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Ulam's stability of impulsive sequential coupled system of mixed order derivatives

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Abstract

This manuscript is devoted to establishing Hyers–Ulam stability for a class of non-linear impulsive coupled sequential fractional differential equations with multi point boundary conditions on a closed interval [0,T] with Caputo fractional derivative having non-instantaneous impulses. Sufficient conditions are introduced that guarantee the existence of a unique solution to the proposed problem. Furthermore, Hyers–Ulam stability of the proposed model is also presented and an example is provided to authenticate the theoretical results.

Keywords: Caputo fractional derivative, boundary conditions, fixed point theorem, Hyers–Ulam stability
2010 MSC: 26A33, 34A08, 34B27.

1. Introduction

Fractional order derivatives are the generalized forms of integer order derivatives. The idea about the fractional order derivative was introduced at the end of sixteenth century (1695) when Leibniz used the notation $\frac{d^n}{dx^n}$ for n^{th} order derivative. By writing a letter to him, L'Hospital asked what we can say about $n = \frac{1}{2}$? Leibniz answered in such words, "An apparent Paradox, a day will come to get benefits of this notion" and this question becomes the foundation of fractional calculus. In that time many mathematicians like Fourier and Laplace contributed in the development of fractional calculus. After that when Riemann and Liouville introduced Riemann-Liouville derivative which is a fundamental concept in fractional calculus, then fractional calculus became the most interested area for researchers. Fractional order derivative is global operator, which is used as a tool for modeling

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different processes and physical phenomenon like mathematical biology [15], electro-chemistry [12], control theory [22], dynamical process [19], image and signal processing [17] etc. For more applications of fractional order differential equations, we refer the reader to [1, 8, 14, 16, 25, 27, 28, 34].

The most preferable research area in the field of Fractional Differential Equations (FDE's) which received great attention from the researchers is the theory regarding the existence of solutions. Many researchers developed some interesting results about the existence of solutions of different boundary value problems (BVP's), using different the approach of fixed point [2, 5, 7, 18]. From the literature, it has been observed that most of the time, the exact solution of nonlinear differential equations is a tough job, in such situation different approximation techniques were introduced. The difference between exact and approximate solutions is now a days dealing with the help of Hyers–Ulam (HU) type stabilities, which was first initiated in 1940 by Ulam [20] and then extended by Hyers in the next year, in the context of Banach spaces. Many researchers investigated HU type stabilities for different problems with different approaches, [3, 6, 11, 13, 23, 29, 30, 31, 32, 33, 35].

Wang *et al.* [24], investigated the existence and HU stability of solutions:

$$\begin{cases} {}^c\mathfrak{D}^{\alpha} p(\mathsf{t}) - \mathcal{F}_1(\mathsf{t})p(\mathsf{t}) = f(\mathsf{t}, p(\mathsf{t}), q(\mathsf{t})), \quad \mathsf{t} \in \mathcal{J}, \mathsf{t} \neq \mathsf{t}_k, \\ {}^c\mathfrak{D}^{\beta} p(\mathsf{t}) - \mathcal{F}_2(\mathsf{t})q(\mathsf{t}) = g(\mathsf{t}, p(\mathsf{t}), q(\mathsf{t})), \quad \mathsf{t} \in \mathcal{J}, \quad \mathsf{t} \neq \mathsf{t}_k, \\ \Delta p(\mathsf{t})|_{\mathsf{t}=\mathsf{t}_k} = I_k(p(\mathsf{t})), \quad \Delta q(\mathsf{t})|_{\mathsf{t}=\mathsf{t}_k} = I_k(q(\mathsf{t})), \\ p(\mathsf{t})|_{\mathsf{t}=\mathsf{t}_k} + \phi(p) = p_0, \quad q(\mathsf{t})|_{\mathsf{t}=\mathsf{t}_k} + \varphi(q) = q_0, \end{cases}$$

where ${}^c\mathfrak{D}^{\alpha}$, ${}^c\mathfrak{D}^{\beta}$ denotes the Caputo derivative of order α and β. Influenced by the above discussion, in this article, we present existence and stability analysis of sequential coupled FDE with non-instantaneous impulses of the form

$$\begin{cases} {}^c\mathfrak{D}^{\alpha}(\mathfrak{D} + \lambda)p(\mathsf{t}) = f(\mathsf{t}, p(\mathsf{t}), q(\mathsf{t})), \quad \mathsf{t} \in (\mathsf{t}_k, s_k], \quad 0 < \alpha < 1, \quad k = 0, 1, \ldots, m, \\ {}^c\mathfrak{D}^{\beta}(\mathfrak{D} + \mu)q(\mathsf{t}) = g(\mathsf{t}, p(\mathsf{t}), q(\mathsf{t})), \quad \mathsf{t} \in (\mathsf{t}_k, s_k], \quad 0 < \beta < 1, \quad k = 0, 1, \ldots, m, \\ p(\mathsf{t}) = \mathcal{N}_k(\mathsf{t}, p(\mathsf{t})), \quad q(\mathsf{t}) = \mathcal{M}_k(\mathsf{t}, q(\mathsf{t})), \quad \mathsf{t} \in (s_{k-1}, \mathsf{t}_k], \quad k = 1, 2, \ldots, m, \\ p(0) = 0, \quad p(s_k) = 0, \quad q(0) = 0, \quad q(s_k) = 0, \quad k = 0, 1, \ldots, m, \end{cases} \tag{1.1}$$

where ${}^c\mathfrak{D}^{\alpha}$, ${}^c\mathfrak{D}^{\beta}$ and $\mathfrak{D}$ denotes Caputo derivatives of order α, β and ordinary derivative, respectively. $0 = \mathsf{t}_0 < s_0 < \mathsf{t}_1 < s_1 < \cdots < \mathsf{t}_m < s_m = T$ for a pre-fixed number $T > 0$ and $\lambda, \mu \in \mathcal{R}_+$. The nonlinear continuous functions are defined as; $f, g : [0, T] \times \mathcal{R}^2 \to \mathcal{R}$ $\forall$ $\mathsf{t} \in [0, T] = \mathcal{J}$ and $\mathcal{M}_k, \mathcal{N}_k : (s_{k-1}, \mathsf{t}_k] \times \mathcal{R} \to \mathcal{R}$ are non-instantaneous impulses such that $\mathcal{M}_k, \mathcal{N}_k$ are continuous for each $k = 1, 2, \ldots, m$.

The rest of the paper have the pattern as: Section 2 is devoted basic notions. In Section 3, the solution of the proposed system is investigated. HU stability is analyzed in Section 4. Finally, an example is provided in section5.

2. Preliminaries and Notions

Here we present the basic notations. Endowing the norms as $\|p\|_{\mathbf{E}_1} = \sup\{|p(\mathsf{t})| \text{ for all } \mathsf{t} \in \mathcal{J}\}$ and $\|q\|_{\mathbf{E}_2} = \sup\{|q(\mathsf{t})| \text{ for all } \mathsf{t} \in \mathcal{J}\}$, where $\mathbf{E}_1$ and $\mathbf{E}_2$ are Banach spaces, respectively. Their product $\mathbf{E} = \mathbf{E}_1 \times \mathbf{E}_2$ is also Banach space with norm $\|(p, q)\|_{\mathbf{E}} = \|p\|_{\mathbf{E}_1} + \|q\|_{\mathbf{E}_2}$.

Let $\mathcal{PC}[\mathcal{J}, \mathbf{E}]$ denotes the space of piecewise continuous functions define as

$$\mathcal{PC}[\mathcal{J}, \mathbf{E}] = \{f : \mathcal{J} \times \mathbf{E} \to \mathcal{R}_+, \mathsf{t} \in \mathcal{J}\}$$

with norms

$$\|f\|_{\mathcal{PC}} = \sup\{|f(\mathsf{t})|, \mathsf{t} \in \mathcal{J}\}.$$

We recall the following definitions from [26].

Definition 2.1. *[9] The fractional order integral of order $\alpha > 0$ for a function $p \in L^1([0,T], \mathcal{R}_+)$ in the sense of Caputo, where the lower limit is zero is defined by*

$$\mathfrak{I}^{\alpha} p(\mathsf{t}) = \frac{1}{\Gamma(\alpha)} \int_0^{\mathsf{t}} (\mathsf{t}-s)^{\alpha-1} p(s) ds \quad \mathsf{t} > 0,$$

provided that the integral on right side exists, where Γ is Euler Gamma function defined as

$$\Gamma(\alpha) = \int_0^{\infty} \mathsf{t}^{\alpha-1} e^{-\mathsf{t}} d\mathsf{t}.$$

Definition 2.2. *[9] The fractional order derivative of order $\alpha \in \mathcal{R}_+$ in the sense of Caputo for a function $p : [0,T] \to \mathcal{R}$ is defined as*

$$^{c}\mathfrak{D}^{\alpha} p(\mathsf{t}) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\mathsf{t}} (\mathsf{t}-s)^{n-\alpha-1} p^{n}(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3. *[10] The sequential fractional order derivative for a function p is defined as:*

$$\mathfrak{D}^{\alpha} p(\mathsf{t}) = \mathfrak{D}^{\alpha_1}\mathfrak{D}^{\alpha_2}\mathfrak{D}^{\alpha_3} \ldots \mathfrak{D}^{\alpha_m} p(\mathsf{t}),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m)$ is any multi-index and the operator $\mathfrak{D}^{\alpha}$ can either be Caputo or Riemann-Liouville or any other kind of integro-differential operator.

Lemma 2.4. *[9] For any $\alpha > 0$, the solution of Caputo fractional differential equation $^{c}\mathfrak{D}^{\alpha} u(\mathsf{t}) = 0$ is of the form*

$$u(\mathsf{t}) = a_0 + a_1\mathsf{t} + a_2\mathsf{t}^2 + \cdots + a_{n-1}\mathsf{t}^{n-1},$$

where $a_i \in \mathcal{R}, i = 0, 1, 2, \ldots, n-1$ and $n = [\alpha] + 1$.

Lemma 2.5. *[9] For any $\alpha > 0$, we have*

$$\mathfrak{I}^{\alpha}(^{c}\mathfrak{D}^{\alpha} u(\mathsf{t})) = u(\mathsf{t}) + a_0 + a_1\mathsf{t} + a_2\mathsf{t}^2 + \cdots + a_{n-1}\mathsf{t}^{n-1},$$

where $a_i \in \mathcal{R}, \{i = 0, 1, 2, \ldots, n-1\}$ and $n = [\alpha] + 1$.

Theorem 2.6. *(Altman [4])*
Let $\mathcal{B}_r \neq \emptyset$ be closed convex subset of Banach space $\mathbf{E}$. Consider $\mathbb{F}$, $\mathbb{G}$ be two operators such that

- $\mathbb{F}(p, q) + \mathbb{G}(\widetilde{p}, \widetilde{q}) \in \mathcal{B}_r$, *where* $(p, q), (\widetilde{p}, \widetilde{q}) \in \mathcal{B}_r$.
- *The operator $\mathbb{F}$ is contractive.*

- *The operator* $\mathbb{G}$ *is completely continuous.*

Then the equation $(p,q) = \mathbb{F}(p,q) + \mathbb{G}(p,q), (p,q) \in \mathbf{E}$ *has a solution* $(p,q) \in \mathcal{B}_r$.

Definition 2.7. *(Urs [21]) The coupled impulsive FDE* (1.1) *is said to be HU stable if there exist* $\mathcal{V}_i (i = 1,2,3,4) > 0$ *such that, for* $\wp_i (i = 1,2) > 0$ *and for every solution* $(\overline{p},\overline{q}) \in \mathbf{E}$ *of the following inequalities*

$$\begin{cases} |{}^c\mathfrak{D}^\alpha(\mathfrak{D}+\lambda)\overline{p}(\mathsf{t}) - f(\mathsf{t},\overline{p}(\mathsf{t}),\overline{q}(\mathsf{t}))| \le \wp_1, & \mathsf{t} \in (\mathsf{t}_k, s_k], \quad 0<\alpha<1, \quad k=0,1,\dots,m, \\ |\overline{p}(\mathsf{t}) - \mathcal{N}_k(\mathsf{t},\overline{p}(\mathsf{t}))| \le \wp_1, & \mathsf{t} \in (s_{k-1}, \mathsf{t}_k], \quad k=1,2,\dots,m, \\ |{}^c\mathfrak{D}^\beta(\mathfrak{D}+\mu)\overline{q}(\mathsf{t}) - g(\mathsf{t},\overline{p}(\mathsf{t}),\overline{q}(\mathsf{t}))| \le \wp_2, & \mathsf{t} \in (\mathsf{t}_k, s_k], \quad 0<\beta<1, \quad k=0,1,\dots,m, \\ |\overline{q}(\mathsf{t}) - \mathcal{M}_k(\mathsf{t},\overline{q}(\mathsf{t}))| \le \wp_2, & k=1,2,\dots,m, \end{cases} \tag{2.1}$$

there exists a solution $(p,q) \in \mathbf{E}$ *with*

$$\begin{cases} \|p-\overline{p}\|_{\mathcal{PC}} \le \mathcal{V}_1\wp_1 + \mathcal{V}_2\wp_2, \\ \|q-\overline{q}\|_{\mathcal{PC}} \le \mathcal{V}_3\wp_1 + \mathcal{V}_4\wp_2. \end{cases}$$

Definition 2.8. *If* η_i *be the (real or complex) eigenvalues of a matrix* $\mathbb{Q} \in \mathcal{C}^{n\times n}$ *for* $i = 1,2,3\dots,n$, *then the spectral radius* $\rho(\mathbb{Q})$ *is defined by*

$$\rho(\mathbb{Q}) = \max\{|\eta_i|, for i = 1,2,\dots,n\}.$$

Further, the system corresponding to the matrix $\mathbb{Q}$ *will converges to zero if* $\rho(\mathbb{Q}) < 1$.

Theorem 2.9. *(Urs[21], Theorem 4)*
Consider $\mathbf{E}$ *be a Banach space with* $\mathcal{Z}_1, \mathcal{Z}_2 : \mathbf{E} \to \mathbf{E}$ *be two operators such that*

$$\begin{cases} \|\mathcal{Z}_1(p,q) - \mathcal{Z}_1(\overline{p},\overline{q})\|_{\mathcal{PC}} \le \mathcal{V}_1\|p-p^*\| + \mathcal{V}_2\|q-q^*\| \\ \|\mathcal{Z}_2(p,q) - \mathcal{Z}_2(\overline{p},\overline{q})\|_{\mathcal{PC}} \le \mathcal{V}_3\|p-p^*\| + \mathcal{V}_4\|q-q^*\| \\ \forall (p,q),(p^*,q^*) \in \mathbf{E} \end{cases}$$

and if the system corresponding to the matrix

$$\mathbb{Q} = \begin{bmatrix} \mathcal{V}_3 & \mathcal{V}_3 \\ \mathcal{V}_3 & \mathcal{V}_3 \end{bmatrix}$$

converges to zero, then the fixed points corresponding to operational system (1.1) *are HU stable.*

3. Existence theory of the proposed problem (1.1)

In this section, we present existence, uniqueness and at least one solution of (1.1).

Lemma 3.1. *Let* $0 < \alpha \le 1$, $0 < \beta \le 1$ *and* $h_1, h_2 : \mathcal{J} \to \mathcal{R}$ *are given continuous functions, a pair (p, q) is a solution of the linear impulsive coupled system*

$$\begin{cases} {}^c\mathfrak{D}^\alpha(\mathfrak{D}+\lambda)p(\mathsf{t}) = h_1(\mathsf{t}), \quad 0<\alpha<1, \quad k=0,1,\dots,m, \quad \mathsf{t}\in\mathcal{J}, \\ {}^c\mathfrak{D}^\beta(\mathfrak{D}+\mu)q(\mathsf{t}) = h_2(\mathsf{t}), \quad 0<\beta<1, \quad k=0,1,\dots,m, \quad \mathsf{t}\in\mathcal{J}, \\ p(\mathsf{t}) = \mathcal{N}_k(\mathsf{t},p(\mathsf{t})), \quad q(\mathsf{t}) = \mathcal{M}_k(\mathsf{t},q(\mathsf{t})), \quad k=1,2,\dots,m, \\ p(0)=0, \quad p(s_k)=0, \quad q(0)=0, \quad q(s_k)=0, \quad k=0,1,\dots,m, \end{cases} \tag{3.1}$$

if and only if (p,q) satisfies the following fractional integral equations.

$$p(\mathsf{t}) = \begin{cases} \int_0^t e^{-\lambda(\mathsf{t}-s)}\mathfrak{J}^\alpha h_1(s)ds + A^\lambda \int_0^{s_0} \mathfrak{J}^\alpha e^{-\lambda(s_0-s)} h_1(s)ds, \quad \mathsf{t} \in [0,s_0], \\ \mathcal{N}_k(\mathsf{t}), \quad \mathsf{t} \in (s_{k-1}, \mathsf{t}_k], \quad k = 1,2,\ldots,m, \\ \int_{\mathsf{t}_k}^t e^{-\lambda(\mathsf{t}-s)}\mathfrak{J}^\alpha h_1(s)ds + B_k^\lambda \int_{\mathsf{t}_k}^{s_k} e^{-\lambda(s_k-s)}\mathfrak{J}^\alpha h_1(s)ds \\ +\delta_k^\lambda \mathcal{N}_k(\mathsf{t}_k), \quad \mathsf{t} \in (\mathsf{t}_k, s_k], \quad k = 1,2,\ldots,m, \end{cases} \tag{3.2}$$

$$q(\mathsf{t}) = \begin{cases} \int_0^t e^{-\mu(\mathsf{t}-s)}\mathfrak{J}^\beta h_2(s)ds + A^\mu \int_0^{s_0} e^{-\beta(s_0-s)}\mathfrak{J}^\beta h_2(s)ds, \quad \mathsf{t} \in [0,s_0], \\ \mathcal{M}_k(\mathsf{t}); \quad \mathsf{t} \in (s_{k-1}, \mathsf{t}_k], \quad k = 1,\ldots,m, \\ \int_{\mathsf{t}_k}^t e^{-\mu(\mathsf{t}-s)}\mathfrak{J}^\beta h_2(s)ds + B_k^\mu \int_{\mathsf{t}_k}^{s_k} e^{-\mu(s_k-s)}\mathfrak{J}^\beta h_2(s)ds \\ +\delta_k^\mu \mathcal{M}_k(\mathsf{t}_k), \quad \mathsf{t} \in (\mathsf{t}_k, s_k], \quad k = 1,2,\ldots,m, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} A^\gamma &= \frac{1-e^{-\gamma \mathsf{t}}}{e^{-\gamma s_0}-1}, \\ B_k^\gamma &= \frac{1-e^{-\gamma(\mathsf{t}-\mathsf{t}k)}}{e^{-\gamma(s_k-\mathsf{t}_k)}-1} \qquad \textit{and} \\ \delta_k^\gamma &= \frac{1-e^{-\gamma(\mathsf{t}-s_k)}}{1-e^{-\gamma(s_k-\mathsf{t}_k)}}. \end{aligned}$$

Proof . Let $(p,q) \in \mathbf{E}$ is a solution of the problem (3.1). To show that $(p,q) \in \mathbf{E}$ satisfies the fractional integral equations (3.2), (3.3) we proceed in the following manner.

For $\mathsf{t} \in [0,s_0]$, we consider

$${}^c\mathfrak{D}^\alpha(\mathfrak{D}+\lambda)p(\mathsf{t}) = h_1(\mathsf{t}). \tag{3.4}$$

Using Lemma 2.4 and ordinary integration, we obtain

$$p(\mathsf{t}) = \int_0^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}\mathfrak{J}^\alpha h_1(s)ds + c_0\left(\frac{1-e^{-\lambda \mathsf{t}}}{\lambda}\right) + d_0 e^{-\lambda \mathsf{t}}. \tag{3.5}$$

For obtaining the arbitrary constants c_0 and d_0, we apply the boundary conditions $p(0) = p(s_0) = 0$ on (3.5), we get

$$c_0 = \frac{\lambda}{-1+e^{-\lambda s_0}} \int_0^{s_0} e^{-\lambda(s_0-s)}\mathfrak{J}^\alpha h_1(s)ds \quad \textit{and} \quad d_0 = 0.$$

Substituting the above c_0 and d_0 values in equation (3.5), we get

$$p(\mathsf{t}) = \int_0^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}\mathfrak{J}^\alpha h_1(s)ds + A^\lambda \int_0^{s_0} e^{-\lambda(s_0-s)}\mathfrak{J}^\alpha h_1(s)ds, \quad \mathsf{t} \in [0,s_0], \tag{3.6}$$

where

$$A^\lambda = \frac{1-e^{-\lambda \mathsf{t}}}{e^{-\lambda s_0}-1}.$$

Now if $\mathsf{t} \in (s_0, \mathsf{t}_1]$ then $p(\mathsf{t}) = \mathcal{N}_1(\mathsf{t})$.

For $\mathsf{t} \in (\mathsf{t}_1, s_1]$, (3.4) gives

$$p(\mathsf{t}) = \int_{\mathsf{t}_1}^t e^{-\lambda(\mathsf{t}-s)}\mathfrak{J}^\alpha h_1(s)ds + c_1\left(\frac{1-e^{-\lambda(\mathsf{t}-\mathsf{t}_1)}}{\lambda}\right) + d_1 e^{-\lambda \mathsf{t}}. \tag{3.7}$$

For obtaining the arbitrary constants c_1 and d_1, we apply the impulsive and boundary conditions $p(\mathsf{t}_1) = \mathcal{N}_1(\mathsf{t}_1)$ and $p(s_1) = 0$ respectively on (3.7), we have

$$c_1 = \frac{\lambda}{e^{-\lambda(s_1-\mathsf{t}_1)} - 1} \int_{\mathsf{t}_1}^{s_1} e^{-\lambda(s_1-s)} \mathfrak{I}^{\alpha} h_1(s) ds + \frac{\lambda(e^{-\lambda(s_1-\mathsf{t}_1)} - 1)}{e^{-\lambda(s_1-\mathsf{t}_1)} - 1} \mathcal{N}_1(\mathsf{t}_1)$$

and

$$d_1 = e^{\lambda \mathsf{t}_1} \mathcal{N}_1(\mathsf{t}_1).$$

By substituting the values of c_1 and d_1 in (3.7), we get

$$p(\mathsf{t}) = \int_{\mathsf{t}_1}^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)} \mathfrak{I}^{\alpha} h_1(s) ds + B_1^{\lambda} \int_{\mathsf{t}_1}^{s_1} e^{-\lambda(s_1-s)} \mathfrak{I}^{\alpha} h_1(s) ds + \delta_1^{\lambda} \mathcal{N}_1(\mathsf{t}_1),$$

where

$$B_1^{\lambda} = \frac{1 - e^{-\lambda(\mathsf{t}-\mathsf{t}_1)}}{e^{-\lambda(s_1-\mathsf{t}_1)} - 1} \quad \textit{and}$$
$$\delta_1^{\lambda} = \frac{1 - e^{-\lambda(\mathsf{t}-s_1)}}{1 - e^{-\lambda(s_1-\mathsf{t}_1)}}.$$

In general: (Repeating the same steps).
For $\mathsf{t} \in (s_{k-1}, \mathsf{t}_k]$, we have

$$p(\mathsf{t}) = \mathcal{N}_k(\mathsf{t}). \tag{3.8}$$

For $\mathsf{t} \in (\mathsf{t}_k, s_k]$ the solution of (3.7) with boundary condition $p(\mathsf{t}_k) = \mathcal{N}_k(\mathsf{t}_k)$ and $p(s_k) = 0$ gives

$$p(\mathsf{t}) = \int_{\mathsf{t}_k}^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)} \mathfrak{I}^{\alpha} h_1(s) ds + B_k^{\lambda} \int_{\mathsf{t}_k}^{s_k} e^{-\lambda(s_k-s)} \mathfrak{I}^{\alpha} h_1(s) ds + \delta_k^{\lambda} \mathcal{N}_k(\mathsf{t}_k), \quad k = 1, 2, \ldots, m, \tag{3.9}$$

where

$$B_k^{\lambda} = \frac{1 - e^{-\lambda(\mathsf{t}-\mathsf{t}_k)}}{e^{-\lambda(s_k-\mathsf{t}_k)} - 1}, \quad k = 1, 2, \ldots, m \ \textit{and}$$
$$\delta_k^{\lambda} = \frac{1 - e^{-\lambda(\mathsf{t}-s_k)}}{1 - e^{-\lambda(s_k-\mathsf{t}_k)}}, \quad k = 1, 2, \ldots, m.$$

Hence, we obtained (3.2), from (3.6), (3.8) and (3.9). In similar way, we can obtain (3.3). □

The following assumptions will be helpful for our results.

($\mathbf{H}_1$) $f, g : \mathcal{J} \times \mathcal{R}^2 \to \mathcal{R}^+$ are continuous functions, for all $(p, q), (\widetilde{p}, \widetilde{q}) \in \mathbf{E}$ and $\mathsf{t} \in \mathcal{J}$, there exist $\mathcal{L}_f, \mathcal{L}_g > 0$ such that

$$|f(\mathsf{t}, p(\mathsf{t}), q(\mathsf{t})) - f(\mathsf{t}, \widetilde{p}(\mathsf{t}), \widetilde{q}(\mathsf{t}))| \leq \mathcal{L}_f |(p - \widetilde{p}, q - \widetilde{q})|,$$

$$|g(\mathsf{t}, p(\mathsf{t}), q(\mathsf{t})) - g(\mathsf{t}, \widetilde{p}(\mathsf{t}), \widetilde{q}(\mathsf{t}))| \leq \mathcal{L}_g |(p - \widetilde{p}, q - \widetilde{q})|.$$

($\mathbf{H}_2$) $f, g : \mathcal{J} \times \mathcal{R}^2 \to \mathcal{R}^+$ are continuous functions, for all $(p,q) \in \mathbf{E}$ and $\mathsf{t} \in \mathcal{J}$ such that

$$|f(\mathsf{t}, p(\mathsf{t}), q(\mathsf{t}))| \leq \phi_f(\mathsf{t}) + \widetilde{\phi}_f(\mathsf{t})|(p,q)|,$$

$$|g(\mathsf{t}, p(\mathsf{t}), q(\mathsf{t}))| \leq \varphi_g(\mathsf{t}) + \widetilde{\varphi}_g(\mathsf{t})|(p,q)|.$$

$$\sup_{\mathsf{t}\in\mathcal{J}} \int_{\mathsf{t}_k}^{\mathsf{t}} I^\alpha \phi_f(s)ds \leq \omega_1 \phi_f^* \quad , \quad \sup_{\mathsf{t}\in\mathcal{J}} \int_{\mathsf{t}_k}^{\mathsf{t}} I^\alpha \widetilde{\phi}_f(s)ds \leq \omega_2 \widetilde{\phi}_f^*,$$
$$\sup_{\mathsf{t}\in\mathcal{J}} \int_{\mathsf{t}_k}^{\mathsf{t}} I^\beta \phi_g(s)ds \leq \omega_3 \phi_g^* \quad \text{and} \quad \sup_{\mathsf{t}\in\mathcal{J}} \int_{\mathsf{t}_k}^{\mathsf{t}} I^\beta \widetilde{\phi}_g(s)ds \leq \omega_4 \widetilde{\phi}_g^*.$$

($\mathbf{H}_3$) $\mathcal{N}_k, \mathcal{M}_k : [s_{k-1}, \mathsf{t}_k] \times \mathcal{R} \to \mathcal{R}$ are continuous for all $k = 1, 2, \ldots, m$ and there exist constants $\mathcal{L}_\mathcal{N}, \mathcal{L}_\mathcal{M} > 0$ such that for any $(p,q), (\widetilde{p}, \widetilde{q}) \in \mathbf{E}$

$$|\mathcal{N}_k(\mathsf{t}, p(\mathsf{t})) - \mathcal{N}_k(\mathsf{t}, \widetilde{p}(\mathsf{t}))| \leq \mathcal{L}_\mathcal{N}|p - \widetilde{p}| \ \ \textit{and} \ \ |\mathcal{M}_k(\mathsf{t}, q(\mathsf{t})) - \mathcal{M}_k(\mathsf{t}, \widetilde{q}(\mathsf{t}))| \leq \mathcal{L}_\mathcal{M}|q - \widetilde{q}|.$$

($\mathbf{H}_4$) $\mathcal{N}_k, \mathcal{M}_k : [s_{k-1}, \mathsf{t}_k] \times \mathcal{R} \to \mathcal{R}$ are continuous for all $k = 1, 2, \ldots, m$ and there are $a_1, b_1 \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$ such that for any $(p,q) \in \mathcal{E}$

$$|\mathcal{N}_k(\mathsf{t}, p(\mathsf{t}))| \leq a_1(\mathsf{t}) + b_1(\mathsf{t})|p(\mathsf{t})| \ \textit{and} \ |\mathcal{M}_k(\mathsf{t}, q(\mathsf{t}))| \leq a_2(\mathsf{t}) + b_2(\mathsf{t})|q(\mathsf{t})|,$$

where $a_1^* = \sup_{\mathsf{t}\in\mathcal{J}} a_1(\mathsf{t})$, $b_1^* = \sup_{\mathsf{t}\in\mathcal{J}} b_1(\mathsf{t})$, $a_2^* = \sup_{\mathsf{t}\in\mathcal{J}} a_2(\mathsf{t})$ and $b_2^* = \sup_{\mathsf{t}\in\mathcal{J}} b_2(\mathsf{t})$.

Consider a closed ball $\mathcal{B}_r = \{(p,q) \in \mathbf{E} : \|(p,q)\| \leq r \ \ \textit{with} \ \ \|p\| \leq \frac{r}{2}, \|q\| \leq \frac{r}{2}\} \subset \mathbf{E}$, where

$$\frac{\psi^\lambda \omega_1 \phi_f^* + \psi^\mu \omega_3 \phi_g^* + a_1^* + c_1^* + |\delta_k^\lambda| a_1^* + |\delta_k^\mu| c_1^*}{1 - \psi^\lambda \omega_2 \phi_f^* - \psi^\mu \omega_4 \phi_g^* - \frac{b_1^* + d_1^* + |\delta_k^\lambda| b_1^* + |\delta_k^\mu| d_1^*}{2}} \leq r.$$

We define two operators $\mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2)$ and $\mathbb{G} = (\mathbb{G}_1, \mathbb{G}_2)$ on closed ball $\mathcal{B}_r$ as

$$\begin{cases} \mathbb{F}_1 p(\mathsf{t}) = \displaystyle\int_0^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)} \mathfrak{I}^\alpha f(s, p(s), q(s))ds + A^\lambda \int_0^{s_0} \mathfrak{I}^\alpha e^{-\lambda(s_0 - s)} f(s, p(s), q(s))ds \\ \qquad + \mathcal{N}_k(\mathsf{t}, p(\mathsf{t})) + \delta_k^\lambda \mathcal{N}_k(\mathsf{t}_k, p(\mathsf{t}_k)), \\ \mathbb{F}_2 q(\mathsf{t}) = \displaystyle\int_0^{\mathsf{t}} e^{-\mu(\mathsf{t}-s)} \mathfrak{I}^\beta g(s, p(s), q(s))ds + A^\mu \int_0^{s_0} \mathfrak{I}^\beta e^{-\mu(s_0 - s)} g(s, p(s), q(s))ds \\ \qquad + \mathcal{M}_k(\mathsf{t}, p(\mathsf{t})) + \delta_k^\mu \mathcal{M}_k(\mathsf{t}_k, p(\mathsf{t}_k)), \end{cases} \tag{3.10}$$

and

$$\begin{cases} \mathbb{G}_1(p,q)(\mathsf{t}) = \displaystyle\int_{\mathsf{t}_k}^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)} \mathfrak{I}^\alpha f(s, p(s), q(s))ds + B_k^\lambda \int_{\mathsf{t}_k}^{s_k} e^{-\lambda(s_k - s)} \mathfrak{I}^\alpha f(s, p(s), q(s))ds, \\ \mathbb{G}_2(p,q)(\mathsf{t}) = \displaystyle\int_{\mathsf{t}_k}^{\mathsf{t}} e^{-\mu(\mathsf{t}-s)} \mathfrak{I}^\alpha g(s, p(s), q(s))ds + B_k^\mu \int_{\mathsf{t}_k}^{s_k} e^{-\mu(s_k - s)} \mathfrak{I}^\alpha g(s, p(s), q(s))ds. \end{cases} \tag{3.11}$$

Theorem 3.2. *Under the assumptions,* $(\mathbf{H}_1) - (\mathbf{H}_4)$*, the coupled* $BVP(1.1)$ *has at least one solution in* $\mathbf{E}$.

Proof . For any $(p,q)\in\mathcal{B}_r$, we have

$$\begin{aligned}\|\mathbb{F}(p,q)+\mathbb{G}(p,q)\|_{\mathcal{PC}} &\le \|\mathbb{F}(p,q)\|_{\mathcal{PC}}+\|\mathbb{G}(p,q)\|_{\mathcal{PC}}\\ &\le \|\mathbb{F}_1(p,q)\|_{\mathcal{PC}}+\|\mathbb{F}_2(p,q)\|_{\mathcal{PC}}+\|\mathbb{G}_1(p,q)\|_{\mathcal{PC}}+\|\mathbb{G}_2(p,q)\|_{\mathcal{PC}}. \end{aligned}\tag{3.12}$$

From (3.10), we get

$$\begin{aligned}|\mathbb{F}_1(p,q)(\mathsf{t})| \le& \Big|\int_0^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}\mathfrak{I}^\alpha f(s,p(s),q(s))ds\Big| + \Big|A^\lambda\int_0^{s_0}\mathfrak{I}^\alpha e^{-\lambda(s_0-s)}f(s,p(s),q(s))ds\Big|\\ &+|\mathcal{N}_k(\mathsf{t},p(\mathsf{t}))|+|\delta_k^\lambda\mathcal{N}_k(\mathsf{t}_k,p(\mathsf{t}_k))|\\ \le&\int_0^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}\mathfrak{I}^\alpha|f(s,p(s),q(s))|ds+|A^\lambda|\int_0^{s_0}e^{-\lambda(s_0-s)}\mathfrak{I}^\alpha|f(s,p(s),q(s))|ds\\ &+|\mathcal{N}_k(\mathsf{t},p(\mathsf{t}))|+|\delta_k^\lambda||\mathcal{N}_k(\mathsf{t}_k,p(\mathsf{t}_k))|\\ \le&\int_0^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}ds\int_0^{\mathsf{t}}\mathfrak{I}^\alpha\phi_f(s)ds+\int_0^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}ds\int_0^{\mathsf{t}}\mathfrak{I}^\alpha\widetilde{\phi}_f(s)ds|(p,q)|\\ &+|A^\lambda|\int_0^{s_0}e^{-\lambda(s_0-s)}ds\int_0^{s_0}\mathfrak{I}^\alpha\phi_f(s)ds\\ &+|A^\lambda|\int_0^{s_0}e^{-\lambda(s_0-s)}ds\int_0^{s_0}\mathfrak{I}^\alpha\widetilde{\phi}_f(s)ds|(p,q)|\\ &+(1+|\delta_k^\lambda|)(a_1(\mathsf{t})+b_1(\mathsf{t})|p(\mathsf{t})|).\end{aligned}$$

Taking $\sup\limits_{\mathsf{t}>0}$, we get

$$\begin{aligned}\|\mathbb{F}_1(p,q)\|_{\mathcal{PC}} \le&\Big(\frac{1-e^{-\lambda\mathsf{t}}}{\lambda}\Big)(\omega_1\phi_f^*+\omega_2\widetilde{\phi}_f^*\|(p,q)\|)+(1+|\delta_k^\lambda|)(a_1^*+b_1^*\|p\|)\\ &+|A^\lambda|\Big(\frac{1-e^{-\lambda s_0}}{\lambda}\Big)(\omega_1\phi_f^*+\omega_2\widetilde{\phi}_f^*\|(p,q)\|)\\ \le&\Big(\frac{1-e^{-\lambda\mathsf{t}}}{\lambda}\Big)(\omega_1\phi_f^*+\omega_2\widetilde{\phi}_f^*r)+|A^\lambda|\Big(\frac{1-e^{-\lambda s_0}}{\lambda}\Big)(\omega_1\phi_f^*+\omega_2\widetilde{\phi}_f^*r)\\ &+(1+|\delta_k^\lambda|)(a_1^*+b_1^*\frac{r}{2}).\end{aligned}\tag{3.13}$$

On the same way, we can obtain

$$\begin{aligned}\|\mathbb{F}_2(p,q)\|_{\mathcal{PC}} \le&\Big(\frac{1-e^{-\mu\mathsf{t}}}{\mu}\Big)(\omega_3\phi_g^*+\omega_4\widetilde{\phi}_g^*r)+|A^\mu|\Big(\frac{1-e^{-\mu s_0}}{\mu}\Big)(\omega_3\phi_g^*+\omega_4\widetilde{\phi}_g^*r)\\ &+(1+|\delta_k^\mu|)(c_1^*+d_1^*\frac{r}{2}).\end{aligned}\tag{3.14}$$

Moreover, we obtain

$$\begin{aligned}|\mathbb{G}_1(p,q)(\mathsf{t})| \leq & \Big|\int_{\mathsf{t}_k}^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}\mathfrak{I}^{\alpha} f(s,p(s),q(s))ds\Big| + |B_k^{\lambda}\int_{\mathsf{t}_k}^{s_k} e^{-\lambda(s_k-s)}\mathfrak{I}^{\alpha} f(s,p(s),q(s))ds| \\ \leq & \int_{\mathsf{t}_k}^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}\mathfrak{I}^{\alpha}|f(s,p(s),q(s))|ds + |B_k^{\lambda}|\int_{\mathsf{t}_k}^{s_k} e^{-\lambda(s_k-s)}\mathfrak{I}^{\alpha}|f(s,p(s),q(s))|ds \\ \leq & \int_{\mathsf{t}_k}^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}ds\int_{\mathsf{t}_k}^{\mathsf{t}}\mathfrak{I}^{\alpha}\phi_f(s)ds + \int_{\mathsf{t}_k}^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}ds\int_{\mathsf{t}_k}^{\mathsf{t}}\mathfrak{I}^{\alpha}\widetilde{\phi}_f(s)ds|(p,q)| \\ & + |B_k^{\lambda}|\int_{\mathsf{t}_k}^{s_k} e^{-\lambda(s_k-s)}ds\int_{\mathsf{t}_k}^{s_k}\mathfrak{I}^{\alpha}\phi_f(s)ds + |B_k^{\lambda}|\int_{\mathsf{t}_k}^{s_k} e^{-\lambda(s_k-s)}ds\int_{\mathsf{t}_k}^{s_k}\mathfrak{I}^{\alpha}\widetilde{\phi}_f(s)ds|(p,q)|.\end{aligned}$$

Taking $\sup_{\mathsf{t}>0}$, we get

$$\begin{aligned}\|\mathbb{G}_1(p,q)\|_{\mathcal{PC}} \leq & \Big(\frac{1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)}}{\lambda}\Big)(\omega_1\phi_f^* + \omega_2\widetilde{\phi}_f^*\|(p,q)\|) + |B_k^{\lambda}|\Big(\frac{1-e^{-\lambda(s_k-\mathsf{t}_k)}}{\lambda}\Big)(\omega_1\phi_f^* + \omega_2\widetilde{\phi}_f^*\|(p,q)\|) \\ \leq & \Big(\frac{1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)}}{\lambda}\Big)(\omega_1\phi_f^* + \omega_2\widetilde{\phi}_f^* r) + |B_k^{\lambda}|\Big(\frac{1-e^{-\lambda(s_k-\mathsf{t}_k)}}{\lambda}\Big)(\omega_1\phi_f^* + \omega_2\widetilde{\phi}_f^* r). \end{aligned} \tag{3.15}$$

Similarly

$$\|\mathbb{G}_2(p,q)\|_{\mathcal{PC}} \leq \Big(\frac{1-e^{-\mu(\mathsf{t}-\mathsf{t}_k)}}{\mu}\Big)(\omega_3\phi_g^* + \omega_4\widetilde{\phi}_g^* r) + |B_k^{\mu}|\Big(\frac{1-e^{-\mu(s_k-\mathsf{t}_k)}}{\mu}\Big)(\omega_3\phi_g^* + \omega_4\widetilde{\phi}_g^* r). \tag{3.16}$$

Using (3.13), (3.14), (3.15) and (3.16) in (3.12), we get

$$\begin{aligned}&\|\mathbb{F}(p,q)+\mathbb{G}(p,q)\|_{\mathcal{PC}} \\ \leq & \Bigg[\frac{(1-e^{-\lambda\mathsf{t}}) + |A^{\lambda}|(1-e^{-\lambda s_0}) + (1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)}) + |B_k^{\lambda}|(1-e^{-\lambda(s_k-\mathsf{t}_k)})}{\lambda}\omega_1\phi_f^* \\ & + \frac{(1-e^{-\mu\mathsf{t}}) + |A^{\mu}|(1-e^{-\mu s_0}) + (1-e^{-\mu(\mathsf{t}-\mathsf{t}_k)}) + |B_k^{\mu}|(1-e^{-\mu(s_k-\mathsf{t}_k)})}{\mu}\omega_3\phi_g^* \\ & + a_1^* + c_1^* + |\delta_k^{\lambda}|a_1^* + |\delta_k^{\mu}|c_1^*\Bigg] \\ & + \Bigg[\Big(\frac{(1-e^{-\lambda\mathsf{t}}) + |A^{\lambda}|(1-e^{-\lambda s_0}) + (1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)}) + |B_k^{\lambda}|(1-e^{-\lambda(s_k-\mathsf{t}_k)})}{\lambda}\omega_2\widetilde{\phi}_f^* \\ & + \frac{(1-e^{-\mu\mathsf{t}}) + |A^{\mu}|(1-e^{-\mu s_0}) + (1-e^{-\mu(\mathsf{t}-\mathsf{t}_k)}) + |B_k^{\mu}|(1-e^{-\mu(s_k-\mathsf{t}_k)})}{\mu}\omega_4\widetilde{\phi}_g^* \\ & + \frac{b_1^* + d_1^* + |\delta_k^{\lambda}|b_1^* + |\delta_k^{\mu}|d_1^*}{2}\Bigg]r \\ \leq & \psi^{\lambda}\omega_1\phi_f^* + \psi^{\mu}\omega_3\phi_g^* + a_1^* + c_1^* + |\delta_k^{\lambda}|a_1^* + |\delta_k^{\mu}|c_1^* \\ & + \Bigg(\psi^{\lambda}\omega_2\phi_f^* + \psi^{\mu}\omega_4\phi_g^* + \frac{b_1^* + d_1^* + |\delta_k^{\lambda}|b_1^* + |\delta_k^{\mu}|d_1^*}{2}\Bigg)r.\end{aligned}$$

Which implies that

$$\|\mathbb{F}(p,q)+\mathbb{G}(p,q)\|_{\mathcal{PC}} \le r,$$

where

$$\psi^{\gamma} = \frac{(1-e^{-\gamma\mathsf{t}}) + |A^{\gamma}|(1-e^{-\gamma s_0}) + (1-e^{-\gamma(\mathsf{t}-\mathsf{t}_k)}) + |B_k^{\gamma}|(1-e^{-\gamma(s_k-\mathsf{t}_k)})}{\gamma}.$$

Hence, this implies that $\mathbb{F}(p,q)+\mathbb{G}(p,q) \in \mathcal{B}_r$.

We need to show that $\mathbb{F}$ is contractive. For this, let (p,q) and $(\bar{p},\bar{q}) \in \mathbf{E}$, we get

$$\begin{aligned}
&|\mathbb{F}_1(p,q)-\mathbb{F}_1(\bar{p},\bar{q})| \\
\le&\Big|\int_0^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}\mathfrak{I}^{\alpha} f(s,p(s),q(s)) - \mathfrak{I}^{\alpha} f(s,\bar{p}(s),\bar{q}(s))ds\Big| \\
&+|A^{\lambda}|\Big|\int_0^{s_0} e^{-\lambda(s_0-s)}\mathfrak{I}^{\alpha} f(s,p(s),q(s)) - \mathfrak{I}^{\alpha} f(s,\bar{p}(s),\bar{q}(s))ds\Big| \\
&+\Big|\mathcal{N}_k(\mathsf{t},p(\mathsf{t}))-\mathcal{N}_k(\mathsf{t},\bar{p}(\mathsf{t}))\Big| + |\delta_k^{\lambda}|\Big|\mathcal{N}_k(\mathsf{t}_k,p(\mathsf{t}_k))-\mathcal{N}_k(\mathsf{t}_k,\bar{p}(\mathsf{t}_k))\Big| \\
\le&\int_0^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}\mathfrak{I}^{\alpha}\Big|f(s,p(s),q(s)) - f(s,\bar{p}(s),\bar{q}(s))\Big|ds \\
&+|A^{\lambda}|\int_0^{s_0} e^{-\lambda(s_0-s)}\mathfrak{I}^{\alpha}\Big|f(s,p(s),q(s)) - f(s,\bar{p}(s),\bar{q}(s))\Big|ds \\
&+\Big|\mathcal{N}_k(\mathsf{t},p(\mathsf{t}))-\mathcal{N}_k(\mathsf{t},\bar{p}(\mathsf{t}))\Big| + |\delta_k^{\lambda}|\Big|\mathcal{N}_k(\mathsf{t}_k,p(\mathsf{t}_k))-\mathcal{N}_k(\mathsf{t}_k,\bar{p}(\mathsf{t}_k))\Big| \\
\le&\int_0^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}ds\int_0^{\mathsf{t}}\mathfrak{I}^{\alpha}\mathcal{L}_f ds|(p-\bar{p},q-\bar{q})| \\
&+|A^{\lambda}|\int_0^{s_0} e^{-\lambda(s_0-s)}ds\int_0^{s_0}\mathfrak{I}^{\alpha}\mathcal{L}_f ds|(p-\bar{p},q-\bar{q})| + \mathcal{L}_{\mathcal{N}}(1+|\delta_k^{\lambda}|)|p-\bar{p}|.
\end{aligned}$$

Applying $\sup\limits_{\mathsf{t}>0}$, we have

$$\begin{aligned}
\|\mathbb{F}_1(p,q)-\mathbb{F}_1(\bar{p},\bar{q})\|_{\mathcal{PC}} \le&\Big[\frac{\mathcal{L}_f(1-e^{-\lambda\mathsf{t}})\mathsf{t}^{\alpha+1}}{\Gamma(\alpha+2)\lambda} + \frac{\mathcal{L}_f|A^{\lambda}|(1-e^{-\lambda s_0})\mathsf{t}^{\alpha+1}}{\Gamma(\alpha+2)\lambda}\Big]\big\|(p-\bar{p},q-\bar{q})\big\| \\
&+\mathcal{L}_{\mathcal{N}}(1+|\delta_k^{\lambda}|)\|p-\bar{p}\|.
\end{aligned}\tag{3.17}$$

Similarly

$$\begin{aligned}
\|\mathbb{F}_2(p,q)-\mathbb{F}_2(\bar{p},\bar{q})\|_{\mathcal{PC}} \le&\Big[\frac{\mathcal{L}_g(1-e^{-\mu\mathsf{t}})\mathsf{t}^{\beta+1}}{\Gamma(\beta+2)\mu} + \frac{\mathcal{L}_g|A^{\mu}|(1-e^{-\mu s_0})\mathsf{t}^{\beta+1}}{\Gamma(\beta+2)\mu}\Big]\big\|(p-\bar{p},q-\bar{q})\big\| \\
&+\mathcal{L}_{\mathcal{M}}(1+|\delta_k^{\mu}|)\|q-\bar{q}\|.
\end{aligned}\tag{3.18}$$

Combining (3.17) and (3.18), we obtain

$$\begin{aligned}\|\mathbb{F}(p,q)-\mathbb{F}(\bar{p},\bar{q})\|_{\mathcal{PC}} \leq & \Bigg[\mathcal{L}_f \frac{(1-e^{-\lambda \mathsf{t}})+|A^{\lambda}|(1-e^{-\lambda s_0})}{\Gamma(\alpha+2)\lambda}\mathsf{t}^{\alpha+1}+\mathcal{L}_{\mathcal{N}}(1+|\delta_k^{\lambda}|)\\ & +\mathcal{L}_g \frac{(1-e^{-\mu \mathsf{t}})+|A^{\mu}|(1-e^{-\mu s_0})}{\Gamma(\beta+2)\mu}\mathsf{t}^{\beta+1}\Bigg]\|p-\bar{p}\|\\ & +\Bigg[\mathcal{L}_f \frac{(1-e^{-\lambda \mathsf{t}})+|A^{\lambda}|(1-e^{-\lambda s_0})}{\Gamma(\alpha+2)\lambda}\mathsf{t}^{\alpha+1}+\mathcal{L}_{\mathcal{M}}(1+|\delta_k^{\mu}|)\\ & +\mathcal{L}_g \frac{(1-e^{-\mu \mathsf{t}})+|A^{\mu}|(1-e^{-\mu s_0})}{\Gamma(\beta+2)\mu}\mathsf{t}^{\beta+1}\Bigg]\|q-\bar{q}\|\\ \leq & \varrho^{*}\|p-\bar{p}\|+\varrho^{**}\|q-\bar{q}\|.\end{aligned}$$

Which implies that

$$\|\mathbb{F}(p,q)-\mathbb{F}(\bar{p},\bar{q})\|_{\mathcal{PC}} \leq \varrho\|(p-\bar{p},q-\bar{q})\|,$$

where $0<\varrho=\max\{\varrho^{*},\varrho^{**}\}<1$. Therefore, $\mathbb{F}$ is contractive.

Next, in order to prove the continuity and compactness of operator $\mathbb{G}$, we consider a sequence $\{X_n=(p_n,q_n)\}$ in $\mathcal{B}_r$ with $(p_n,q_n)\to(p,q)$ as $n\to\infty$ in $\mathcal{B}_r$. Thus, we have

$$\begin{aligned}&|\mathbb{G}(p_n,q_n)(\mathsf{t})-\mathbb{G}(p,q)(\mathsf{t})|\\ \leq & |\mathbb{G}_1(p_n,q_n)(\mathsf{t})-\mathbb{G}_1(p,q)(\mathsf{t})|+|\mathbb{G}_2(p_n,q_n)(\mathsf{t})-\mathbb{G}_2(p,q)(\mathsf{t})|\\ \leq & \int_{\mathsf{t}_k}^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}\mathfrak{I}^{\alpha}|f(s,p_n(s),q_n(s))-f(s,p(s),q(s))|ds\\ & +|B_k^{\lambda}|\int_{\mathsf{t}_k}^{s_k} e^{-\lambda(s_k-s)}\mathfrak{I}^{\alpha}|f(s,p_n(s),q_n(s))-f(s,p(s),q(s))|ds\\ & +\int_{\mathsf{t}_k}^{\mathsf{t}} e^{-\mu(\mathsf{t}-s)}\mathfrak{I}^{\beta}|g(s,p_n(s),q_n(s))-g(s,p(s),q(s))|ds\\ & +|B_k^{\mu}|\int_{\mathsf{t}_k}^{s_k} e^{-\mu(s_k-s)}\mathfrak{I}^{\beta}|g(s,p_n(s),q_n(s))-g(s,p(s),q(s))|ds\\ \leq & \Bigg[|B_k^{\lambda}|\int_{\mathsf{t}_k}^{s_k} e^{-\lambda(s_k-s)}ds\int_{\mathsf{t}_k}^{s_k}\mathfrak{I}^{\alpha}\mathcal{L}_f ds+|B_k^{\mu}|\int_{\mathsf{t}_k}^{s_k} e^{-\mu(s_k-s)}ds\int_{\mathsf{t}_k}^{s_k}\mathfrak{I}^{\beta}\mathcal{L}_g ds\\ & +\int_{\mathsf{t}_k}^{\mathsf{t}} e^{-\lambda(\mathsf{t}-s)}ds\int_{\mathsf{t}_k}^{\mathsf{t}}\mathfrak{I}^{\alpha}\mathcal{L}_f ds+\int_{\mathsf{t}_k}^{\mathsf{t}} e^{-\mu(\mathsf{t}-s)}ds\int_{\mathsf{t}_k}^{\mathsf{t}}\mathfrak{I}^{\beta}\mathcal{L}_g ds\Bigg]|(p_n-p,q_n-q)|.\end{aligned}$$

Now applying $\sup\limits_{\mathsf{t}>0}$, we get

$$\begin{aligned}&\|\mathbb{G}(p_n,q_n)(\mathsf{t})-\mathbb{G}(p,q)(\mathsf{t})\|_{\mathcal{PC}}\\ \leq & \Bigg[\mathcal{L}_f \frac{|B_k^{\lambda}|(s_k-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(s_k-\mathsf{t}_k)})+(\mathsf{t}-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\\ & +\mathcal{L}_g \frac{|B_k^{\mu}|(s_k-\mathsf{t}_k)^{\beta+1}(1-e^{-\mu(s_k-\mathsf{t}_k)})+(\mathsf{t}-\mathsf{t}_k)^{\beta+1}(1-e^{-\mu(\mathsf{t}-\mathsf{t}_k)})}{\Gamma(\beta+2)\mu}\Bigg]\|(p_n-p,q_n-q)\|.\end{aligned}$$

This implies that, $\|\mathbb{G}(p_n,q_n)(\mathsf{t})-\mathbb{G}(p,q)(\mathsf{t})\|_{\mathcal{PC}}\to 0$ as $n\to\infty$. Therefore, the operator $\mathbb{G}=(\mathbb{G}_1,\mathbb{G}_2)$ is continuous.
Here, we have to show that $\mathbb{G}$ is uniformly bounded on $\mathcal{B}_r$. From (3.15) and (3.16), we obtain

$$\begin{aligned}\|\mathbb{G}(p,q)\|_{\mathcal{PC}}\leq&\|\mathbb{G}_1(p,q)\|_{\mathcal{PC}}+\|\mathbb{G}_2(p,q)\|_{\mathcal{PC}}\\ \leq&\Bigg[\omega_1\phi_f^*\frac{(1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)})+|B_k^\lambda|(1-e^{-\lambda(s_k-\mathsf{t}_k)})}{\lambda}\\ &+\omega_3\phi_g^*\frac{(1-e^{-\mu(\mathsf{t}-\mathsf{t}_k)})+|B_k^\mu|(1-e^{-\mu(s_k-\mathsf{t}_k)})}{\mu}\Bigg]\\ &+\Bigg[\omega_2\widetilde{\phi}_f^*\frac{(1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)})+|B_k^\lambda|(1-e^{-\lambda(s_k-\mathsf{t}_k)})}{\lambda}\\ &+\omega_4\widetilde{\phi}_g^*\frac{(1-e^{-\mu(\mathsf{t}-\mathsf{t}_k)})+|B_k^\mu|(1-e^{-\mu(s_k-\mathsf{t}_k)})}{\mu}\Bigg]r.\end{aligned}$$

Thus, $\mathbb{G}$ is uniformly bounded operator on $\mathcal{B}_r$.

Now for equi-continuity, take $\tau_1,\tau_2\in\mathcal{J}$ with $\tau_2<\tau_1$ and for any $(p,q)\in\mathcal{B}_r\subset\mathbf{E}$, where $\mathcal{B}_r$ is clearly bounded, we have

$$\begin{aligned}&\Big|\mathbb{G}_1(p,q)(\tau_1)-\mathbb{G}_1(p,q)(\tau_2)\Big|\\ \leq&\Big|\int_{\mathsf{t}_k}^{\tau_1}e^{-\lambda(\tau_1-s)}\mathfrak{I}^\alpha f(s,p(s),q(s))ds-\int_{\mathsf{t}_k}^{\tau_2}e^{-\lambda(\tau_2-s)}\mathfrak{I}^\alpha f(s,p(s),q(s))ds\Big|.\end{aligned}$$

This implies that $\|\mathbb{G}_1(p,q)(\tau_1)-\mathbb{G}_1(p,q)(\tau_2)\|_{\mathcal{PC}}\to 0$ as $\tau_1\to\tau_2$. On the same way, we have $\|\mathbb{G}_2(p,q)(\tau_1)-\mathbb{G}_2(p,q)(\tau_2)\|_{\mathcal{PC}}\to 0$ as $\tau_1\to\tau_2$. Hence $\|\mathbb{G}(p,q)(\tau_1)-\mathbb{G}(p,q)(\tau_2)\|_{\mathcal{PC}}\to 0$ as $\tau_1\to\tau_2$. Therefore, $\mathbb{G}$ is relatively compact on $\mathcal{B}_r$. By *Arzelä-Ascolli* theorem, $\mathbb{G}$ is compact and hence completely continuous operator. So there exist at least one solution of coupled BVP (1.1). □

Theorem 3.3. *Suppose that the assumptions* $(H_1)-(H_2)$ *holds and if* $\vartheta^*<1$. *Then the coupled problem* (1.1) *has a unique solution.*

Proof . Define the operator $\mathcal{Z}=(\mathcal{Z}_1,\mathcal{Z}_2):\mathbf{E}\to\mathbf{E}$, i.e. $\mathcal{Z}(p,q)(\mathsf{t})=\big(\mathcal{Z}_1(p,q)(\mathsf{t}),\mathcal{Z}_2(p,q)(\mathsf{t})\big)$, for each $\mathsf{t}\in\mathcal{J}$, where

$$\begin{aligned}\mathcal{Z}_1(p,q)(\mathsf{t})=&\int_0^{\mathsf{t}}e^{-\lambda(\mathsf{t}-s)}\mathfrak{I}^\alpha f(s,p(s),q(s))ds+A^\lambda\int_0^{s_0}e^{-\lambda(s_0-s)}\mathfrak{I}^\alpha f(s,p(s),q(s))ds\\ &+B_k^\lambda\int_{\mathsf{t}_k}^{s_k}e^{-\lambda(s_k-s)}\mathfrak{I}^\alpha f(s,p(s),q(s))ds+\int_{\mathsf{t}_k}^{\mathsf{t}}e^{-\lambda(\mathsf{t}-s)}\mathfrak{I}^\alpha f(s,p(s),q(s))ds\\ &+\mathcal{N}_k(\mathsf{t},p(\mathsf{t}))+\delta_k^\lambda\mathcal{N}_k(\mathsf{t}_k,p(\mathsf{t}_k))\end{aligned}$$

and

$$\begin{aligned}\mathcal{Z}_2(p,q)(\mathsf{t})=&\int_0^{\mathsf{t}}e^{-\mu(\mathsf{t}-s)}\mathfrak{I}^\beta g(s,p(s),q(s))ds+A^\mu\int_0^{s_0}e^{-\mu(s_0-s)}\mathfrak{I}^\beta g(s,p(s),q(s))ds\\ &+B_k^\mu\int_{\mathsf{t}_k}^{s_k}e^{-\mu(s_k-s)}\mathfrak{I}^\beta g(s,p(s),q(s))ds+\int_{\mathsf{t}_k}^{\mathsf{t}}e^{-\mu(\mathsf{t}-s)}\mathfrak{I}^\beta g(s,p(s),q(s))ds\\ &+\mathcal{N}_k(\mathsf{t},p(\mathsf{t}))+\delta_k^\mu\mathcal{N}_k(\mathsf{t}_k,p(\mathsf{t}_k)).\end{aligned}$$

In view of Theorem 3.2, we have

$$\begin{aligned}
&\|\mathcal{Z}_1(p,q)-\mathcal{Z}_1(\bar{p},\bar{q})\|_{\mathcal{PC}}\\
\leq&\mathcal{L}_f\Bigg[\frac{(1-e^{-\lambda \mathsf{t}})\mathsf{t}^{\alpha+1}+(\mathsf{t}-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\\
&+\frac{|A^\lambda|(1-e^{-\lambda s_0})\mathsf{t}^{\alpha+1}+|B_k^\lambda|(s_k-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(s_k-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\Bigg]\big\|(p-\bar{p},q-\bar{q})\big\|\\
&+\mathcal{L}_{\mathcal{N}}(1+|\delta_k^\lambda|)\|p-\bar{p}\|\\
\leq&\vartheta_1^*\big\|(p-\bar{p},q-\bar{q})\big\|
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathcal{Z}_2(p,q)-\mathcal{Z}_2(\bar{p},\bar{q})\|_{\mathcal{PC}}||\\
\leq&\mathcal{L}_g\Bigg[\frac{(1-e^{-\mu \mathsf{t}})\mathsf{t}^{\beta+1}+(\mathsf{t}-\mathsf{t}_k)^{\beta+1}(1-e^{-\mu(\mathsf{t}-\mathsf{t}_k)})}{\Gamma(\beta+2)\mu}\\
&+\frac{|A^\mu|(1-e^{-\mu s_0})\mathsf{t}^{\beta+1}+|B_k^\mu|(s_k-\mathsf{t}_k)^{\beta+1}(1-e^{-\mu(s_k-\mathsf{t}_k)})}{\Gamma(\beta+2)\mu}\Bigg]\big\|(p-\bar{p},q-\bar{q})\big\|\\
&+\mathcal{L}_{\mathcal{M}}(1+|\delta_k^\mu|)\|q-\bar{q}\|\\
\leq&\vartheta_2^*\big\|(p-\bar{p},q-\bar{q})\big\|.
\end{aligned}$$

Hence

$$\|\mathcal{Z}(p,q)-\mathcal{Z}(\bar{p},\bar{q})\|_{\mathcal{PC}}\leq\vartheta^*\|(p,q)-(\bar{p},\bar{q})\|,$$

where $\vartheta^*=\max\{\vartheta_1^*,\vartheta_2^*\}$. This implies that $\mathcal{Z}$ is contractive operator. Therefore, (1.1) has a unique solution. □

4. Hyers–Ulam stability analysis

In this portion, we analyze HU stability for the oupled system (1.1).

Theorem 4.1. *Suppose that the assumption* $(\mathbf{H}_1)$ *to* $(\mathbf{H}_4)$ *holds and* $\vartheta^*<1$ *along with the condition that the system corresponding to the matrix* $\mathbb{Q}$ *is converging to zero. Then the solution of* (1.1) *is HU stable.*

Proof . From Theorem 3.2, we have

$$\begin{aligned}
&\|\mathcal{Z}_1(p,q)-\mathcal{Z}_1(p^*,q^*)\|_{\mathcal{PC}}\\
\leq&\mathcal{L}_f\Bigg[\frac{|A^\lambda|(1-e^{-\lambda s_0})\mathsf{t}^{\alpha+1}+|B_k^\lambda|(s_k-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(s_k-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\\
&+\frac{(1-e^{-\lambda \mathsf{t}})\mathsf{t}^{\alpha+1}+(\mathsf{t}-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\Bigg]\big(\|p-p^*\|+\|q-q^*\|\big)\\
&+\mathcal{L}_{\mathcal{N}}(1+|\delta_k^\lambda|)\|p-p^*\|
\end{aligned}$$

$$\begin{aligned}
\leq&\Bigg[\mathcal{L}_f\Bigg(\frac{|A^\lambda|(1-e^{-\lambda s_0})\mathsf{t}^{\alpha+1}+|B_k^\lambda|(s_k-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(s_k-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\\
&+\frac{(1-e^{-\lambda\mathsf{t}})\mathsf{t}^{\alpha+1}+(\mathsf{t}-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\Bigg)+\mathcal{L}_{\mathcal{N}}(1+|\delta_k^\lambda|)\Bigg]\|p-p^*\|\\
&+\mathcal{L}_f\Bigg(\frac{|A^\lambda|(1-e^{-\lambda s_0})\mathsf{t}^{\alpha+1}+|B_k^\lambda|(s_k-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(s_k-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\\
&+\frac{(1-e^{-\lambda\mathsf{t}})\mathsf{t}^{\alpha+1}+(\mathsf{t}-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\Bigg)\|q-q^*\|\\
\leq&\mathcal{V}_1\|p-p^*\|+\mathcal{V}_2\|q-q^*\|, \qquad (4.1)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{V}_1=&\mathcal{L}_f\Bigg(\frac{|A^\lambda|(1-e^{-\lambda s_0})\mathsf{t}^{\alpha+1}+|B_k^\lambda|(s_k-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(s_k-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\\
&+\frac{(1-e^{-\lambda\mathsf{t}})\mathsf{t}^{\alpha+1}+(\mathsf{t}-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\Bigg)+\mathcal{L}_{\mathcal{N}}(1+|\delta_k^\lambda|)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}_2=&\mathcal{L}_f\Bigg(\frac{|A^\lambda|(1-e^{-\lambda s_0})\mathsf{t}^{\alpha+1}+|B_k^\lambda|(s_k-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(s_k-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\\
&+\frac{(1-e^{-\lambda\mathsf{t}})\mathsf{t}^{\alpha+1}+(\mathsf{t}-\mathsf{t}_k)^{\alpha+1}(1-e^{-\lambda(\mathsf{t}-\mathsf{t}_k)})}{\Gamma(\alpha+2)\lambda}\Bigg).
\end{aligned}$$

In the same fashion, we can obtain

$$\|\mathcal{Z}_2(p,q)-\mathcal{Z}_2(p^*,q^*)\|_{\mathcal{PC}}\leq\mathcal{V}_3\|p-p^*\|+\mathcal{V}_4\|q-q^*\|. \qquad (4.2)$$

where

$$\begin{aligned}
\mathcal{V}_3=&\mathcal{L}_g\Bigg(\frac{(1-e^{-\mu\mathsf{t}})\mathsf{t}^{\beta+1}+(\mathsf{t}-\mathsf{t}_k)^{\beta+1}(1-e^{-\mu(\mathsf{t}-\mathsf{t}_k)})}{\Gamma(\beta+2)\mu}\\
&+\frac{|A^\mu|(1-e^{-\mu s_0})\mathsf{t}^{\beta+1}+|B_k^\mu|(s_k-\mathsf{t}_k)^{\beta+1}(1-e^{-\mu(s_k-\mathsf{t}_k)})}{\Gamma(\beta+2)\mu}\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}_4=&\mathcal{L}_g\Bigg(\frac{(1-e^{-\mu\mathsf{t}})\mathsf{t}^{\beta+1}+(\mathsf{t}-\mathsf{t}_k)^{\beta+1}(1-e^{-\mu(\mathsf{t}-\mathsf{t}_k)})}{\Gamma(\beta+2)\mu}\\
&+\frac{|A^\mu|(1-e^{-\mu s_0})\mathsf{t}^{\beta+1}+|B_k^\mu|(s_k-\mathsf{t}_k)^{\beta+1}(1-e^{-\mu(s_k-\mathsf{t}_k)})}{\Gamma(\beta+2)\mu}\Bigg)+\mathcal{L}_{\mathcal{M}}(1+|\delta_k^\mu|).
\end{aligned}$$

Thus from the above two equations (4.1) and (4.2), we obtain the following inequalities

$$\begin{aligned}
\|\mathcal{Z}_1(p,q)-\mathcal{Z}_1(p^*,q^*)\|_{\mathcal{PC}}\leq&\mathcal{V}_1\|p-p^*\|+\mathcal{V}_2\|q-q^*\|\\
\|\mathcal{Z}_2(p,q)-\mathcal{Z}_2(p^*,q^*)\|_{\mathcal{PC}}\leq&\mathcal{V}_3\|p-p^*\|+\mathcal{V}_4\|q-q^*\|.
\end{aligned}$$

From these inequalities, we get

$$\|\mathcal{Z}(p,q)-\mathcal{Z}(p^*,q^*)\|_{\mathcal{PC}}\leq\|(p,q)-(p^*,q^*)\|\mathbb{Q},$$

where

$$\mathbb{Q} = \begin{pmatrix} \mathcal{V}_1 & \mathcal{V}_2 \\ \mathcal{V}_3 & \mathcal{V}_4 \end{pmatrix}.$$

With the help of Definition 2.8 and Theorem 2.9, we conclude that coupled BVP (1.1) is Hyers–Ulam stable. □

5. Example

In this section, we are illustrating our main result by an example.

Example 5.1. *Consider the BVP*

$$\begin{cases} {}^c\mathcal{D}^{\frac{1}{2}}(\mathcal{D}+2)p(\mathsf{t}) = \dfrac{e^{-\mathsf{t}}\sin|p(\mathsf{t})| + |q(\mathsf{t})|}{20+t^2}, & \mathsf{t} \in (0,1)\cup(2,3), \\ {}^c\mathcal{D}^{\frac{2}{3}}(\mathcal{D}+1)q(\mathsf{t}) = \dfrac{1+|p(\mathsf{t})| + \cos|q(\mathsf{t})|}{30+e^{\mathsf{t}}+\mathsf{t}^2}, & \mathsf{t} \in (0,1)\cup(2,3), \\ p(\mathsf{t}) = \dfrac{|p(\mathsf{t})|}{(5+\mathsf{t}^2)(1+|p(\mathsf{t})|)}, & (1,2], \\ q(\mathsf{t}) = \dfrac{1+|q(\mathsf{t})|}{(9+\mathsf{t}^3)(2+3|q(\mathsf{t})|)}, & (1,2], \\ p(0)=p(3)=0, \quad q(0)=q(3)=0. \end{cases} \tag{5.1}$$

From the above system (5.1)*, we see that* $\alpha = \frac{1}{2}$*,* $\beta = \frac{2}{3}$*,* $\lambda = 2$*,* $\mu = 1$ *and the nonlinear functions* $f(\mathsf{t}, p(\mathsf{t}), q(\mathsf{t})) = \frac{e^{-\mathsf{t}}\sin|p(\mathsf{t})|+|q(\mathsf{t})|}{20+t^2}$ *and* $g(\mathsf{t}, p(\mathsf{t}), q(\mathsf{t})) = \frac{1+|p(\mathsf{t})|+\cos|q(\mathsf{t})|}{30+e^{\mathsf{t}}+\mathsf{t}^2}$.

By Lemma 3.1, we get the following integral equations

$$p(\mathsf{t}) = \begin{cases} \int_0^{\mathsf{t}} e^{-2(\mathsf{t}-s)}\mathfrak{I}^{\frac{1}{2}}\frac{e^{-s}\sin|p(s)|+|q(s)|}{20+s^2}ds + A^2\int_0^1 e^{-2(1-s)}\mathfrak{I}^{\frac{1}{2}}\frac{e^{-s}\sin|p(s)|+|q(s)|}{20+s^2}ds; & \mathsf{t}\in[0,1) \\ \frac{|p(\mathsf{t})|}{(5+t^2)(1+|p(\mathsf{t})|)}; \quad \mathsf{t}\in(1,2] \\ \int_2^{\mathsf{t}} e^{-2(\mathsf{t}-s)}\mathfrak{I}^{\frac{1}{2}}\frac{e^{-s}\sin|p(s)|+|q(s)|}{20+s^2}ds + B_1^2\int_2^3 e^{-2(3-s)}\mathfrak{I}^{\frac{1}{2}}\frac{e^{-s}\sin|p(s)|+|q(s)|}{20+s^2}ds \\ +\delta_1^2\frac{|p(2)|}{(5+2^2)(1+|p(2)|)}; \quad \mathsf{t}\in(2,3), \end{cases} \tag{5.2}$$

$$q(\mathsf{t}) = \begin{cases} \int_0^{\mathsf{t}} e^{-(\mathsf{t}-s)}\mathfrak{I}^{\frac{2}{3}}\frac{1+|p(s)|+\cos|q(s)|}{30+e^s+s^2}ds + A\int_0^1 e^{-(1-s)}\mathfrak{I}^{\frac{2}{3}}\frac{1+|p(s)|+\cos|q(s)|}{30+e^s+s^2}ds; & \mathsf{t}\in[0,1) \\ \frac{1+|q(\mathsf{t})|}{(9+t^3)(2+3|q(\mathsf{t})|)}; \quad \mathsf{t}\in(1,2] \\ \int_2^{\mathsf{t}} e^{-(\mathsf{t}-s)}\mathfrak{I}^{\frac{2}{3}}\frac{1+|p(s)|+\cos|q(s)|}{30+e^s+s^2}ds + B_1\int_2^3 e^{-(3-s)}\mathfrak{I}^{\frac{2}{3}}\frac{1+|p(s)|+\cos|q(s)|}{30+e^s+s^2}ds \\ +\delta_1\frac{1+|q(2)|}{(9+2^3)(2+3|q(2)|)}; \quad \mathsf{t}\in(2,3]. \end{cases} \tag{5.3}$$

where

$$\begin{aligned} A^2 &= \frac{1-e^{-2\mathsf{t}}}{e^{-2}-1}, & A &= \frac{1-e^{-\mathsf{t}}}{e^{-1}-1} \\ B_1^2 &= \frac{1-e^{-2(\mathsf{t}-2)}}{e^{-2(3-2)}-1}, & B_1 &= \frac{1-e^{-(\mathsf{t}-2)}}{e^{-(3-2)}-1} \\ \delta_1^2 &= \frac{1-e^{-2(\mathsf{t}-3)}}{1-e^{-2(3-2)}} \ \textit{and}\ & \delta_1 &= \frac{1-e^{-(\mathsf{t}-3)}}{1-e^{-(3-2)}}. \end{aligned}$$

For Theorem 3.3, we find $\mathcal{L}_f = \frac{1}{20}$, $\mathcal{L}_g = \frac{1}{30}$, $\mathcal{L}_{\mathcal{N}} = \frac{1}{4}$ *and* $\mathcal{L}_{\mathcal{M}} = \frac{1}{9}$. *We easily get* $\vartheta^* = 0.2876$, *which shows that* (5.1) *has a unique solution. By using Theorem 4.1, we find* $\mathcal{V}_1 = 0.5366$, $\mathcal{V}_2 = -0.0054$, $\mathcal{V}_3 = -0.1639$ *and* $\mathcal{V}_4 = 0.5928$. *On calculation, we get the eigenvalues as* 0.60572, 0.56133, *which shows that the matrix* $\mathbb{Q}$ *converges to* 0, *and by applying Theorem 4.1, the solution of the coupled BVP* (5.1) *is HU stable.*

Conclusion

With the help of Krasnoselskii's fixed point theorem and Banach contraction principle, we presented sufficient conditions for the existence and uniqueness of solution of proposed BVP(1.1). Likewise under specific assumptions and conditions, we gave the HU stability of mentioned coupled BVP(1.1).

Conflict of interests

The author says publicly that there is no contending interest concerning the paper.

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