By_n)||^q = 0. \tag{3.21}$$

Also, we have from (3.20) that

$$\begin{aligned} t_n||Ax_n - By_n||^p &\leq \alpha_n(D_p(u, x^*) + D_p(v, y^*)) - (1-\alpha_n)\Theta_n(x^*, y^*) \\ &\quad -\Theta_{n+1}(x^*, y^*) + \frac{t_n^q}{q}M_n \to 0, \quad n \to \infty. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} ||Ax_n - By_n||^p = 0. \tag{3.22}$$

Let $a_n = J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n))$ and $b_n = J_q^{E_2}(J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n))$, then $u_n = R_{\lambda T_1} a_n$ and $v_n = R_{\lambda T_2} b_n$. Following similar argument as in (3.6) (3.7), (3.8),(3.9) and (3.10) we obtain

$$D_p(a_n, x^*) + D_p(b_n, y^*) \leq D_p(x_n, x^*) + D_p(y_n, y^*).$$

It follows from Lemma 2.3 that

$$\begin{aligned} D_p(a_n, u_n) + D_p(b_n, v_n) &= D_p(a_n, R_{\lambda T_1} a_n) + D_p(b_n, R_{\lambda T_2} b_n) \\ &\leq (D_p(a_n, x^*) + D_p(b_n, y^*)) - (D_p(u_n, x^*) + D_p(v_n, y^*)) \\ &\leq (D_p(x_n, x^*) + D_p(y_n, y^*)) - (D_p(u_n, x^*) + D_p(v_n, y^*)) \\ &= (D_p(x_n, x^*) + D_p(y_n, y^*)) - (D_p(x_{n+1}, x^*) + D_p(y_{n+1}, y^*)) \\ &\quad + (D_p(x_{n+1}, x^*) + D_p(y_{n+1}, y^*)) - (D_p(u_n, x^*) + D_p(v_n, y^*)) \\ &\leq (D_p(x_n, x^*) + D_p(y_n, y^*)) - (D_p(x_{n+1}, x^*) + D_p(y_{n+1}, y^*)) \\ &\quad + \alpha_n (D_p(u, x^*) + D_p(v, y^*)) + (1 - \alpha_n)(D_p(u_n, x^*) + D_p(v_n, y^*)) \\ &\quad - (D_p(u_n, x^*) + D_p(v_n, y^*)) \to 0, \ n \to \infty. \end{aligned} \tag{3.23}$$

Hence,

$$\lim_{n\to\infty} D_p(a_n, u_n) = 0, \quad \text{and} \quad \lim_{n\to\infty} D_p(b_n, v_n) = 0.$$

Thus,

$$\lim_{n\to\infty} ||a_n - u_n|| = 0, \quad \text{and} \quad \lim_{n\to\infty} ||b_n - v_n|| = 0. \tag{3.24}$$

Since E_1 and E_2 are uniformly smooth, then $J_p^{E_1}$ and $J_p^{E_2}$ are uniformly continuous on bounded subsets of E_1 and E_2, respectively. Thus

$$\lim_{n\to\infty} ||J_p^{E_1} a_n - J_p^{E_1} u_n|| = 0, \quad \text{and} \quad \lim_{n\to\infty} ||J_p^{E_2} b_n - J_p^{E_2} v_n|| = 0. \tag{3.25}$$

It follows from the definition of a_n that

$$\begin{aligned} 0 &\leq ||J_p^{E_1}(a_n) - J_p^{E_1}(x_n)|| \\ &\leq t_n ||A^*|| ||J_p^{E_3}(Ax_n - By_n)|| \\ &= t_n ||A^*|| ||Ax_n - By_n||^{p-1} \to 0, \ n \to \infty. \end{aligned}$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} ||a_n - x_n|| = 0, \ n \to \infty. \tag{3.26}$$

Similarly, we can show that

$$\lim_{n\to\infty} ||b_n - y_n|| = 0, \ n \to \infty. \tag{3.27}$$

It follows therefore from (3.24) that

$$||u_n - x_n|| \leq ||u_n - a_n|| + ||a_n - x_n|| \to 0, \ n \to \infty, \ \text{and} \ ||v_n - y_n|| \leq ||v_n - b_n|| + ||b_n - y_n|| \to 0, \ n \to \infty. \tag{3.28}$$

Furthermore, from (3.1), we have

$$\begin{aligned} D_p(x_{n+1}, u_n) &\leq \alpha_n D_p(u, u_n) + (1 - \alpha_n) D_p(u_n, u_n) \\ &= \alpha_n D_p(u, u_n) \to 0, \ n \to \infty, \end{aligned}$$

and

$$\begin{aligned} D_p(y_{n+1}, v_n) &\leq \alpha_n D_p(v, v_n) + (1-\alpha_n) D_p(v_n, v_n) \\ &\leq \alpha_n D_p(v, v_n) \to 0, \ n \to \infty. \end{aligned}$$

Hence

$$\lim_{n\to\infty} ||x_{n+1} - u_n|| = \lim_{n\to\infty} ||y_{n+1} - v_n|| = 0. \tag{3.29}$$

This together with (3.28) implies that

$$||x_{n+1}-x_n|| \leq ||x_{n+1}-u_n|| + ||u_n - x_n|| \to 0, \text{ and } ||y_{n+1}-y_n|| \leq ||y_{n+1}-v_n|| + ||v_n - y_n|| \to 0, \ n \to \infty. \tag{3.30}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, there exist subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{y_{n_i}\}$ of $\{y_n\}$ such that $x_{n_i} \rightharpoonup \bar{x} \in \omega(x_n)$ and $y_{n_i} \rightharpoonup \omega(y_n)$ respectively. Now, since $\lim_{n\to\infty} ||u_n - x_n|| = 0$ and $\lim_{n\to\infty} ||v_n - y_n|| = 0$, we obtain $u_{n_i} \rightharpoonup \bar{x}$ and $v_{n_i} \rightharpoonup \bar{y}$. Let $(z, u) \in G(T_1)$, that is $z \in T_1 u$. Since $u_{n_i} = R_{\lambda T_1} a_{n_i}$ for all $\lambda > 0$, we have

$$J_p^{E_1} a_{n_i} \in (J_p^{E_1} + \lambda T_1) u_{n_i},$$

which implies that

$$\frac{1}{\lambda}(J_p^{E_1} a_{n_i} - J_p^{E_1} u_{n_i}) \in T_1 u_{n_i}.$$

By the maximal monotonicity of T_1, we have

$$\langle z - \frac{1}{\lambda}(J_p^{E_1} a_{n_i} - J_p^{E_1} u_{n_i}), u - u_{n_i} \rangle \geq 0,$$

which implies that

$$\langle z, u - u_{n_i} \rangle \geq \frac{1}{\lambda} \langle u - n_{n_i}, J_p^{E_1} a_{n_i} - J_p^{E_1} u_{n_i} \rangle.$$

It follows from (3.25) and the fact that $u_{n_i} \rightharpoonup \bar{x}$ that

$$\langle z, u - \bar{x} \rangle \geq 0.$$

Since T_1 is maximal monotone, we have $0 \in T_1 \bar{x}$.
Following similar analysis as above, we obtain $0 \in T_2 \bar{y}$.
Now, since $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators, we have $Ax_{n_i} \rightharpoonup A\bar{x}$ and $By_{n_i} \rightharpoonup B\bar{y}$. By the weak lower semicontinuity of the norm and (3.22), we have

$$||A\bar{x} - B\bar{y}|| \leq \liminf_{i\to\infty} ||Ax_{n_i} - By_{n_i}|| = 0.$$

Hence, $A\bar{x} = B\bar{y}$.
We now show the sequence $\{(x_n, y_n)\}$ strongly converges to $(x^*, y^*) = (\Pi_{\Gamma_1} u, \Pi_{\Gamma_2} v)$. From (3.17), we have

$$\begin{aligned} D_p(x_{n+1}, x^*) + D(y_{n+1}, y^*) \leq\ & (1-\alpha_n)(D_p(x_n, x^*) + D_p(y_n, y^*)) \\ & +\alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^* \rangle \\ & +\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^* \rangle). \end{aligned} \tag{3.31}$$

Choose subsequences $\{x_{n_j}\}$ of $\{x_n\}$ and $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^* \rangle = \lim_{j\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_j+1} - x^* \rangle,$$

and

$$\limsup_{n\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle = \lim_{j\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n_j+1} - y^*\rangle.$$

Since $x_{n_j} \rightharpoonup \bar{x}$ and $y_{n_j} \rightharpoonup \bar{y}$, it follows from (2.7) that

$$\begin{aligned}\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle &= \lim_{j\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_j+1} - x^*\rangle \\ &= \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), \bar{x} - x^*\rangle \leq 0, \end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle &= \lim_{j\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n_j+1} - y^*\rangle \\ &= \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), \bar{y} - y^*\rangle \leq 0. \end{aligned} \tag{3.33}$$

Using Lemma 2.4 in (3.31), we conclude that

$$D_p(x_n, x^*) + D_p(y_n, y^*) \to 0, \;\; n \to \infty. \tag{3.34}$$

Thus, $D_p(x_n, x^*) \to 0$ and $D_p(y_n, y^*) \to 0$, $n \to \infty$. Therefore $x_n \to x^*$ and $y_n \to y^*$.
Case 2: Assume that $\{\Theta_n(x^*, y^*)\}$ is not monotonically decreasing. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by

$$\tau(n) = \max\{k \in \mathbb{N} : k \leq n, \tau_k \leq \tau_{k+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$0 \leq \Theta_{\tau(n)}(x^*, y^*) \leq \Theta_{\tau(n)+1}(x^*, y^*), \quad \forall n \geq n_0.$$

Following similar analysis as in Case 1, we conclude that $\lim_{n\to\infty}||Ax_{\tau(n)} - By_{\tau(n)}|| = 0$; $\lim_{n\to\infty}||x_{\tau(n)+1} - x_{\tau(n)}|| = 0$ and $\lim_{n\to\infty}||y_{\tau(n)+1} - y_{\tau(n)}|| = 0$. Also we have that

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{\tau(n)+1} - x^*\rangle \leq 0 \;\text{ and }\; \limsup_{n\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{\tau(n)+1} - y^*\rangle \leq 0. \tag{3.35}$$

Now, since $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ are bounded, there exist subsequences of $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ still denoted as $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ which converge weakly to $\bar{x} \in E_1$ and $\bar{y} \in E_2$ respectively. From (3.17), we have

$$\begin{aligned}\Theta_{\tau(n)+1}(x^*, y^*) \;\leq\; & (1 - \alpha_{\tau(n)})\Theta_{\tau(n)}(x^*, y^*) + \alpha_{\tau(n)}(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{\tau(n)+1} - x^*\rangle \\ & + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{\tau(n)+1} - y^*\rangle). \end{aligned} \tag{3.36}$$

Since $\Theta_{\tau(n)}(x^*, y^*) \leq \Theta_{\tau(n)+1}(x^*, y^*)$, it follows from (3.36) that

$$\Theta_{\tau(n)}(x^*, y^*) \leq \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{\tau(n)+1} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{\tau(n)+1} - y^*\rangle.$$

Then from (3.35), we have that

$$\lim_{n\to\infty}\Theta_{\tau(n)}(x^*, y^*) = \lim_{n\to\infty}(D_p(x_{\tau(n)}, x^*) + D_p(y_{\tau(n)}, y^*)) = 0.$$

Hence, $\lim_{n\to\infty}D_p(x_{\tau(n)}, x^*) = 0$ and $\lim_{n\to\infty}D_p(y_{\tau(n)}, y^*) = 0$.
Thus we have $\lim_{n\to\infty}||x_{\tau(n)} - x^*|| = 0$ and $\lim_{n\to\infty}||y_{\tau(n)} - y^*|| = 0$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Theta_n(x^*, y^*) \leq \max\{\Theta_{\tau(n)}(x^*, y^*), \Theta_{\tau(n)+1}(x^*, y^*)\} = \Theta_{\tau(n)+1}(x^*, y^*).$$

Hence, $\lim_{n\to\infty}\Theta_n(x^*,y^*) = \lim_{n\to\infty}(D_p(x_n,x^*)+D_p(y_n,y^*)) = 0.$
Thus,

$$\lim_{n\to\infty} D_p(x_n,x^*) = \lim_{n\to\infty} D_p(y_n,y^*) = 0.$$

Therefore, we have

$$\lim_{n\to\infty}||x_n - x^*|| = 0 \ \text{ and } \ \lim_{n\to\infty}||y_n - y^*|| = 0.$$

This implies that the sequences $\{(x_n,y_n)\}$ strongly converges to $(x^*,y^*) = (\Pi_{\Gamma_1}u, \Pi_{\Gamma_2}v)$.
□

4. Applications and Numerical Example

4.1. Applications

Next, we obtain the following consequences from our main theorem.

4.1.1. Split Equality Feasibility Problem:

Let E be a p-uniformly real Banach space which is also uniformly smooth. Given a proper, convex and lower semicontinuous function $f : E \to \mathbb{R}\cup\{+\infty\}$, the subdifferential of such function is the mapping $\partial f : E \to 2^{E^*}$ defined by

$$\partial f(x) = \{x^* \in E^* : f(x) - f(u) \leq \langle x-u, x^*\rangle, \ \forall\ u \in E\}.$$

Let C be a nonempty closed and convex subset of E and i_C be the indicator function of C, defined by

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{if } x \notin C. \end{cases} \tag{4.1}$$

We define the normal cone $N_C(x)$ to C at a point $x \in C$ as follows:

$$N_C(x) = \{x^* \in E^* : \langle x^*, u - x\rangle \leq 0 \ \forall\ u \in E\}.$$

Then i_C is proper, lower and semicontinuous function on E and the subdifferential ∂_{i_C} of i_C is a maximal monotone operator (see [49]). We also define the resolvent $R_{\lambda\partial_{i_C}}$ of ∂_{i_C} for $\lambda > 0$ as

$$R_{\lambda\partial_{i_C}}x = (J_p^E + \lambda\partial_{i_C})^{-1}J_p^E x,$$

for all $x \in E$. By definitions, we obtain

$$\begin{aligned} \partial_{i_C}x &= \{x^* \in E^* : i_C x + \langle x^*, u-x\rangle \leq i_C u, \ \forall\ u \in E\} \\ &= \{x^* \in E^* : \langle x^*, u-x\rangle \leq 0, \ \forall\ u \in C\} \\ &= N_C x, \end{aligned} \tag{4.2}$$

for all $x \in C$. Hence, for $\lambda > 0$, we have that

$$\begin{aligned} u = R_{\lambda\partial_{i_C}}x &\Leftrightarrow \ J_p^E x \in J_p^E u + \lambda\partial i_C u \Leftrightarrow J_p^E(x-u) \in \lambda N_C u \\ &\Leftrightarrow \ \langle J_p^E(x-u), z-u\rangle \leq 0, \ \forall\ z \in C \\ &\Leftrightarrow \ u = \Pi_C x. \end{aligned} \tag{4.3}$$

Now, let E_1, E_2 and E_3 be p-uniformly convex Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively and let $A : E_1 \to E_3$

and $B : E_2 \to E_3$ be bounded linear operators. The Split Equality Feasibility Problem (SEFP) is defined as

$$\text{find } x^* \in C \text{ and } y^* \in Q \text{ such that } Ax^* = By^*. \tag{4.4}$$

When $E_2 = E_3$ and $B = I$ the identity mapping in (4.4), the SEFP reduces to the Split Feasibility Problem SFP introduced by Censor and Elfving [15]. Setting $T_1 = \partial_{i_C}$ and $T_2 = \partial_{i_Q}$ in Theorem 3.1, then the algorithm (3.1) becomes

$$\begin{cases} u_n = \Pi_C J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), \\ x_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(u_n)), \\ v_n = \Pi_Q J_q^{E_2^*}(J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)), \\ y_{n+1} = J_p^{E_2^*}(\alpha_n J_p^{E_2}(v) + (1-\alpha_n) J_p^{E_2}(v_n)), \end{cases} \tag{4.5}$$

and we obtain a strong convergence for approximation of solution of split equality feasibility problems in Banach spaces.

4.1.2. Split Equality Convex Minimization Problem:

Let E be a p-uniformly convex real Banach space which is also uniformly smooth and C be nonempty closed convex subset of E. Let $\phi : C \to \mathbb{R}$ be a proper convex lower semicontinuous function. We know that the subdifferential $\partial\phi$ is maximal monotone and the resolvent operator $R_{\lambda\partial\phi} = prox_{\lambda\phi}$ where

$$prox_{\lambda\phi} x = \underset{u \in E}{argmin}\{\phi(u) + \frac{1}{2\lambda} D_p(u, x)\},$$

for each $x \in E$ (see [44] for more details).
Let E_1, E_2 and E_3 be p-uniformly convex real Banach spaces which are also uniformly smooth and C and Q be nonempty closed convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. The Split Equality Convex Minimization Problem (SECMP) is define as: find $x^* \in E_1$ and $y^* \in E_2$ such that

$$\begin{cases} x^* = \underset{x \in E_1}{argmin}\ \phi(x) \text{ and } y^* = \underset{y \in E_2}{argmin}\ \psi(y) \\ Ax^* = By^*, \end{cases} \tag{4.6}$$

where $\phi : C \to \mathbb{R}$ and $\psi : Q \to \mathbb{R}$ are proper convex lower semicontinuous functions. Now, by setting $T_1 = \partial\phi$ and $T_2 = \partial\psi$ in Theorem 3.1, then the algorithm (3.1) becomes

$$\begin{cases} u_n = prox_{\lambda\phi} J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), \\ x_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(u_n)), \\ v_n = prox_{\lambda\psi} J_q^{E_2^*}(J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)), \\ y_{n+1} = J_p^{E_2^*}(\alpha_n J_p^{E_2}(v) + (1-\alpha_n) J_p^{E_2}(v_n)), \end{cases} \tag{4.7}$$

and we can obtain a strong convergence result for approximating solutions of SECMP in Banach spaces.

4.1.3. Split Equality Equilibrium Problem:

Let E be a p-uniformly convex Banach space which is also uniformly smooth and C be a nonempty closed convex subset of E. Let $F : C \times C \to \mathbb{R}$ be a bifunction, the equilibrium problem introduced by Blum and Oettli [11] in 1994 is defined as: find $x \in C$ such that

$$F(x, y) \geq 0, \ \forall\ y \in C.$$

We denote the set of solution of the equilibrium problem as $EP(F)$. For solving the equilibrium problem, it is assumed that the bifunction F satisfied the following conditions:
(A1) $F(x, x) = 0$ for all $x \in C$,
(A2) F is monotone, that is $F(x, y) + F(y, x) \leq 0$, forall $x, y \in C$,
(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$,
(A4) for all $x \in C$, $F(x, \dot{)}$ is convex and lower semicontinuous. The resolvent operator T_r^F associated with the bifunction F for $r > 0$ is defined as

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, J_p^E z - J_p^E x\rangle \geq 0,\ \forall\ y \in C\}.$$

It is well known that T_r^F satisfy following properties:
(a) T_r^F is single-valued,
(b) T_r^F is a firmly nonexpansive mapping, that is,

$$\langle T_r^F z - T_r^F y, J_p^E T_r^F z - J_P^E T_r^F y\rangle \leq \langle T_r^F z - T_r^F y, J_p^E z - J_p^E y\rangle\ \forall z, y \in E,$$

(c)$F(T_r^F) = EP(F)$,
(d) $EP(F)$ is closed and convex.
Now, define a multi-valued mapping $A_F : E \to 2^{E^*}$ by

$$A_F(x) := \begin{cases} \{z \in E^* : F(x, y) \geq \frac{1}{r}\langle J_p^E y - J_p^E x, z\rangle,\ \forall\ y \in C\}; & x \in C, \\ \emptyset; \quad x \notin C, \end{cases} \tag{4.8}$$

then, we know that $EP(F) = A_F^{-1}0$ and A_F is a maximal monotone operator with dom$(A_F) \subset C$ (see [56]). Further, for any $x \in E$ and $r > 0$, the resolvent T_r^F of F coincides with the resolvent of A_F, that is

$$T_r^F x = (J_p^E + rA_F)^{-1} J_p^E x.$$

Let E_1, E_2 and E_3 be p-uniformly convex real Banach spaces which are also uniformly smooth and C and Q be nonempty closed convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bifunctions. The Split Equality Equilibrium Problem (SEEP) is defined as: Find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{cases} F(x^*, x) \geq 0\ \ \forall\ x \in C, \quad G(y^*, y) \geq 0\ \ \forall\ y \in Q \\ \text{and } Ax^* = By^*. \end{cases} \tag{4.9}$$

Setting $R_{\lambda T_1} = T_r^F$ and $R_{\lambda T_2} = T_r^G$ in Theorem 3.1, then the algorithm (3.1) becomes

$$\begin{cases} u_n = T_{r_n}^F J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), \\ x_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(u_n)), \\ v_n = T_{r_n}^G J_q^{E_2^*}(J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)), \\ y_{n+1} = J_p^{E_2^*}(\alpha_n J_p^{E_2}(v) + (1-\alpha_n) J_p^{E_2}(v_n)), \end{cases} \tag{4.10}$$

for $r_n > 0$, and we obtain a strong convergence result for approximation of solution of the SEEP in Banach spaces.

4.1.4. Saddle Points Problem

Let X and Y be two Hilbert spaces and $E = X \times Y$. Let $L : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function such that $L(x, y)$ is convex in $x \in X$ and concave in $y \in Y$, (convex-concave function). To such a function, Rockafellar [48] associated the operator T_L defined by

$$T_L = \partial_1 L \times \partial_2(-L),$$

where ∂_1 (resp. ∂_2) stands for the subdifferential of L with respect to the first (resp. the second) variable. T_L is a maximal monotone operator if and only if L is closed and convex in Rockafellar sense (see [48]).
Moreover, it is well known that (x^*, y^*) is a saddle point of L, namely:

$$L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*), \ \forall \ (x, y) \in E,$$

if and only if the following inclusion holds

$$(0, 0) \in T_L(x^*, y^*).$$

The proximal operator associated with T_L is define by

$$prox_{\lambda L}(x, y) = arg \underset{(u,v)}{minmax}\{L(u, v) + \frac{1}{2\lambda}||x - u||^2 - \frac{1}{2\lambda}||y - v||^2\},$$

for all $(x, y) \in E$. Now, if in problem (1.6), we set $E_1 = X_1 \times Y_1$, $E_2 = X_2 \times Y_2$, $E_3 = X_3 \times Y_3$ and $T_1 = T_{L_1}$, $T_2 = T_{L_2}$, where L_i $(i = 1, 2)$ are convex-concave functions on E_i for $i = 1, 2$, respectively. Then, we have the following split equality saddle point problem: find $(x_1^*, y_1^*) \in E_1$ and $(x_2^*, y_2^*) \in E_2$ such that

$$\begin{cases} (x_1^*, y_1^*) = arg \underset{(x_1,y_1)}{minmax} \ L_1(x_1, y_1) \\ (x_2^*, y_2^*) = arg \underset{(x_2,y_2)}{minmax} \ L_2(x_2, y_2) \\ \text{and } A(x_1^*, y_1^*) = B(x_2^*, y_2^*), \end{cases} \tag{4.11}$$

where $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators. Then we can obtain the following strong convergence result from Theorem 3.1.

Theorem 4.1. *Let X_i and Y_i be real Hilbert spaces for $i = 1, 2, 3$. Let $E_1 = X_1 \times Y_1$, $E_2 = X_2 \times Y_2$, $E_3 = X_3 \times Y_3$. Let C and Q be nonempty closed convex subset of E_1 and E_2 respectively, $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. Let $L_i : E_i \to \mathbb{R} \cup \{-\infty, +\infty\}$ be convex-concave functions, for $i = 1, 2, 3$. and $\Gamma := \{\bar{x} = (x_1, x_2) \in T_{L_1}^{-1}(0, 0), \ \bar{y} = (y_1, y_2) \in T_{L_2}^{-1}(0, 0) \ ; A\bar{x} = B\bar{y}\}$ is nonempty. For fixed $\bar{u} = (u_1, u_2) \in E_1$ and $\bar{v} = (v_1, v_2) \in E_2$, choose an initial guess $\bar{x}_1 \in E_1$ and $\bar{y}_1 \in E_2$ arbitrarily. Let $\{\alpha_n\} \subset [0, 1]$. Assume that the nth iterate $\bar{x}_n = (x_{n,1}, x_{n,2}) \in E_1$ and $\bar{y}_n = (y_{n,1}, y_{n,2}) \in E_2$ have been constructed; then we calculate the $(n + 1)$th iterate $(\bar{x}_{n+1}, \bar{y}_{n+1})$ via the formula*

$$\begin{cases} \bar{u}_n = prox_{\lambda L_1}(\bar{x}_n) - t_n A^*(A\bar{x}_n - B\bar{y}_n), \\ \bar{x}_{n+1} = \alpha_n(\bar{u}) + (1 - \alpha_n)\bar{u}_n, \\ \bar{v}_n = prox_{\lambda L_2}(\bar{y}_n) + t_n B^*(A\bar{x}_n - B\bar{y}_n), \\ \bar{y}_{n+1} = \alpha_n(\bar{v}) + (1 - \alpha_n)\bar{v}_n, \end{cases} \tag{4.12}$$

where $\lambda > 0$, A^* and B^* are the adjoints of A and B respectively and the stepsize t_n is choosen in such a way that

$$t_n \in \Big(\epsilon, \frac{2||A\bar{x}_n - B\bar{y}_n||^2}{||A^*(A\bar{x}_n - B\bar{y}_n)||^2 + ||B^*(A\bar{x}_n - B\bar{y}_n)||^2} - \epsilon\Big),\ n \in \Omega,$$

for small enough ϵ, otherwise $t_n = t$ (t being any nonegative value), where the set of indices $\Omega = \{n : A\bar{x}_n - B\bar{y}_n \neq 0\}$. Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequences $\{(\bar{x}_n, \bar{y}_n)\}$ strongly converges to $(\bar{x}, \bar{y}) = (P_{\Gamma_1}\bar{u}, P_{\Gamma_2}\bar{v})$, where $\Gamma_i = \{\bar{z} \in E_i : 0 \in T_{L_i}(\bar{z})\}$ for $(i = 1, 2)$, P_{Γ_1} and P_{Γ_2} are the metric projections onto Γ_1 and Γ_2 respectively.

4.2. Numerical Example

For simplicity, we take $E_1 = E_2 = E_3 = \mathbb{R}$, with $p = 2$. Let $A(x) = x,\ B(x) = 2x,\ T_1(x) = 2x$ and $T_2(x) = 3x$. Choose $\lambda = 2$ and $\alpha_n = \frac{1}{\sqrt{n}}$, then algorithm (3.1) becomes

$$\begin{cases} x_{n+1} = \frac{1}{\sqrt{n}}u + (\frac{\sqrt{n}-1}{5\sqrt{n}})(x_n - t_n(x_n - 2y_n)) \\ y_{n+1} = \frac{1}{\sqrt{n}}v + (\frac{\sqrt{n}-1}{7\sqrt{n}})(y_n + 2t_n(x_n - 2y_n)), \end{cases} \tag{4.13}$$

where the step size t_n is chosen in such a way that

$$t_n \in \Big(\epsilon, \frac{2||Ax_n - By_n||^2}{||A^T(Ax_n - By_n)||^2 + ||B^T(Ax_n - By_n)||^2} - \epsilon\Big),\ n \in \Omega,$$

for small enough ϵ, otherwise $t_n = t$ (t being any nonegative value), where the set of indices $\Omega = \{n : Ax_n - By_n \neq 0\}$.
We make different choice of u, v, x_1, and y_1 and use $\epsilon < 10^{-2}$, for the stopping criterion.
Case 1:
(i) Take $x_1 = 1, y_1 = -1, u = 0.5$ and $v = 1$.
(ii) Take $x_1 = 0.25, y_1 = 0.005, u = -0.0675$ and $v = 0.001$.
Case 2:
(i) Take $x_1 = -0.02, y_1 = -0.005, u = 0.1$ and $v = 1$.
(ii) Take $x_1 = -0.0005, y_1 = -0.12, u = 1$ and $v = 0.001$.
We note that the choice of t_n, as long as it is in the range, does not have any significant effect on both the number of iterations and cpu time. Mathlab version R2014a is used to obtain the graphs of errors against number of iterations, execution time against accuracy and number of iterations against accuracy.

Declaration
The authors declare that they have no competing interests.
Acknowledgement: The second author acknowledges with thanks the bursary and financial support from Department of Science and Innovation and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

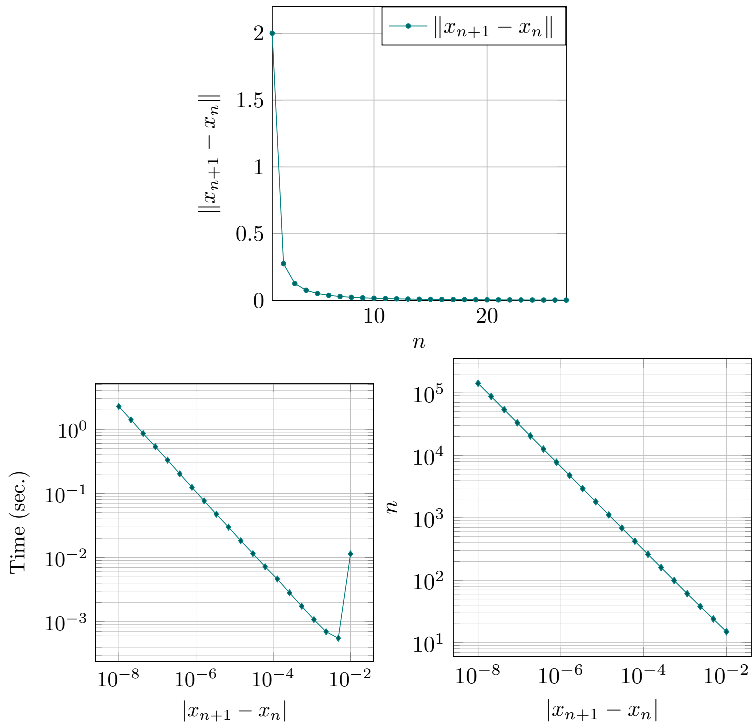

Figure 1: Case 1(i): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

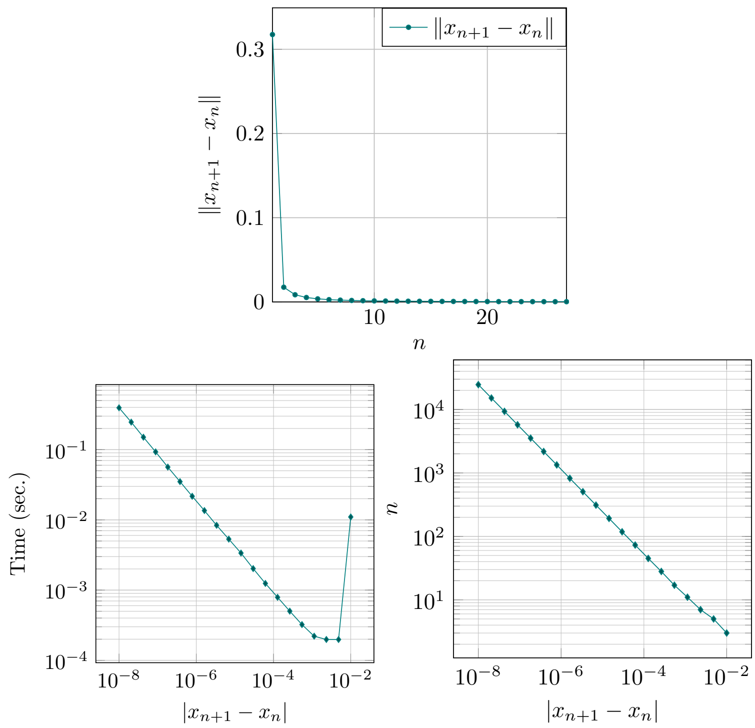

Figure 2: Case 1(ii): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

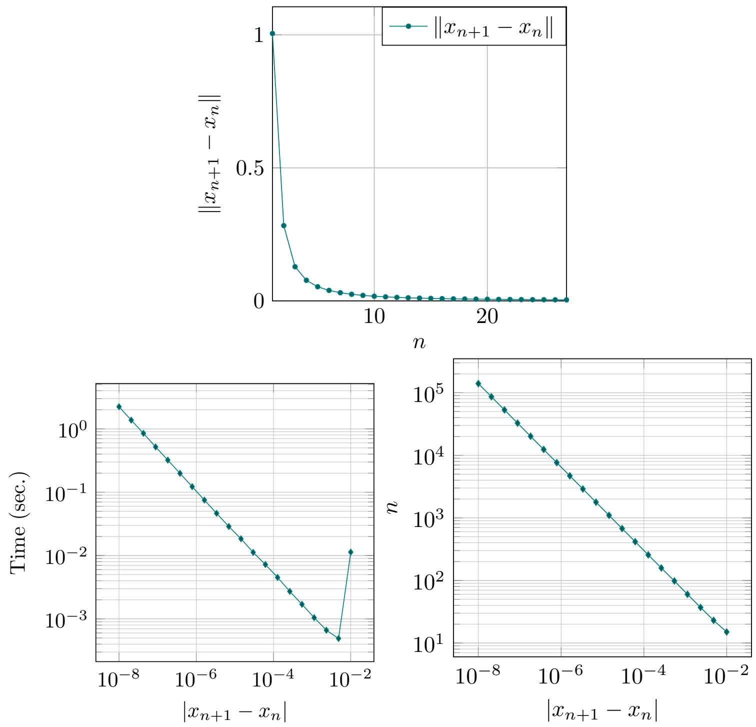

Figure 3: Case 2(i): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

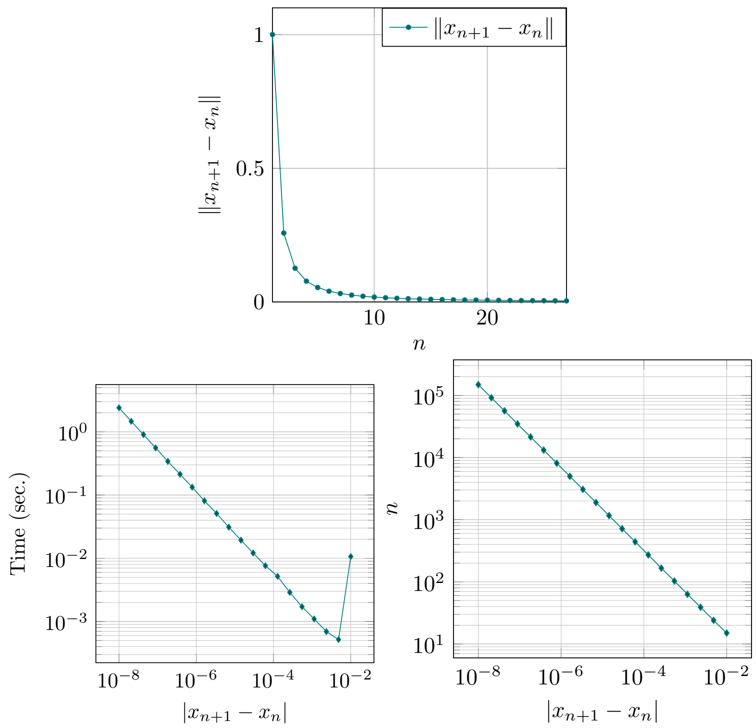

Figure 4: Case 2(ii): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

References

[1] T.O. Alakoya, L.O. Jolaoso, O.T. Mewomo, *A self adaptive inertial algorithm for solving split variational inclusion and fixed point problems with applications*, J. Ind. Manag. Optim., (2020), doi:10.3934/jimo.2020152.

[2] T.O. Alakoya, L.O. Jolaoso, O.T. Mewomo, *Two modifications of the inertial Tseng extragradient method with self-adaptive step size for solving monotone variational inequality problems*, Demonstr. Math., 53, (2020),208-–224.

[3] T.O. Alakoya, L.O. Jolaoso, O.T. Mewomo, *Modified inertia subgradient extragradient method with self adaptive stepsize for solving monotone variational inequality and fixed point problems*, Optimization, (2020), DOI:10.1080/02331934.2020.1723586.

[4] T.O. Alakoya, L.O. Jolaoso, O.T. Mewomo, *Strong convergence theorems for finite families of pseudomonotone equilibrium and fixed point problems in Banach spaces,* Afr. Mat., (2021), DOI:10.1007/s13370-020-00869-z.

[5] T.O. Alakoya, A. Taiwo, O.T. Mewomo, Y.J. Cho, *An iterative algorithm for solving variational inequality, generalized mixed equilibrium, convex minimization and zeros problems for a class of nonexpansive-type mappings*, Ann. Univ. Ferrara Sez. VII Sci. Mat. (2020) DOI: 10.1007/s11565-020-00354-2.

[6] T.O. Alakoya, A. Taiwo, O.T. Mewomo, *On system of split generalised mixed equilibrium and fixed point problems for multivalued mappings with no prior knowledge of operator norm*, Fixed Point Theory, (2020), (accepted, to appear).

[7] Y.I. Alber, *Metric and generalized projection operators in Banach spaces: Properties and applications in Theory and Applications of Nonlinear Operators of Accretive and Monotone type*, Lecture Notes in Pure and Applied Mathematics, Dekker, New York, NY,USA, Vol. 178,(1996), 15-50,

[8] K. O. Aremu, H. A. Abass, C. Izuchukwu and O. T. Mewomo, *A viscosity-type algorithm for an infinitely countable family of* (f, g)*-generalized k-strictly pseudononspreading mappings in CAT(0) spaces*, Analysis, 40 (1), (2020), 19–37.

[9] K.O. Aremu, C. Izuchukwu, G.N. Ogwo, O.T. Mewomo, *Multi-step Iterative algorithm for minimization and fixed point problems in p-uniformly convex metric spaces*, J. Ind. Manag. Optim., (2020), DOI:10.3934/jimo.2020063.

[10] V. Barbu, T. Precupanu, *Convexity and Optimization in Banach spaces*, Spinger Monographs in Mathematics, (2012), DOI:10.1007/978-94-007-2247-7-2.

[11] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Stud., 63, (1994), 123–145.

[12] J.Y. Bello Cruz and Y. Shehu, *An iterative method for split inclusion problems without prior knowledge of operator norms*, J. Fixed Point Theory Appl., 19 (3), (2017), 2017-–2036.

[13] C. Bryne, Y. Censor, A. Gibali, S. Reich, *Weak and strong convergence of algorithms for the split common null point problem*, J. Nonlinear Convex Anal., 13 (2012), 759–775.

[14] Y. Censor, A. Lent, *An iterative row-action method for interval convex programming*, J. Optim. Theory Appl., 34, (1981), 321–353.

[15] Y. Censor, T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms, 8, (1994), 221–239.

[16] Y. Censor, A. Gibali and S. Reich, *Algorithms for the split variational inequality problem*, Numer. Algorithms, 59 (2), (2012), 301–323.

[17] H. Dehghan, C. Izuchukwu, O.T. Mewomo, D.A. Taba, G.C. Ugwunnadi, *Iterative algorithm for a family of monotone inclusion problems in CAT(0) spaces*, Quaest. Math., 43 (7), (2020), 975-–998.

[18] A. Gibali, L.O. Jolaoso, O.T. Mewomo, A. Taiwo, *Fast and simple Bregman projection methods for solving variational inequalities and related problems in Banach spaces*, *Results Math.*, 75, (2020), Art. No. 179, 36 pp.

[19] E.C. Godwin, C. Izuchukwu, O.T. Mewomo, *An inertial extrapolation method for solving generalized split feasibility problems in real Hilbert spaces*, Boll. Unione Mat. Ital., (2020), DOI 10.1007/s40574-020-00.

[20] K. Goebel, S. Reich, *Uniformly convexity, Hyperbolic Geometry and Nonexpansive mappings*, Marcel Dekker, New York, NY, USA, (1984).

[21] H. Guo, H. He and R. Chen, *Convergence theorems for the split equality variational inclusion problem and fixed point problem in Hilbert spaces*, Fixed Point Theory and Appl., 2015 (2015), Art. ID 223.

[22] J.M. Hendrickx, A. Olshevsky, *Matrix P-Norms are NP-Hard to approximate if* $p = 1, 2, \infty$, SIAM J. Matrix Anal. Appl., 31 (2012), 2802–2812.

[23] C. Izuchukwu, A.A. Mebawondu, O.T. Mewomo, *A New Method for Solving Split Variational Inequality Problems without Co-coerciveness*, J. Fixed Point Theory Appl., (2020), 22 (4), (2020), Art. No. 98, 23 pp.

[24] C. Izuchukwu, G.N. Ogwo, O.T. Mewomo, *An Inertial Method for solving Generalized Split Feasibility Problems over the solution set of Monotone Variational Inclusions*, Optimization, (2020), DOI 10.1080/02331934.2020.1808648.

[25] C. Izuchukwu, C. C. Okeke, O. T. Mewomo, *Systems of Variational Inequalities and multiple-set split equality fixed point problems for countable families of multivalued type-one demicontractive-type mappings*, Ukraïn. Mat. Zh., 71 (11), (2019), 1480–1501.

[26] L.O. Jolaoso, T.O. Alakoya, A. Taiwo and O.T. Mewomo, *Inertial extragradient method via viscosity approximation approach for solving Equilibrium problem in Hilbert space*, Optimization, (2020), DOI:10.1080/02331934.2020.1716752.

[27] L.O. Jolaoso, K.O. Oyewole, K.O. Aremu, O.T. Mewomo, *A new efficient algorithm for finding common fixed points of multivalued demicontractive mappings and solutions of split generalized equilibrium problems in Hilbert spaces*, Int. J. Comput. Math., (2020), https://doi.org/10.1080/00207160.2020.1856823.

[28] L.O. Jolaoso, A. Taiwo, T.O. Alakoya, O.T. Mewomo, *A unified algorithm for solving variational inequality and fixed point problems with application to the split equality problem*, Comput. Appl. Math., 39 (1), (2020), Art. No. 38, 28 pp.

[29] L.O. Jolaoso, A. Taiwo, T.O. Alakoya, O.T. Mewomo, *Strong convergence theorem for solving pseudo-monotone variational inequality problem using projection method in a reflexive Banach space*, J. Optim. Theory Appl., 185 (3), (2020), 744–766.

[30] K.R. Kazmi, S.H. Rizvi, *An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping*, Optim. Lett., 8 (3) (2014), 1113-–1124.

[31] S.H. Khan, T.O. Alakoya, O.T. Mewomo, *Relaxed projection methods with self-adaptive step size for solving variational inequality and fixed point problems for an infinite family of multivalued relatively nonexpansive mappings in Banach spaces,* Math. Comput. Appl., 25, (2020), Art. 54.

[32] B. Martinet, *Regularisation d'inequation variationelles par approximations successives*, Rev. Francaise inf. Rech. Oper., (1970), pp. 154-159.

[33] A. Moudafi, *A relaxed alternating CQ algorithm for the split feasibility problems*, Nonlinear Anal., 79, (2013), 117–121.

[34] A. Moudafi, *Split Monotone variational inclusion problem*, J. Optim. Theory Appl., 150, (2011), 275–283.

[35] A. Moudafi, E. Al-Shemas, *Simultaneous iterative methods for split equality problems and applications*, Trans. Math. Prog. Appl., 1, (2013), 1–11.

[36] G. N. Ogwo, C. Izuchukwu, K.O. Aremu, O.T. Mewomo, *A viscosity iterative algorithm for a family of monotone inclusion problems in an Hadamard space*, Bull. Belg. Math. Soc. Simon Stevin, 27, (2020), 127–152.

[37] G. N. Ogwo, C. Izuchukwu, K.O. Aremu, O.T. Mewomo, *On θ-generalized demimetric mappings and monotone operators in Hadamard spaces*, Demonstr. Math., 53 (1), (2020), 95-–111.

[38] G. N. Ogwo, C. Izuchukwu, O.T. Mewomo, *Inertial methods for finding minimum-norm solutions of the split variational inequality problem beyond monotonicity*, Numer. Algorithms, (2021), (accepted, to appear).

[39] A. O.-E. Owolabi, T.O. Alakoya, A. Taiwo, O.T. Mewomo, *A new inertial-projection algorithm for approximating common solution of variational inequality and fixed point problems of multivalued mappings*, Numer. Algebra Control Optim., (2021) DOI:10.3934/naco.2021004.

[40] O.K. Oyewole, H.A. Abass, O.T. Mewomo, *A Strong convergence algorithm for a fixed point constrainted split null point problem*, Rend. Circ. Mat. Palermo II, (2020), DOI:10.1007/s12215-020-00505-6.

[41] K.O. Oyewole, C. Izuchukwu, C.C. Okeke, O.T. Mewomo, *Inertial approximation method for split variational inclusion problem in Banach spaces,* Int. J. Nonlinear Anal. Appl., 11 (2), (2020), 285–304.

[42] K.O. Oyewole, O.T. Mewomo, L.O. Jolaoso and Safeer H. Khan, *An extragradient algorithm for split generalized equilibrium problem over the set of fixed points of quasi-phi-nonexpansive mappings in Banach spaces*, Turkish J. Math., 44 (4), (2020), 1146–1170.

[43] S. Reich and S. Sabach, *Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces*, Nonlinear Anal., 73 (1) (2010) 122–135.

[44] S. Reich, S. Sabach, *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numer. Funct. Anal. Optim., 31 (1-3) (2010), 22–44.

[45] F. Schopfer, T. Schuster, A.K. Louis, *An iterative regularization method for the solution of the split feasibility problem in Banach spaces*, Inverse Problems, 24 (5) (2008), Art. I, 055008, 20 pp.

[46] Y. Shehu, O.S. Iyiola, C.D. Enyi, *Iterative algorithm for split feasibility problems and fixed point problems in Banach spaces*, Numer. Algorithms, 72, (2016), 835–854.

[47] Y. Shehu, F.U. Ogbuisi and O.S. Iyiola, *Convergence analysis of an iterative algorithm for fixed point problems and split feasibility problems in certain Banach spaces*, Optimization, 65 (2), (2016), 299–323.

[48] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control and Optimization, 14 (5), (1976), 877-–898.

[49] R.T. Rockafellar, *On the maximal monotonicity of subdifferentials mappings*, Pac. J. Math., 33, (1970), 209-216.

[50] A. Taiwo, T.O. Alakoya, O.T. Mewomo, *Halpern-type iterative process for solving split common fixed point and monotone variational inclusion problem between Banach spaces*, Numer. Algorithms, (2020), DOI: 10.1007/s11075-020-00937-2.

[51] A. Taiwo, T.O. Alakoya, O.T. Mewomo, *Strong convergence theorem for fixed points of relatively nonexpansive multi-valued mappings and equilibrium problems in Banach spaces*, Asian-Eur. J. Math., (2020), https://doi.org/10.1142/S1793557121501370.

[52] A. Taiwo, L.O. Jolaoso and O.T. Mewomo, *Inertial-type algorithm for solving split common fixed-point problem in Banach spaces*, J. Sci. Comput. (2020) DOI: 10.1007/s10915-020-01385-9.

[53] A. Taiwo, L.O. Jolaoso, O.T. Mewomo, *Viscosity approximation method for solving the multiple-set split equality common fixed-point problems for quasi-pseudocontractive mappings in Hilbert Spaces*, J. Ind. Manag. Optim., (2020), DOI:10.3934/jimo.2020092.

[54] A. Taiwo, L.O. Jolaoso, O.T. Mewomo, A. Gibali, *On generalized mixed equilibrium problem with α-β-μ bifunction and μ-τ monotone mapping*, J. Nonlinear Convex Anal., 21 (6), (2020), 1381-1401.

[55] A. Taiwo, A. O.-E. Owolabi, L.O. Jolaoso, O.T. Mewomo, A. Gibali, *A new approximation scheme for solving various split inverse problems*, Afr. Mat., (2020), DOI:https://doi.org/10.1007/s13370-020-00832-y.

[56] S. Takahashi, W. Takahashi, M. Toyoda, *Strong convergence theorem for maximal monotone operators with nonlinear mappings in Hilbert spaces*, J. Optim. Theory Appl., 147, (2010), 27–41.

[57] H.K. Xu, *Inequality in Banach spaces with applications*, Nonlinear Anal., 16, (1991), 1127-1138.

[58] J. Zhao, Solving split equality fixed point problem of quasi-nonexpanive mappings without prior knowledge of operator norms, Optimization, 64, (12) 2015), 2619–2630.

[59] J. Zhao, Q. Yang, *A simple feasibility problem for solving the multiple set split feasibility problem*, Inverse Prob. Sci. Eng., 21, (2013), 537–546.