Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 2013-2020
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5338

Comparing three estimators of fuzzy reliability for One scale parameter rayleigh distribution

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(Communicated by Madjid Eshaghi Gordji)

Abstract

This paper deals with comparing three different estimators of fuzzy reliability estimator of one scale parameter Rayleigh distribution were first of all the one scale parameter Rayleigh is defined. Afterwards, the cumulative distribution function is derived, as well as the reliability function is also found. The parameters θ is estimated by three different methods, which are maximum likelihood, and moments, as well as the third method of estimation which is called percentile method or Least Square method, where the estimator $(\hat{\vartheta}_{pec})$ obtained from Minimizing the total sum of the square between given C DF, and one non-parametric estimator like $\hat{F}(ti,\theta) = \frac{i}{n+1}$ after the estimator of (θ), which $(\hat{\theta})$ is obtained. We work on comparing different fuzzy reliability estimators and all the results are explained besides different sets of taking four sample sizes $(n = 20, 40, 60, and 80)$.

Keywords: Fuzzy Reliability Estimator (FRE), Least Square Estimators (LSE), Maximum Likelihood Estimator (MLE), Moments estimator (MOM), Rayleigh Distribution.
2010 MSC: Please write mathematics subject classification of your paper here.

1. Introduction

Rayleigh distribution is one of the continuous probability distribution given a positive value for random variables. In [3], the authors discuss the reliability allocation optimization in fuzzy environment to find optimal allocation for optimization problem. In [5], the authors attempted to estimate

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Received: March 2021 *Accepted:* July 2021

the parameters of a Rayleigh distribution numerical iterative methods or routines are frequently employed. In this work, an exact method on the constant minimization of the goal function is proposed. A new family of slash distributions, the modified Slashed-Rayleigh distribution, is proposed and studied. This family is an extension of the ordinary Rayleigh distribution, being more flexible in terms of distributional kurtosis which reviewed its most important aspects [2]. Some important applications of this distribution were made by installing some parameters in a field that allows for the creation of powerful estimators that allow meaningful results such as application of three-parameter Rayleigh distribution model in estimating the probability distribution of linear and non-linear responses of turret moored is studied [1]. Some studies have been undertaken to extend an important aspect of this variation. A new two-parameter Rayleigh distribution motivated by Aliya known as Alpha Power Rayleigh distribution [4].
There are some works found in the literature on transmuted Rayleigh distribution and its variants such as the slashed Rayleigh distribution [8]. In [7], the authors discuss fuzzy, Exponential distribution and discuss how to compute reliability in case of stress- strength model, and Ranked set sampling. The importance of such a distribution is due to its applications in testing lifetime of an object whose lifetime depends on its age, and because of the fact that the life of the model theory reliability plays an important role in modeling the life of the random phenomenon [9]. The discusses an extended form of Rayleigh model distributions and some important cases of these distributions. A new four-parameters extended Rayleigh model is proposed, which generalizes the model of Rayleigh distributions. The applicability of the new models is well justified by means of two real data sets [6].

2. Theoretical Aspect

definition of distribution:
The PDF of one scale parameter Rayleigh failure model is defined in equation (1) as:

$$g_T(t,\theta) = \frac{2}{\theta} t e^{-\frac{t^2}{\theta}} \qquad t>0, \theta>0 \tag{2.1}$$

While equation (2) represent the cumulative distribution function (C.D.F) is given in:

$$G_T(t) = 1 - e^{-\frac{t^2}{\theta}} \tag{2.2}$$

For the class of distribution in equation (1), Reliability Function.
$R(t) = 1 - F(t)$ represent the probability of failure- free operation until time (t), or survival until time(t), then the stochastic behavior of the failure time can be studied through either one of the four function f(t), F(t), R(t), μ(t)
$R(t) = 1 - F(t)$

3. Reliability Estimation

The term (reliable) used in various contexts in everyday life, such as reliable service station. In the research of life testing we may concern with a quantitative measure of the reliability of an item or advice which we are interested in.
The reliability of a unit or system is defined to be the probability it will perform satisfactory, at least for a specified period of time without major break down. let x be $r.v$ represent life time of the unit, then the reliability of a unit at time(t) is denoted by:

$$R(t) = pr(x \geq t) = 1 - F(t) = 1 - \int_0^t f(u)du \tag{3.1}$$

$R(T) = 1 - F(t) = 1 - \int_T^\infty f_T(t)dt$
And $f(t) = -\frac{R(t)}{dt}$
The probability of failure in a given time interval between (t_1, t_2),can be represented interms of reliability as:
$R_1(t) - R_2(t_2) = \int_{t_1}^\infty f(t)dt - \int_{t_2}^\infty f(t)dt$
Also the rate of failure occurs in the interval (t_1, t_2) is $\lambda(t)$ define by:

$$\begin{aligned}\lambda(t) &= \frac{R(t) - R(t+\Delta t)}{\Delta t R(t)}\\ &= \frac{R(t_1) - R(t_2)}{(t_2 - t_1)R(t_1)}\end{aligned} \tag{3.2}$$

The fuzzy reliability is defined as the probability of advice or tool performing its work at different disparity degrees of success for a period of time, under operating conditions encountered; it is denoted by $(\tilde{R})$ which is a function of fuzzy setes $(\tilde{A}_i)$,
Let $\mu_{\tilde{A}_i}(R)$ be the degree of membership(R function) in $(\tilde{A}_i)$
Then:
$\tilde{R} = \mu_{\tilde{A}_i}(R).R$
And since $R = \int_t^\infty f(t)dt$, then

$$\tilde{R} = \mu_{\tilde{A}}(R) \int_t^\infty f(t)dt \tag{3.3}$$

4. Fuzzy Probability Function and its Reliability Function

The studied PDF is one scale parameter Rayleigh defined in equation (1):

$$g_T(t) = \frac{2}{\theta} t e^{-\frac{t^2}{\theta}} \qquad t > 0 \ \theta > 0 \tag{4.1}$$

And the cumulative distribution function (C.D.F) is:

$$G_T(t) = \left(1 - e^{-\frac{t^2}{\theta}}\right) \tag{4.2}$$

While the reliability function is:

$$R_T(t) = 1 - F_T(t) = e^{-\frac{t^2}{\theta}} \tag{4.3}$$

The fuzzy values of $r.v.T$ is $\tilde{T}$
Were $\tilde{t} = \{[0, \infty), \mu_{\hat{t}}\}$
Where $\tilde{t} = \tilde{k}t \qquad t \in T$
Then the vagueness of triangular fuzzy number
$\hat{k} = \{[0, \infty), \mu_{\hat{k}}\}$

$$\begin{cases} \mu_{\hat{k}}(k) = \frac{k - k\min}{1 - k_{\min}}, & k \in (k_{\min}, 1) \\ \frac{k - k\max}{1 - k_{\max}}, & k \in (1, k_{\max}) \end{cases} \tag{4.4}$$

where $0 < k_{\min} < k_{\max}$
The membership $\mu_{\tilde{k}}(k)$ can be represented by:

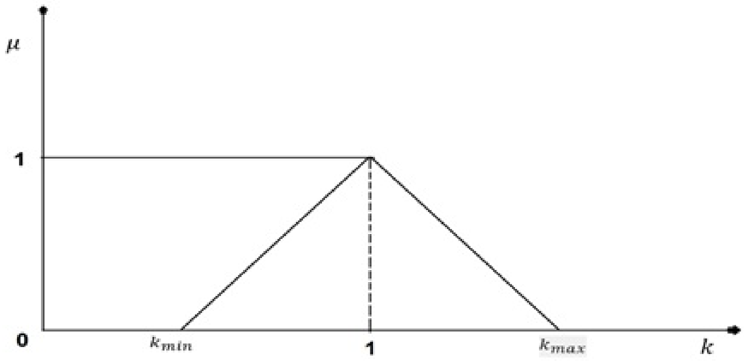

Figure 1: fuzzy reliability reference [10]

The fuzzy random variable $\tilde{T}$,have fuzzy one scale parameter Rayleigh, with fuzzy C.D.F: $\tilde{G}_T(t) = \left(1 - e^{-\frac{\tilde{k}}{\theta}t^2}\right)$, so the α-cut fuzzy C.D.F $\alpha \in [0,1]$

$$\begin{aligned}\tilde{G}_T(t) &= [G_{1\alpha}(t), G_{2\alpha}(t)] \\ &= \{\left(1 - e^{-\frac{1}{\theta}(1-k_{\max})\alpha + k_{\max}\frac{t^2}{\theta}}\right) \\ &= \left(1 - e^{-\frac{1}{\theta}(1-k_{\min})\alpha + k_{\min}\frac{t^2}{\theta}}\right)\}\end{aligned} \tag{4.5}$$

And fuzzy reliability is

$$\tilde{R}(t) = e^{-\frac{\tilde{k}}{\theta}t^2} \tag{4.6}$$

Were $\forall\alpha \in [0,1]$ the α −cuts of fuzzy reliability function is written as $R_\alpha(t) = [R_{1\alpha}(t), R_{2\alpha}(t)]$

5. Estimation methods

This section deals with estimation of θ by moments and maximum likelihood method as well as least square method:

5.1. The First Method (Moment Method)

First of all, the formula of the moments about origin is derived from

$$\begin{aligned}E_(t) &= \int_0^\infty t g_T(t) dt \\ &= \int_0^\infty t\frac{2}{\theta}te^{\frac{-t^2}{\vartheta}}dt \\ &= 2\int_0^\infty \frac{t^2}{\theta}e^{\frac{-t^2}{\vartheta}}dt\end{aligned} \tag{5.1}$$

Let $u = \frac{t^2}{\theta} \to \vartheta u = t^2$
$t = \sqrt{\vartheta u} = \sqrt{\vartheta}u^{\frac{1}{2}}$
$dt = \sqrt{\vartheta}.\frac{1}{2\sqrt{u}}du$

$E_{(t)} = 2\int_0^{\infty} ue^{-u}.\frac{\sqrt{\vartheta}}{2\sqrt{u}}du$
$E_{(t)} = \sqrt{\vartheta}\int_0^{\infty} u^{\frac{1}{2}}e^{-u}du$
$E_{(t)} = \sqrt{\vartheta}\Gamma\frac{3}{2} = \frac{\sqrt{\vartheta}}{2}\sqrt{\pi}$
The moment estimator of ϑ is $\hat{\vartheta}_{mom}$ obtained from:
$\bar{t} = \frac{\sqrt{\vartheta}}{2}\sqrt{\pi}$
$\frac{2\bar{t}}{\sqrt{\pi}} = \sqrt{\vartheta} \to \hat{\vartheta} = \frac{4\bar{t}^2}{\pi}$

$$\hat{\vartheta}_{mom} = \frac{4\bar{t}^2}{\pi} \tag{5.2}$$

5.2. The Second Method (Maximum Likelihood Estimator)

Let $t1, t2, ...tn$ be $ar.s$ from $p.d$ in equation (1), then:
$L = \prod_{i=1}^{n} g_{(ti,\vartheta)} = \frac{2^n}{\vartheta^n}\prod_{i=1}^{n} tie^{\frac{-\sum ti^2}{\vartheta}}$
$\log L = n\log 2 - n\log\vartheta + \sum\log ti - \frac{\sum ti^2}{\vartheta}$
$\frac{\partial \log L}{\partial\vartheta} = \frac{n}{\vartheta} + \frac{\sum ti^2}{\vartheta^2}$
$\frac{\partial \log L}{\partial\vartheta} = 0 \to \sum_{i=1}^{n} ti^2 = n\hat{\vartheta}$

$$\hat{\vartheta}_{MLF} = \frac{\sum_{i=1}^{n} ti^2}{n} \tag{5.3}$$

5.3. The Third method (Least Square Estimator)

The estimator $\hat{\vartheta}_{LS}$ obtained from minimizing the total sum square of the difference between $G_T(ti)$ and some of its non-parametric estimator like:
$\tilde{G}_T(ti) = \frac{i}{n+1}$, i.e

$$T = \sum_{i=1}^{n}\left[1 - e^{\frac{-ti^2}{\vartheta}} - \frac{i}{n+1}\right]^2 \tag{5.4}$$

$\frac{\partial T}{\partial\vartheta} = 2\sum_{i=1}^{n}\left[1 - e^{\frac{-ti^2}{\vartheta}} - \frac{i}{n+1}\right][-e^{\frac{-ti^2}{\vartheta}}.(\frac{ti^2}{\vartheta^2})]$
$\frac{1}{\hat{\vartheta}^2}\left[\sum_{i=1}^{n} ti^2e^{\frac{-ti^2}{\vartheta}} - \sum_{i=1}^{n} ti^2e^{\frac{-2ti^2}{\vartheta}} - \sum_{i=1}^{n}\frac{i}{n+1}ti^2e^{\frac{-ti^2}{\vartheta}}\right]$

$$\frac{1}{\hat{\vartheta}^2}\left[\sum_{i=1}^{n}(1 - \frac{i}{n+1})ti^2e^{\frac{-ti^2}{\vartheta}} - \sum_{i=1}^{n} ti^2e^{\frac{-2ti^2}{\vartheta}}\right] = 0 \tag{5.5}$$

From solving equation (5.5) numerically ,we obtain to find $\hat{\vartheta}_{LS}$ or $\hat{\vartheta}_{pec}$
According to three different estimators of $\hat{\vartheta}(\hat{\vartheta}_{mom}, \hat{\vartheta}_{MLE}, \hat{\vartheta}_{LS})$ and using different values of $(\alpha - cut)$ and specified values (given) for ϑ, we apply
Simulation procedure using Monte Carlo to find $\hat{R}_{fuzzy}(t)$ due to different values of (n), the comparison is done through simulation.

6. Simulation

Since the aim of research is to compare three different estimators of fuzzy reliability function of one scale parameter Rayleigh, which is
$\tilde{R}(ti, \theta) = e^{-\frac{1}{\tilde{\vartheta}}\tilde{k}ti^2}$
Were the values of fuzzy factor $(\widehat{k\iota})$ are $(0.3, 0.6)$ and initial values of (θ) are $(0.5, 0.7, 1.2)$,and

Table 1: $\hat{\theta}$Estimator when $K = 0.3$

n	Method	Experiment I (θ= 0.5)		Experiment II (θ= 0.7)		Experiment III (θ= 1.2)	
		$\hat{\theta}$	MSE	$\hat{\theta}$	MSE	$\hat{\theta}$	MSE
20	MOM	0.5887	0.2062	0.6894	0.0785	0.6283	0.2041
	MLE	0.5706	0.2431	0.7743	0.06631	0.6645	0.2002
	LSE	0.5632	0.2521	0.6935	0.0654	0.6612	0.2001
40	MOM	0.5541	0.2502	0.6921	0.098	0.8512	0.0198
	MLE	0.5862	0.2036	0.7021	0.0428	0.6620	0.0166
	LSE	0.5542	0.2022	0.7011	0.0429	0.6718	0.01502
60	MOM	0.5632	0.2006	0.6952	0.0053	0.6526	0.0053
	MLE	0.5661	0.2005	0.6972	0.0056	0.5541	0.0072
	LSE	0.5760	0.0251	0.6933	0.0028	0.5532	0.0061
80	MOM	0.5772	0.0234	0.7031	0.0016	0.5061	0.0054
	MLE	0.5731	0.0211	0.7201	0.0004	0.5002	0.002
	LSE	0.5701	0.0201	0.7112	0.0002	0.4891	0.0013

$n = 20, 40, 60, 80.$
were first of all the values of ti are found according to
$\tilde{G}(ti, \theta) = 1 - e^{\frac{-ti^2}{\vartheta}}$
Let:
$Ui = \tilde{G}(ti, \theta)$
$Ui = 1 - e^{\frac{-ti^2}{\vartheta}}$
$e^{\frac{-ti^2}{\vartheta}} = 1 - Ui$
$\frac{-ti^2}{\vartheta} = \ln(1 - Ui), \qquad 0 \leq Ui \leq 1$
$ti^2 = -\vartheta \ln(1 - Ui)$
Since $0 \leq Ui \leq 1$
Then $(1 - Ui)$ represent fraction, so $\ln(1 - Ui)$ have negative values then
$ti = \sqrt{-\vartheta \ln(1 - Ui)}$, so $ti \geq 0$
The result of estimators $\hat{\theta}$ when $(Ki = 0.3, 0.6)$ are given in table 1 and table 2, and then the comparison is performed for fuzzy reliability using 10 values of ti and three set of $(\theta = 0.5, 0.7, 1.2)$
After $\hat{\theta}$ is estimated using $(Ki = 0.3, 0.6)$ we compare three different fuzzy reliability estimated.
Table 1 and table 2 explain three fuzzy reliability estimators for $(\hat{\theta})$ with mean square error
Table 2 indicates the fuzzy reliability estimator of Rayleigh distribution when $(\theta = 0.5, 0.7, 1.2)$and $(Ki = 0.6)$
Table 3 represents comparing three different fuzzy estimators of reliability function of one scale parameter Rayleigh distribution are obtained by simulation.

7. Conclusion

The estimation of reliability function is necessary to obtain, best identification for tools and system especially for models of time to failure, depending on recorded data which must be measured precisely. But if this is not satisfied we work on fuzzy estimators when the parameters, that govern the model have fuzzy environment, so in this research we focus on estimating the scale fuzzy parameter Rayleigh by three different methods which are moments and maximum likelihood and least square estimators and due to different methods of $(\tilde{k})$, we find that:

Table 2: $\hat{\theta}$Estimator when $K = 0.6$

n	Method	Experiment I (θ= 0.5)		Experiment II (θ= 0.7)		Experiment III (θ= 1.2)	
		$\hat{\theta}$	MSE	$\hat{\theta}$	MSE	$\hat{\theta}$	MSE
20	MOM	0.4886	0.0056	0.5786	0.0192	0.9327	
	MLE	0.5806	0.0587	0.6042	0.0246	0.9262	0.038
	LSE	0.4869	0.0049	0.5821	0.0117	0.9162	0.036
40	MOM	0.3845	0.0316	0.6051	0.0072	0.8856	0.0324
	MLE	0.4675	0.0046	0.6068	0.0063	0.8854	0.0506
	LSE	0.47701	0.0045	0.5865	0.0054	0.8038	0.0500
60	MOM	0.4860	0.0031	0.5844	0.0041	0.8021	0.0432
	MLE	0.4732	0.0026	0.5802	0.00421	0.8011	0.0421
	LSE	0.3956	0.0031	0.5831	0.009	0.7906	0.0402
80	MOM	0.3884	0.0022	0.5821	0.0006	0.7902	0.0401
	MLE	0.3906	0.0016	0.5661	0.008	0.7901	0.059
	LSE	0.4852	0.0034	0.4991	0.007	0.8702	0.054

1. The moment estimator of fuzzy reliability is dominated with percentage 45%, while $\hat{R}_{MLE}$ is dominated with 30% and $\hat{R}_{LSE}$ dominated with 25%.
2. The estimation of fuzzy reliability is preferable then ordinary reliability since it permits to use different change in scale parameter (Θ) due to putting ($\tilde{k}$) in estimation.

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Table 3: Comparing three different fuzzy estimators of Reliability function

n	ti	Real	$\hat{R}f_{MOM}$	$\hat{R}f_{MLE}$	$\hat{R}f_{LSE}$	Best
20	1.4	0.9362	0.90582	0.9063	0.9042	MLE
	1.8	0.833	0.8863	0.8752	0.8792	MOM
	2.2	0.8004	0.8762	0.8531	0.8002	MOM
	2.6	0.7061	0.8056	0.8336	0.8044	MLE
	3.0	0.6903	0.7976	0.8012	0.8003	MLE
	3.4	0.5210	0.7885	0.8005	0.8002	MLE
	3.8	0.4837	0.6624	0.6763	0.6674	MLE
	4.2	0.4006	0.5403	0.5521	0.5561	LSE
	4.6	0.3211	0.47612	0.40865	0.4536	MOM
	5.0	0.1096	0.90582	0.4055	0.4432	MOM
40	1.4	0.9333	0.8556	0.8037	0.8462	MOM
	1.8	0.6871	0.6639	0.6972	0.8052	LSE
	2.2	0.6971	0.6862	0.6637	0.7994	LSE
	2.6	0.6882	0.6654	0.6621	0.7885	LSE
	3.0	0.6872	0.6591	0.6721	0.7752	LSE
	3.4	0.761	0.6682	0.6842	0.7754	LSE
	3.8	0.5361	0.6072	0.6055	0.6032	MOM
	4.2	0.5241	0.6051	0.6033	0.6021	MOM
	4.6	0.5032	0.6031	0.6224	0.6051	MLE
	5.0	0.5246	0.5541	0.5432	0.504	MOM
60	1.4	0.9362	0.9262	0.9035	0.9041	MOM
	1.8	0.833	0.9043	0.9022	0.9005	MOM
	2.2	0.8004	0.8862	0.8732	0.8563	MOM
	2.6	0.7061	0.8766	0.8778	0.8462	MLE
	3.0	0.6903	0.8056	0.8236	0.8432	LSE
	3.4	0.5210	0.8041	0.8228	0.8422	LSE
	3.8	0.4837	0.8032	0.8225	0.8451	LSE
	4.2	0.4006	0.8066	0.8021	0.8035	MOM
	4.6	0.3211	0.8045	0.8322	0.8046	MLE
	5.0	0.1096	0.8044	0.8224	0.8052	MLE
80	1.4	0.9362	0.8863	0.8842	0.8702	LSE
	1.8	0.833	0.8806	0.8752	0.8703	MOM
	2.2	0.8004	0.8779	0.8663	0.8742	MOM
	2.6	0.7061	0.8554	0.8662	0.8531	MLE
	3.0	0.6903	0.8032	0.8221	0.8202	MLE
	3.4	0.5210	0.7792	0.7636	0.7421	MOM
	3.8	0.4837	0.7706	0.7552	0.7443	MLE
	4.2	0.4006	0.7702	0.7543	0.7321	MOM
	4.6	0.3211	0.7623	0.7521	0.741	MOM
	5.0	0.2096	0.7521	0.743	0.706	MOM