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System of generalized nonlinear variational-like inclusion problems in 2-uniformly smooth Banach spaces

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Abstract

In this manuscript, we introduce and study the existence of a solution of a system of generalized nonlinear variational-like inclusion problems in 2-uniformly smooth Banach spaces by using $H(.,.)$-η-proximal mapping. The method used in this paper can be considered as an extension of methods for studying the existence of solutions of various classes of variational inclusions considered and studied by many authors in 2-uniformly smooth Banach spaces. Some important results, theorems and the existence of solution of the proposed system of generalized nonlinear variational-like inclusion problems have been derived.

Keywords: (System of generalized nonlinear variational-like inclusion problems, $H(.,.)$-η-Proximal mapping method, 0-Diagonally Quasi-concave (0-$DQCV$), 2-uniformly smooth Banach spaces, Iterative algorithm, Convergence analysis.)
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1. Introduction

A widely studied problem known as variational inclusion problem has many applications in the fields of optimization and control, economics and transportation equilibrium, engineering sciences, etc. Several researchers used different approaches to develop iterative algorithms for solving various classes of variational inequality and variational inclusion problems. For details, we refer to [1-3,9-13,16,18,20,22,24,25,28,29]. In 2014, Sahu *et al.*[21] proved the existence of solution for a class of variational inclusions using fixed point theory in 2-uniformly smooth Banach spaces.

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From the above results, in this manuscript we intend to define $H(.,.)$-η-proximal mapping for a nonconvex, proper, lower semicontinuous and subdifferentiable functional in 2-uniformly smooth Banach spaces. We suggest an iterative algorithm for detecting the approximate solution of the system of variational inclusions and examine the convergence of sequences generated by iterative algorithm. The results presented in this paper generalize many known results in the literature [1,13,16,18,28,29].

2. $H(.,.)$-η-Proximal Mapping and Formulation of Problem

Let X be a real 2-uniformly smooth Banach space equipped with norm $\|.\|$ and a semi-inner product $[.,.]$. Let $C(X)$ be the family of all nonempty compact subsets of X and 2^X be the power set of X. We need the following definitions and results from the literature.

Definition 2.1[17]. Let X be a vector space over the field F of real or complex numbers. A functional $[.,.]: X \times X \to F$ is called a semi-inner product if it satisfies the following:

(i) $[x+y,z] = [x,z] + [y,z], \quad \forall x,y,z \in X$;

(ii) $[\lambda x, y] = \lambda[x,y], \quad \forall \lambda \in F$ and $x, y \in X$;

(iii) $[x,x] > 0$, for $x \neq 0$;

(iv) $|[x,y]|^2 \leq [x,x][y,y]$.

The pair $(X, [.,.])$ is called a semi-inner product space.

We observe that $||x|| = [x,x]^{\frac{1}{2}}$ is a norm on X. Hence every semi-inner product space is a normed linear space. On the other hand, in a normed linear space, one can generate semi-inner product in infinitely many different ways. Giles [7] had proved that if the underlying space X is a uniformly convex smooth Banach space then it is possible to find a semi-inner product, uniquely. Also the unique semi-inner product has the following nice properties:

(i) $[x,y] = 0$ if and only if y is orthogonal to x, that is if and only if $||y|| \leq ||y + \lambda x||, \quad \forall$ scalars λ.

(ii) Generalized Riesz representation theorem: If f is a continuous linear functional on X then there is a unique vector $y \in X$ such that $f(x) = [x,y], \quad \forall x \in X$.

(iii) The semi-inner product is continuous, that is for each $x, y \in X$, we have $\text{Re}[y, x+\lambda y] \to \text{Re}[y,x]$ as $\lambda \to 0$.

The sequence space $l^p,\ p > 1$ and the function space $L^p,\ p > 1$ are uniformly convex smooth Banach spaces. So one can define semi-inner product on these spaces, uniquely.

Example 2.2[21]. The real sequence space l^p for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[x,y] = \frac{1}{||y||_p^{p-2}} \sum_i x_i y_i |y_i|^{p-2}, \quad x, y \in l^p.$$

Example 2.3[7,21]. The real Banach space $L^p(X,\mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[f,g] = \frac{1}{||g||_p^{p-2}} \int_X f(x)|g(x)|^{p-1}\text{sgn}(g(x))d\mu, \quad f, g \in L^p.$$

Definition 2.4[21,26]. Let X be a real Banach space. Then:

(i) The modulus of smoothness of X is defined as

$$\rho_X(t) = \sup\left\{ \frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1, ||y|| = t,\ t > 0 \right\}.$$

(ii) X is said to be uniformly smooth if $\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0$.

(iii) X is said to be p-uniformly smooth if there exists a positive real constant c such that $\rho_X(t) \leq c\,t^p,\ p > 1$. Clearly, X is 2-uniformly smooth if there exists a positive real constant c such that $\rho_X(t) \leq c\,t^2$.

Lemma 2.5[21,26]. Let $p > 1$ be a real number and X be a smooth Banach space. Then the following statements are equivalent:

(i) X is 2-uniformly smooth.

(ii) There is a constant $c > 0$ such that for every $x, y \in X$, the following inequality holds

$$||x+y||^2 \leq ||x||^2 + 2\langle y, f_x\rangle + c||y||^2,$$

where $f_x \in J(x)$ and $J(x) = \{x^\star \in X^\star : \langle x, x^\star\rangle = ||x||^2 \text{ and } ||x^\star|| = ||x||\}$ is the normalized duality mapping.

Remark 2.6[21]. Every normed linear space is a semi-inner product space (see[17]). In fact by Hahn Banach theorem, for each $x \in X$, there exists atleast one functional $f_x \in X^\star$ such that $\langle x, f_x\rangle = ||x||^2$. Given any such mapping f from X into $X^\star$, we can verify that $[y, x] = \langle y, f_x\rangle$ defines a semi-inner product. Hence we can write (ii) of above Lemma as

$$||x+y||^2 \leq ||x||^2 + 2[y, x] + c||y||^2, \quad \forall x, y \in X.$$

The constant c is chosen with best possible minimum value. We call c, as the constant of smoothness of X.

Definition 2.7. Let X be a real 2-uniformly smooth Banach space. Let $\eta : X \times X \to X$ be a single-valued mapping. A proper functional $\phi : X \to R \cup \{+\infty\}$ is said to be η-subdifferentiable at a point $x \in X$, if there exists a point $f^\star \in X$ such that

$$\phi(y) - \phi(x) \geq (f^\star, \eta(y, x)), \quad \forall y \in X,$$

where $f^\star$ is called η-subgradient of ϕ at x. The set of all η-subgradients of ϕ at x is denoted by $\partial\phi(x)$. The mapping $\partial\phi : X \to 2^X$ defined by

$$\partial\phi(x) = \{f^\star \in X : \phi(y) - \phi(x) \geq (f^\star, \eta(y, x)), \quad \forall y \in X\}$$

is called η-subdifferential of ϕ at x.

Definition 2.8[18,21]. Let X be a real 2-uniformly smooth Banach space. A mapping $T : X \to X$ is said to be:

(i) monotone, if $[Tx - Ty, x - y] \geq 0, \quad \forall x, y \in X$.

(ii) strictly monotone, if $[Tx - Ty, x - y] \geq 0, \quad \forall x, y \in X$. and equality holds if and only if $x = y$.

(iii) r-strongly monotone if there exists a positive constant $r > 0$ such that

$$[Tx - Ty, x - y] \geq r||x - y||^2, \quad \forall x, y \in X.$$

(iv) δ-Lipschitz continuous, if there exists a constant $\delta > 0$ such that

$$\|T(x) - T(y)\| \leq \delta \|x - y\|, \quad \forall x, y \in X.$$

(v) η-monotone, if $[Tx - Ty, \eta(x, y)] \geq 0, \quad \forall x, y \in X.$

(vi) strictly η-monotone, if $[Tx - Ty, \eta(x, y)] \geq 0, \quad \forall x, y \in X.$ and equality holds if and only if $x = y$.

(vii) r-strongly η-monotone if there exists a positive constant $r > 0$ such that

$$[Tx - Ty, \eta(x, y)] \geq r||x - y||^2, \quad \forall x, y \in X.$$

(viii) μ-cocoercive if there exists a constant $\mu > 0$ such that

$$[Tx - Ty, x - y] \geq \mu||Tx - Ty||^2, \quad \forall x, y \in X.$$

Definition 2.9[18,21]. Let X be a real 2-uniformly smooth Banach space. The mapping $M : X \to 2^X$ is said to be

(i) monotone if

$$[u - v, x - y] \geq 0, \quad \forall x, y \in X, u \in M(x), v \in M(y).$$

(ii) r-strongly monotone if there exists a positive constant $r > 0$ such that

$$[u - v, x - y] \geq r||x - y||^2, \quad \forall x, y \in X, u \in M(x), v \in M(y).$$

(iii) η-monotone if

$$[u - v, \eta(x, y)] \geq 0, \quad \forall x, y \in X, u \in M(x), v \in M(y).$$

(iv) r-strongly η-monotone if

$$[u - v, \eta(x, y)] \geq r||x - y||^2, \quad \forall x, y \in X, u \in M(x), v \in M(y).$$

Definition 2.10. Let $A, B : X \to X$ and $H, \eta : X \times X \to X$ be a single-valued mappings. Then

(i) $H(A, .)$ is said to be α-strongly η-monotone w.r.t A if there exists a constant $\alpha > 0$ satisfying

$$\langle H(Ax, u) - H(Ay, u), \eta(y, x) \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y, u \in X.$$

(ii) $H(., B)$ is said to be β-relaxed η-monotone w.r.t B if there exists a constant $\beta > 0$ satisfying

$$\langle H(u, Bx) - H(u, By), \eta(y, x) \rangle \geq -\beta \|x - y\|^2, \quad \forall x, y, u \in X.$$

(iii) $H(., .)$ is said to be $\alpha\beta$-symmetric η-monotone w.r.t A and B if $H(A, .)$ is α-strongly η-monotone w.r.t A and $H(., B)$ is β-relaxed η-monotone w.r.t B with $\alpha > \beta$ and $\alpha = \beta$ if and only if $x = y, \quad \forall x, y \in X$.

Definition 2.11[27]. A functional $f : X \times X \to R \cup \{+\infty\}$ is said to be 0-diagonally quasi-concave (in short, 0-$DQCV$) in the first argument, if for any finite set $\{x_1, \cdots, x_n\} \subset X$ and for any $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, $\min_{1 \leq i \leq n} f(x_i, y) \leq 0$ holds.

Lemma 2.12[6]. Let G be a nonempty convex subset of a topological vector space and $f : G \times G \to [-\infty, +\infty]$ be such that

(i) for each $x \in G,\ y \to f(x,y)$ is lower semicontinuous on each compact subset of G;

(ii) for each $y \in G,\ f(x,y)$ is 0-$DQCV$ in x;

(iii) there exists a nonempty convex subset G_0 of G and a nonempty compact subset K of G such that for each $y \in G \backslash K$, there exists $x \in c_0(G_0 \cup \{y\})$ satisfying $f(x,y) > 0$, where $c_0(X)$ denotes the convex hull of set X.

Then there exists $\tilde{y} \in G$ such that $f(x,\tilde{y}) \leq 0,\ \ \forall x \in G$.

Definition 2.13. Let $\eta : X \times X \to X$ and $A, B : X \to X$ be single-valued mappings. Let $\phi : X \to R \cup \{+\infty\}$ be a proper and η-subdifferentiable (not necessarily convex) functional and $H : X \times X \to X$ be a nonlinear mapping. If for any given point $t \in X$ and $\rho > 0$, there exists a unique point $x \in X$ satisfying

$$\langle H(Ax,Bx) - t, \eta(y,x)\rangle + \rho\phi(y) - \rho\phi(x) \geq 0,\quad \forall y \in X,$$

then the mapping $t \to x$, denoted by $R^{\partial\phi,\eta}_{H(.,.),\rho}(t)$, is called $H(.,.)$-η-proximal mapping of ϕ. Clearly, we have $t - H(Ax,Bx) \in \rho\partial\phi(x)$ and then it follows that

$$R^{\partial\phi,\eta}_{H(.,.),\rho}(t) = (H(A,B) + \rho\partial\phi)^{-1}(t).$$

Now we prove the following result which guarantees the existence of $H(.,.)$-η-proximal mapping of a proper, lower semicontinuous and η-subdifferentiable functional ϕ on 2-uniformly smooth Banach spaces.

Theorem 2.14. Let $\eta : X \times X \to X$ be τ-Lipschitz continuous such that $\eta(y,y') + \eta(y',y) = 0\ \ \forall y, y' \in X$; let $H : X \times X \to X$ be $\alpha\beta$-symmetric η-monotone continuous with respect to A and B, let for any given $t \in X$, the function $h(y,x) = \langle t - H(Ax,Bx), \eta(y,x)\rangle$ be $0 - DQCV$ in y and let $\phi : X \to R \cup \{+\infty\}$ be a proper, lower semicontinuous and η-subdifferentiable functional, which may not be convex. Then for any given constant $\rho > 0$ and $t \in X$, there exists a unique $x \in X$ such that

$$\langle H(Ax,Bx) - t, \eta(y,x)\rangle \geq \rho\phi(x) - \rho\phi(y),\quad \forall y \in X, \tag{2.1}$$

that is, $x = R^{\partial\phi,\eta}_{H(.,.),\rho}(t)$.

Proof. For any given $H : X \times X \to X$, $\rho > 0$ and $t \in X$, define a functional $f : X \times X \to R \cup \{+\infty\}$ by $f(y,x) = \langle t - H(Ax,Bx), \eta(y,x)\rangle + \rho\phi(x) - \rho\phi(y)\ \ \forall y, x \in X$. Since $H(.,.)$ and η are continuous and ϕ is lower semicontinuous, then for any $y \in X$, the mapping $x \mapsto f(y,x)$ is lower semicontinuous on X. Next, claim that $f(y,x)$ satisfies condition (ii) of Lemma 2.12. Indeed, let there exist a finite set $\{y_1, \cdots, y_m\} \subset X$ and $x_0 = \sum_{i=1}^{m} \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$ such that

$$\langle t - H(Ax_0,Bx_0), \eta(y_i,x_0)\rangle + \rho\phi(x_0) - \rho\phi(y_i) > 0,\quad \forall i = 1, 2, \cdots, m.$$

Since ϕ is η-subdifferentiable at x_0, there exists a point $f^\star \in X$ such that

$$\rho\phi(y_i) - \rho\phi(x_0) \geq \rho\langle f^\star, \eta(y_i,x_0)\rangle,\quad \forall i = 1, 2, \cdots, m.$$

Hence we must have

$$\langle t - H(Ax_0,Bx_0) - \rho f^\star, \eta(y_i,x_0)\rangle > 0.$$

On the other hand, since $h(y,x_0) = \langle t - H(Ax_0,Bx_0) - \rho f^\star, \eta(y,x_0)\rangle$ is $0 - DQCV$ in y, we have

$$0 < \sum_{n=0}^{\infty} \lambda_i \langle t - H(Ax_0,Bx_0) - \rho f^\star, \eta(y_i,x_0)\rangle$$

$$= \langle t - H(Ax_0, Bx_0) - \rho f^\star, \eta(x_0, x_0)\rangle = 0,$$

which is a contradiction. Hence $f(y, x)$ satisfies condition (ii) of Lemma 2.12. Now, take a fixed $\tilde{y} \in$ dom ϕ. Since ϕ is η-subdifferentiable at $\tilde{y}$, there exists $f^\star \in X$ such that

$$\begin{aligned} f(\tilde{y}, x) &= \langle t - H(Ax, Bx), \eta(\tilde{y}, x)\rangle + \rho\phi(x) - \rho\phi(\tilde{y}) \\ &\geq \langle H(A\tilde{y}, B\tilde{y}) - H(Ax, B\tilde{y}), \eta(\tilde{y}, x)\rangle \\ &\quad + \langle H(Ax, B\tilde{y}) - H(Ax, Bx), \eta(\tilde{y}, x)\rangle \\ &\quad + \langle t - H(A\tilde{y}, B\tilde{y}), \eta(\tilde{y}, x)\rangle + \rho\langle f^\star, \eta(x, \tilde{y})\rangle \\ &\geq \alpha\|\tilde{y} - x\|^2 - \beta\|\tilde{y} - x\|^2 - (\|t\| + \|H(A\tilde{y}, B\tilde{y})\| + \rho\|f^\star\|)\|\eta(\tilde{y}, x)\| \\ &\geq (\alpha - \beta)\|\tilde{y} - x\|^2 - \tau(\|t\| + \|H(A\tilde{y}, B\tilde{y})\| + \rho\|f^\star\|)\|\tilde{y} - x\| \\ &= \|\tilde{y} - x\|\{(\alpha - \beta)\|\tilde{y} - x\| - \tau(\|t\| + \|H(A\tilde{y}, B\tilde{y})\| + \rho\|f^\star\|)\}. \end{aligned}$$

Let $r = \dfrac{\tau}{(\alpha - \beta)}(\|t\| + \|H(A\tilde{y}, B\tilde{y})\| + \rho\|f^\star\|)$, and $K = \{x \in X, \|\tilde{y} - x\| \leq r\}$. Then $G_0 = \{\tilde{y}\}$ and K are both weakly compact convex subsets of X and for each $x \in X \backslash K$, there exists $\tilde{y} \in c_0(G_0 \cup \{\tilde{y}\})$ such that $f(\tilde{y}, x) > 0$. Hence all the conditions of Lemma 2.12 are satisfied. By Lemma 2.12, there exists $\tilde{x} \in X$ such that $f(y, \tilde{x}) \leq 0 \ \ \forall y \in X$, that is, for any given $t \in X$,

$$\langle H(A\tilde{x}, B\tilde{x}) - t, \eta(y, \tilde{x})\rangle \geq \rho\phi(\tilde{x}) - \rho\phi(y) \ \ \forall y \in X.$$

Next, we show that $\tilde{x}$ is a unique solution of (2.1). Suppose that $\tilde{x}_1, \tilde{x}_2 \in X$ are any two solutions of (2.1). Then we have, for any given $t \in X$,

$$\langle H(A\tilde{x}_1, B\tilde{x}_1) - t, \eta(y, \tilde{x}_1)\rangle \geq \rho\phi(\tilde{x}_1) - \rho\phi(y) \ \ \forall y \in X, \tag{2.2}$$

and

$$\langle H(A\tilde{x}_2, B\tilde{x}_2) - t, \eta(y, \tilde{x}_2)\rangle \geq \rho\phi(\tilde{x}_2) - \rho\phi(y) \ \ \forall y \in X. \tag{2.3}$$

Taking $y = \tilde{x}_2$ in (2.2) and $y = \tilde{x}_1$ in (2.3) and then adding the resulting inequalities, we obtain

$$\langle H(A\tilde{x}_1, B\tilde{x}_1) - H(A\tilde{x}_2, B\tilde{x}_2), \eta(\tilde{x}_2, \tilde{x}_1)\rangle \geq 0,$$

Since $\eta(y, y') + \eta(y', y) = 0 \ \ \forall y, y' \in X$ and $H : X \times X \to X$ is $\alpha\beta$-symmetric η-monotone continuous with respect to A and B, we have

$$\langle H(A\tilde{x}_1, B\tilde{x}_2) - H(A\tilde{x}_2, B\tilde{x}_2), \eta(\tilde{x}_1, \tilde{x}_2)\rangle + \langle H(A\tilde{x}_1, B\tilde{x}_1) - H(A\tilde{x}_1, B\tilde{x}_2), \eta(\tilde{x}_1, \tilde{x}_2)\rangle \leq 0,$$

thus

$$\alpha\|\tilde{x}_1 - \tilde{x}_2\|^2 - \beta\|\tilde{x}_1 - \tilde{x}_2\|^2 \leq 0.$$

That is $(\alpha - \beta)\|\tilde{x}_1 - \tilde{x}_2\|^2 \leq 0$, and hence we have $\tilde{x}_1 = \tilde{x}_2$. This completes the proof.

Remark 2.15. Theorem 2.14 shows that for any $\alpha\beta$-symmetric η-monotone mapping $H : X \times X \to X$ and $\rho > 0$, the $H(.,.)$-η-proximal mapping $R^{\partial\phi,\eta}_{H(.,.),\rho} : X \to X$ of a proper, lower semicontinuous and η-subdifferentiable functional ϕ is well defined and for each $t \in X, x = R^{\partial\phi,\eta}_{H(.,.),\rho}(t)$ is the unique solution of (2.1).

Now, we give the following important result which guarantees the Lipschitz continuity of the $H(.,.)$-η-proximal mapping.

Theorem 2.16. Let $\eta : X \times X \to X$ be τ-Lipschitz continuous such that $\eta(y, y') + \eta(y', y) = 0\ \ \forall y, y' \in X$, $H : X \times X \to X$ be $\alpha\beta$-symmetric η-monotone continuous with respect to A and B and for any given $t \in X$, the function $h(y, x) = \langle t - H(Ax, Bx), \eta(y, x)\rangle$ be $0 - DQCV$ in y. Suppose that $\phi : X \to R \cup \{+\infty\}$ be a proper, lower semicontinuous and η-subdifferentiable functional and $\rho > 0$ be any given constant. Then the $H(.,.)$-η-proximal mapping $R^{\partial\phi,\eta}_{H(.,.),\rho}$ of ϕ is $\frac{\tau}{(\alpha-\beta)}$-Lipschitz continuous, that is, for any $t_1, t_2 \in X$,

$$\|R^{\partial\phi,\eta}_{H(.,.),\rho}(t_1) - R^{\partial\phi,\eta}_{H(.,.),\rho}(t_2)\| \le \frac{\tau}{(\alpha-\beta)}\|t_1 - t_2\|.$$

Proof. For any given $t_1, t_2 \in X$, we have $x_1 = R^{\partial\phi,\eta}_{H(.,.),\rho}(t_1)$ and $x_2 = R^{\partial\phi,\eta}_{H(.,.),\rho}(t_2)$ such that

$$\langle H(Ax_1, Bx_1) - t_1, \eta(y, x_1)\rangle \ge \rho\phi(x_1) - \rho\phi(y), \quad \forall y \in X, \tag{2.4}$$

and

$$\langle H(Ax_2, Bx_2) - t_2, \eta(y, x_2)\rangle \ge \rho\phi(x_2) - \rho\phi(y), \quad \forall y \in X. \tag{2.5}$$

Taking $y = x_2$ in (2.4) and $y = x_1$ in (2.5) and then adding the resulting inequalities, we obtain

$$\langle H(Ax_1, Bx_1) - H(Ax_2, Bx_2), \eta(x_1, x_2)\rangle \le \langle t_1 - t_2, \eta(x_1, x_2)\rangle,$$

which implies

$$\begin{aligned}&\langle H(Ax_1, Bx_2) - H(Ax_2, Bx_2), \eta(x_1, x_2)\rangle \\ &\qquad + \langle H(Ax_1, Bx_1) - H(Ax_1, Bx_2), \eta(x_1, x_2)\rangle \le \|t_1 - t_2\|\|\eta(x_1, x_2)\|.\end{aligned}$$

Since $H(.,.)$ is $\alpha\beta$-symmetric η-monotone continuous with respect to A and B,

$$\alpha\|x_1 - x_2\|^2 - \beta\|x_1 - x_2\|^2 \le \tau\|t_1 - t_2\|\ \|x_1 - x_2\|$$

which implies

$$\|x_1 - x_2\| \le \frac{\tau}{(\alpha-\beta)}\|t_1 - t_2\|. \tag{2.6}$$

Thus

$$\|R^{\partial\phi,\eta}_{H(.,.),\rho}(t_1) - R^{\partial\phi,\eta}_{H(.,.),\rho}(t_2)\| \le \frac{\tau}{(\alpha-\beta)}\|t_1 - t_2\|.$$

Lemma 2.17[15]. Let $\{\zeta^n\}, \{\hbar^n\}$ and $\{c^n\}$ be nonnegative sequences satisfying

$$\zeta^{n+1} \le (1 - \omega^n)\zeta^n + \omega^n\hbar^n + c^n, \quad \forall n \ge 0,$$

where $\{\omega^n\}_{n=0}^{\infty} \subset [0, 1]$, $\sum_{n=0}^{\infty}\omega^n = +\infty$, $\lim_{n\to\infty}\hbar^n = 0$ and $\sum_{n=0}^{\infty} c^n < \infty$. Then $\lim_{n\to\infty}\zeta^n = 0$.

Definition 2.18. The Hausdorff metric $\mathcal{D}(\cdot,\cdot)$ on $CB(X)$, is defined by

$$\mathcal{D}(A, B) = \max\left\{\sup_{u\in A}\inf_{v\in B} d(u, v),\ \sup_{v\in B}\inf_{u\in A} d(u, v)\right\},\ A, B \in CB(X),$$

where $d(\cdot,\cdot)$ is the induced metric on X and $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X.

Definition 2.19[4]. A set-valued mapping $T : X \to CB(X)$ is said to be γ-$\mathcal{D}$-Lipschitz continuous, if there exists a constant $\gamma > 0$ such that

$$\mathcal{D}(T(x), T(y)) \le \gamma\|x - y\|, \quad \forall x, y \in X.$$

Theorem 2.20[19]. Let $T : X \to CB(X)$ be a set-valued mapping on X and (X, d) be a complete metric space. Then:

(i) For any given $\xi > 0$ and for any given $u, v \in X$ and $x \in T(u)$, there exists $y \in T(v)$ such that

$$d(x, y) \le (1 + \xi)\mathcal{D}(T(u), T(v)).$$

(ii) If $T : X \to C(X)$, then (i) holds for $\xi = 0$, (where $C(X)$ denotes the family of all nonempty compact subsets of X).

Example 2.21[21]. The function space L^p is 2-uniformly smooth for $p \ge 2$ and it is p-uniformly smooth for $1 < p < 2$. If $2 \le p < \infty$, then we have for all $x, y \in L^p$,

$$||x + y||^2 \le ||x||^2 + 2[y, x] + (p - 1)||y||^2.$$

Here the constant of smoothness is $p - 1$.

Now, we formulate our main problem.

For each $i = 1, 2, j \in \{1, 2\} \setminus i$, let X_i be a real 2-uniformly smooth Banach space. Let $g_i : X_i \to X_i, \eta_i : X_i \times X_i \to X_i,\ N_i : X_i \times X_j \to X_i, Q_i : X_i \times X_i \to X_i, E_i : X_i \to X_i, P_i : X_j \to X_i$ be single-valued mappings, let $S_i, T_i, G_i : X_i \to C(X_i)$ be multi-valued mappings such that $u_i \in S_i(x_i), v_i \in T_i(x_i), z_i \in G_i(x_i)$, let $\phi_i : X_i \times X_i \to R \cup \{+\infty\}$ be a proper, lower semicontinuous and η_i-subdifferentiable and $g_i(X_i) \cap \text{dom}\ \partial\phi_i(., z_i) \neq \emptyset$. We consider the following system of generalized nonlinear variational-like inclusion problems (SGNVLIP): Find (x_i, u_i, v_i, z_i), where $x_i, u_i, v_i, z_i \in X_i$ such that

$$\left.\begin{array}{l} \Big[N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2)), \eta_1(y_1, g_1(x_1))\Big]_1 \\ \qquad\qquad \ge \phi_1(g_1(x_1), z_1) - \phi_1(y_1, z_1), \quad \forall y_1 \in X_1, \\ \\ \Big[N_2(u_2, v_1) - Q_2(E_2(x_2), P_2(x_1)), \eta_2(y_2, g_2(x_2))\Big]_2 \\ \qquad\qquad \ge \phi_2(g_2(x_2), z_2) - \phi_2(y_2, z_2), \quad \forall y_2 \in X_2. \end{array}\right\} \tag{2.7}$$

Special Cases:

I. If in problem (2.7) $X_1 = X_2 = H$, a real Hilbert space, $N_1 = N_2 = N, Q_1 = Q_2 = Q$ such that $N, Q : H \to H$, $\phi_1 = \phi_2 = \phi : H \times H \to R \cup \{+\infty\}$ be a proper functional such that for each fixed $y \in H$, $\phi(., y)$ is a lower semicontinuous and η-subdifferentiable on H and $g(H) \cap \text{dom}\partial\phi(., y) \neq \emptyset$, then problem (2.7) reduces to the following problem: Find $x \in H$ such that $g(x) \in \text{dom}\partial\phi(., x)$ and

$$\langle N(x) - Q(x), \eta(y, g(x))\rangle \ge \phi(g(x), x) - \phi(y, x), \quad \forall y \in H. \tag{2.8}$$

Problem (2.8) has been considered and studied by Ding and Luo [5].

We remark that for the appropriate and suitable choices of mappings $N_i, Q_i, E_i, P_i, g_i, \eta_i, \phi_i, S_i, T_i, G_i$ and the underlying spaces X_i, one can obtain from SGNVLIP (2.7) many known and new classes of systems of generalized variational inequalities, see for example, [1,8,9,13,14,23] and the relevant references cited therein.

3. Existence of Solution

First, we give the following technical lemma which guarantees the existence of solution of SGNVLIP (2.7).

Lemma 3.1. For each $i = 1, 2$, let X_i be a 2-uniformly smooth Banach space, let $\eta_i : X_i \times X_i \to X_i$ be a continuous mapping such that $\eta_i(y_i, y_i') + \eta_i(y_i', y_i) = 0, \quad \forall y_i, y_i' \in X_i$, let $A_i,\ B_i : X_i \to X_i$ be nonlinear mappings, let the mapping $H_i : X_i \times X_i \to X_i$ be $\alpha_i\beta_i$-symmetric η_i-monotone continuous with respect to A_i and B_i, let for any given $t_i \in X_i$, the function $h_i(y_i, x_i) = \langle t_i - H_i(A_i x_i, B_i x_i), \eta_i(y_i, x_i)\rangle$ be $0 - DQCV$ in y_i and let $\phi_i : X_i \times X_i \to R \cup \{\infty\}$ be a proper, lower semicontinuous and η_i-subdifferential functional. Then (x_i, u_i, v_i, z_i) is a solution of SGNVLIP (2.7) if and only if

$$\left.\begin{aligned} g_1(x_1) &= R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1, B_1)\circ g_1)(x_1) - \rho_1\{N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2))\}\Big\},\\ g_2(x_2) &= R^{\partial\phi_2(.,z_2),\eta_2}_{H_2(.,.),\rho_2}\Big\{(H_2(A_2, B_2)\circ g_2)(x_2) - \rho_2\{N_2(u_2, v_1) - Q_2(E_2(x_2), P_2(x_1))\}\Big\}.\end{aligned}\right\} \quad (3.1)$$

Proof. Let (x_i, u_i, v_i, z_i) where $x_i \in X_i, u_i \in S_i(x_i), v_i \in T_i(x_i), z_i \in G_i(x_i)$ satisfies (3.1), then we have

$$g_1(x_1) = R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1, B_1)\circ g_1)(x_1) - \rho_1\{N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2))\}\Big\}$$

$$\begin{aligned}\iff\ & g_1(x_1) = \Big(H_1(A_1, B_1) + \rho_1\partial\phi_1(., z_1)\Big)^{-1}\Big\{(H_1(A_1, B_1)\circ g_1)(x_1)\\ & -\rho_1\{N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2))\}\Big\}\end{aligned}$$

$$\begin{aligned}\iff\ & (H_1(A_1, B_1)\circ g_1)(x_1) + \rho_1\partial\phi_1(g_1(x_1), z_1)\\ & = (H_1(A_1, B_1)\circ g_1)(x_1) - \rho_1\{N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2))\}\end{aligned}$$

$$\iff\ -\{N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2))\} \in \partial\phi_1(g_1(x_1), z_1)$$

$$\begin{aligned}\iff\ & \phi_1(y_1, z_1) - \phi_1(g_1(x_1), z_1)\\ & \geq \Big[- \{N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2))\}, \eta_1(y_1, g_1(x_1))\Big]_1\end{aligned}$$

$$\begin{aligned}\iff\ & \Big[\{N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2))\}, \eta_1(y_1, g_1(x_1))\Big]_1\\ & \geq \phi_1(g_1(x_1), z_1) - \phi_1(y_1, z_1).\end{aligned}$$

Proceeding likewise, we have

$$g_2(x_2) = R^{\partial\phi_2(.,z_2),\eta_2}_{H_2(.,.),\rho_2}\Big\{(H_2(A_2, B_2)\circ g_2)(x_2) - \rho_2\{N_2(u_2, v_1) - Q_2(E_2(x_2), P_2(x_1))\}\Big\}$$

$$\begin{aligned}\iff & \Big[N_2(u_2, v_1) - Q_2(E_2(x_2), P_2(x_1)), \eta_2(y_2, g_2(x_2))\Big]_2\\ & \geq \phi_2(g_2(x_2), z_2) - \phi_2(y_2, z_2).\end{aligned}$$

Now, we give the following result for the existence of solution of SGNVLIP (2.7).

Theorem 3.2. For $i \in \{1, 2\}, j \in \{1, 2\}\backslash\{i\}$. Let X_i be a real 2-uniformly smooth Banach space, $g_i : X_i \to X_i$ be q_i-strongly monotone and l_{g_i}-Lipschitz continuous, $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschitz continuous. Let the mapping $H_i : X_i \times X_i \to X_i$ be $\alpha_i\beta_i$-symmetric η_i-monotone continuous w.r.t A_i and B_i, let $\phi_i : X_i \times X_i \to R \cup \{+\infty\}$ be a proper, lower semicontinuous and η_i-subdifferential functional and $(H_i(A_i, B_i)\circ g_i)$ be l_{H_i}-Lipschitz continuous. Suppose that N_i be δ_i-strongly monotone w.r.t $(H_i(A_i, B_i)\circ g_i)$ in the first argument and ξ_i-strongly monotone w.r.t Q_i in the second argument and E_i, P_i be l_{E_i} and l_{P_i}-Lipschitz continuous, respectively. Let N_i be $l_{N_{i_1}}, l_{N_{i_2}}$-Lipschitz continuous in the first and second arguments, respectively, Q_i be $l_{Q_{i_1}}, l_{Q_{i_2}}$-Lipschitz continuous in the first and second arguments, respectively and $S_i, T_i, G_i : X_i \to C(X_i)$ be such that S_i is $l_{S_i} - \mathcal{D}$-Lipschitz continuous, T_i is $l_{T_i} - \mathcal{D}$-Lipschitz continuous, G_i is $l_{G_i} - \mathcal{D}$-Lipschitz continuous. In addition, if

$$\|R^{\partial\phi(.,z_i),\eta_i}_{H_i(.,.),\rho_i}(x_i) - R^{\partial\phi_i(.,z'_i),\eta_i}_{H_i(.,.),\rho_i}(x_i)\| \leq r_i\|z_i - z'_i\|, \quad \forall x_i, z_i, z'_i \in X_i, \tag{3.2}$$

and

$$k_i = b_i + d_j < 1, \tag{3.3}$$

where

$$b_i := \sqrt{1 - 2q_i + cl^2_{g_i}} + \frac{\tau_i}{\alpha_i - \beta_i}\left(\sqrt{l^2_{H_i} - 2\rho_i\delta_i + c\rho_i^2 l^2_{N_{i_1}} l^2_{S_i}} + \rho_i l_{Q_{i_1}} l_{E_i}\right) + r_i l_{G_i};$$

$$d_i := \frac{\tau_i}{\alpha_i - \beta_i}\rho_i\sqrt{l^2_{Q_{i_2}} l^2_{P_i} - 2\rho_i\xi_i + c\rho_i^2 l^2_{N_{i_2}} l^2_{T_j}}$$

and c is a constant of smoothness of Banach space X_i, then SGNVLIP (2.7) has a solution.

Proof. For each $(x_1, x_2) \in X_1 \times X_2$, define a mapping $V : X_1 \times X_2 \to X_1 \times X_2$ by

$$V(x_1, x_2) = (K_1(x_1, x_2), K_2(x_1, x_2)), \quad \forall (x_1, x_2) \in X_1 \times X_2, \tag{3.4}$$

where $K_1 : X_1 \times X_2 \to X_1$ and $K_2 : X_1 \times X_2 \to X_2$ are defined by

$$K_1(x_1, x_2) = x_1 - g_1(x_1) + R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1, B_1) \circ g_1)(x_1) - \rho_1\{N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2))\}\Big\}, \tag{3.5}$$

and

$$K_2(x_1, x_2) = x_2 - g_2(x_2) + R^{\partial\phi_2(.,z_2),\eta_2}_{H_2(.,.),\rho_2}\Big\{(H_2(A_2, B_2) \circ g_2)(x_2) - \rho_2\{N_2(u_2, v_1) - Q_2(E_2(x_2), P_2(x_1))\}\Big\}, \tag{3.6}$$

for $\rho_1, \rho_2 > 0$, respectively.
For any $(x_1, x_2), (x'_1, x'_2) \in X_1 \times X_2$, it follows from (3.5),(3.6) and Lipschitz continuity of $R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}$ and $R^{\partial\phi_2(.,z_2),\eta_2}_{H_2(.,.),\rho_2}$ that

$$\begin{aligned}
&\Big\|K_1(x_1, x_2) - K_1(x'_1, x'_2)\Big\|_1 \\
&\leq \|(x_1 - x'_1) - (g_1(x_1) - g_1(x'_1))\|_1 \\
&\quad + \Big\|R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1, B_1) \circ g_1)(x_1) - \rho_1\{N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2))\}\Big\} \\
&\quad - R^{\partial\phi_1(.,z'_1),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1, B_1) \circ g_1)(x'_1) - \rho_1\{N_1(u'_1, v'_2) - Q_1(E_1(x'_1), P_1(x'_2))\}\Big\}\Big\|_1.
\end{aligned} \tag{3.7}$$

Since g_i is q_i-strongly monotone and l_{g_i}-Lipschitz continuous, X_i is a real 2-uniformly smooth Banach space, by using Remark 2.6, we have

$$\|(x_1 - x'_1) - (g_1(x_1) - g_1(x'_1))\|^2_1$$

$$\leq \quad ||x_1 - x_1'||_1^2 - 2\Big[g_1(x_1) - g_1(x_1'), x_1 - x_1'\Big]_1 + c||g_1(x_1) - g_1(x_1')||_1^2$$

$$\leq \quad ||x_1 - x_1'||_1^2 - 2q_1||x_1 - x_1'||_1^2 + cl_{g_1}^2||x_1 - x_1'||_1^2$$

$$= \quad (1 - 2q_1 + cl_{g_1}^2)||x_1 - x_1'||_1^2$$

$$\Longrightarrow ||(x_1 - x_1') - (g_1(x_1) - g_1(x_1'))||_1 \leq \sqrt{1 - 2q_1 + cl_{g_1}^2}\ ||x_1 - x_1'||_1. \tag{3.8}$$

Now,

$$\begin{aligned}
&\Big\|R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1) - \rho_1\{N_1(u_1,v_2) - Q_1(E_1(x_1),P_1(x_2))\}\Big\}\\
&\quad -R^{\partial\phi_1(.,z_1'),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1') - \rho_1\{N_1(u_1',v_2') - Q_1(E_1(x_1'),P_1(x_2'))\}\Big\}\Big\|_1\\
&\leq \Big\|R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1) - \rho_1\{N_1(u_1,v_2) - Q_1(E_1(x_1),P_1(x_2))\}\Big\}\\
&\quad -R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1') - \rho_1\{N_1(u_1',v_2') - Q_1(E_1(x_1'),P_1(x_2'))\}\Big\}\\
&\quad +R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1') - \rho_1\{N_1(u_1',v_2') - Q_1(E_1(x_1'),P_1(x_2'))\}\Big\}\\
&\quad -R^{\partial\phi_1(.,z_1'),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1') - \rho_1\{N_1(u_1',v_2') - Q_1(E_1(x_1'),P_1(x_2'))\}\Big\}\Big\|_1\\
&\leq \frac{\tau_1}{\alpha_1 - \beta_1}\Big\|\Big\{(H_1(A_1,B_1)\circ g_1)(x_1) - \rho_1\{N_1(u_1,v_2) - Q_1(E_1(x_1),P_1(x_2))\}\Big\}\\
&\quad -\Big\{(H_1(A_1,B_1)\circ g_1)(x_1') - \rho_1\{N_1(u_1',v_2') - Q_1(E_1(x_1'),P_1(x_2'))\}\Big\}\Big\|_1\\
&\quad +r_1\|z_1 - z_1'\|_1\\
&\leq \frac{\tau_1}{\alpha_1 - \beta_1}\Bigg\{\Big\|(H_1(A_1,B_1)\circ g_1)(x_1) - (H_1(A_1,B_1)\circ g_1)(x_1')\\
&\quad -\rho_1\{N_1(u_1,v_2) - N_1(u_1',v_2)\}\Big\|_1\\
&\quad +\rho_1\Big\|Q_1(E_1(x_1),P_1(x_2)) - Q_1(E_1(x_1'),P_1(x_2))\Big\|_1\\
&\quad +\rho_1\Big\|Q_1(E_1(x_1'),P_1(x_2)) - Q_1(E_1(x_1'),P_1(x_2'))\\
&\quad -\rho_1\{N_1(u_1',v_2) - N_1(u_1',v_2')\}\Big\|_1\Bigg\} + r_1\|z_1 - z_1'\|_1.
\end{aligned} \tag{3.9}$$

Since $(H_i(A_i,B_i)\circ g_i)$ is l_{H_i}-Lipschitz continuous, N_i is δ_i-strongly monotone w.r.t $(H_i(A_i,B_i)\circ g_i)$ in the first argument and ξ_i-strongly monotone w.r.t Q_i in the second argument, E_i, P_i is l_{E_i} and l_{P_i}-Lipschitz continuous, respectively and N_i is $l_{N_{i_1}}, l_{N_{i_2}}$-Lipschitz continuous in the first and second arguments, respectively, Q_i is $l_{Q_{i_1}}, l_{Q_{i_2}}$-Lipschitz continuous in the first and second arguments, respectively and $S_i, T_i, G_i : X_i \to C(X_i)$ is such that S_i is $l_{S_i} - \mathcal{D}$-Lipschitz continuous, T_i is $l_{T_i} - \mathcal{D}$-Lipschitz continuous, G_i is $l_{G_i} - \mathcal{D}$-Lipschitz continuous, therefore by using Remark 2.6, we have

$$\begin{aligned}
&\Big\|(H_1(A_1,B_1)\circ g_1)(x_1) - (H_1(A_1,B_1)\circ g_1)(x_1') - \rho_1\{N_1(u_1,v_2) - N_1(u_1',v_2)\}\Big\|_1^2 \\
&\leq \Big\|(H_1(A_1,B_1)\circ g_1)(x_1) - (H_1(A_1,B_1)\circ g_1)(x_1')\Big\|_1^2 \\
&\quad -2\rho_1\Big[N_1(u_1,v_2) - N_1(u_1',v_2), (H_1(A_1,B_1)\circ g_1)(x_1) - (H_1(A_1,B_1)\circ g_1)(x_1')\Big]_1 \\
&\quad +c\rho_1^2\Big\|N_1(u_1,v_2) - N_1(u_1',v_2)\Big\|_1^2 \\
&\leq l_{H_1}^2\|x_1 - x_1'\|_1^2 - 2\rho_1\delta_1\|x_1 - x_1'\|_1^2 + c\rho_1^2 l_{N1_1}^2\|u_1 - u_1'\|_1^2 \\
&\leq l_{H_1}^2\|x_1 - x_1'\|_1^2 - 2\rho_1\delta_1\|x_1 - x_1'\|_1^2 + c\rho_1^2 l_{N1_1}^2\mathcal{D}(S_1(x_1), S_1(x_1'))_1^2 \\
&\leq \Big(l_{H_1}^2 - 2\rho_1\delta_1 + c\rho_1^2 l_{N1_1}^2 l_{S_1}^2\Big)\|x_1 - x_1'\|_1^2 \\
\Longrightarrow\ &\Big\|(H_1(A_1,B_1)\circ g_1)(x_1) - (H_1(A_1,B_1)\circ g_1)(x_1') - \rho_1\{N_1(u_1,v_2) - N_1(u_1',v_2)\}\Big\|_1 \\
\leq\ &\sqrt{l_{H_1}^2 - 2\rho_1\delta_1 + c\rho_1^2 l_{N1_1}^2 l_{S_1}^2}\ \|x_1 - x_1'\|_1. \qquad (3.10)
\end{aligned}$$

Again,

$$\begin{aligned}
&\Big\|Q_1(E_1(x_1), P_1(x_2)) - Q_1(E_1(x_1'), P_1(x_2))\Big\|_1 \\
&\qquad \leq l_{Q1_1}\Big\|E_1(x_1) - E_1(x_1')\Big\|_1 \\
&\qquad \leq l_{Q1_1} l_{E_1}\|x_1 - x_1'\|_1 \\
\Longrightarrow\ &\Big\|Q_1(E_1(x_1), P_1(x_2)) - Q_1(E_1(x_1'), P_1(x_2))\Big\|_1 \\
&\qquad \leq l_{Q1_1} l_{E_1}\|x_1 - x_1'\|_1. \qquad (3.11)
\end{aligned}$$

Again by using Remark 2.6, we have

$$\Big\|Q_1(E_1(x_1'), P_1(x_2)) - Q_1(E_1(x_1'), P_1(x_2'))$$

$$
\begin{aligned}
&\quad -\rho_1\{N_1(u_1', v_2) - N_1(u_1', v_2')\}\Big\|_1^2 \\
&\leq \Big\|Q_1(E_1(x_1'), P_1(x_2)) - Q_1(E_1(x_1'), P_1(x_2'))\Big\|_1^2 \\
&\quad -2\rho_1\Big[N_1(u_1', v_2) - N_1(u_1', v_2'), \\
&\quad Q_1(E_1(x_1'), P_1(x_2)) - Q_1(E_1(x_1'), P_1(x_2'))\Big]_1 \\
&\quad +c\rho_1^2\Big\|N_1(u_1', v_2) - N_1(u_1', v_2')\Big\|_1^2 \\
&\leq l_{Q_{1_2}}^2 l_{P_1}^2\|x_2 - x_2'\|_2^2 - 2\rho_1\xi_1\|x_2 - x_2'\|_2^2 + c\rho_1^2 l_{N_{1_2}}^2\|v_2 - v_2'\|_2^2 \\
&\leq l_{Q_{1_2}}^2 l_{P_1}^2\|x_2 - x_2'\|_2^2 - 2\rho_1\xi_1\|x_2 - x_2'\|_2^2 + c\rho_1^2 l_{N_{1_2}}^2\mathcal{D}(T_2(x_2), T_2(x_2'))_2^2 \\
&\leq l_{Q_{1_2}}^2 l_{P_1}^2\|x_2 - x_2'\|_2^2 - 2\rho_1\xi_1\|x_2 - x_2'\|_2^2 + c\rho_1^2 l_{N_{1_2}}^2 l_{T_2}^2\|x_2 - x_2'\|_2^2 \\
&\leq \Big(l_{Q_{1_2}}^2 l_{P_1}^2 - 2\rho_1\xi_1 + c\rho_1^2 l_{N_{1_2}}^2 l_{T_2}^2\Big)\|x_2 - x_2'\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \Big\|&Q_1(E_1(x_1'), P_1(x_2)) - Q_1(E_1(x_1'), P_1(x_2')) \\
&\quad -\rho_1\{N_1(u_1', v_2) - N_1(u_1', v_2')\}\Big\|_1 \\
&\quad \leq \sqrt{l_{Q_{1_2}}^2 l_{P_1}^2 - 2\rho_1\xi_1 + c\rho_1^2 l_{N_{1_2}}^2 l_{T_2}^2}\ \|x_2 - x_2'\|_2. \qquad (3.12)
\end{aligned}
$$

Also,

$$\|z_1 - z_1'\|_1 \leq \mathcal{D}(G_1(x_1), G_1(x_1')) \leq l_{G_1}\|x_1 - x_1'\|_1. \qquad (3.13)$$

From (3.7)-(3.13), we have

$$
\begin{aligned}
&\Big\|K_1(x_1, x_2) - K_1(x_1', x_2')\Big\|_1 \\
&\leq \sqrt{1 - 2q_1 + cl_{g_1}^2}\ \|x_1 - x_1'\|_1 + \frac{\tau_1}{\alpha_1 - \beta_1}\Bigg\{\sqrt{l_{H_1}^2 - 2\rho_1\delta_1 + c\rho_1^2 l_{N_{1_1}}^2 l_{S_1}^2}\ \|x_1 - x_1'\|_1 \\
&\quad +\rho_1 l_{Q_{1_1}} l_{E_1}\|x_1 - x_1'\|_1 + \rho_1\sqrt{l_{Q_{1_2}}^2 l_{P_1}^2 - 2\rho_1\xi_1 + c\rho_1^2 l_{N_{1_2}}^2 l_{T_2}^2}\ \|x_2 - x_2'\|_2\Bigg\} \\
&\quad +r_1 l_{G_1}\|x_1 - x_1'\|_1 \\
&\leq b_1\|x_1 - x_1'\|_1 + d_1\|x_2 - x_2'\|_2. \qquad (3.14)
\end{aligned}
$$

Similarly, we infer that

$$
\begin{aligned}
&\Big\|K_2(x_1, x_2) - K_2(x_1', x_2')\Big\|_2 \\
&\quad \leq b_2\|x_2 - x_2'\|_2 + d_2\|x_1 - x_1'\|_1. \qquad (3.15)
\end{aligned}
$$

From (3.14) and (3.15), we have

$$\begin{aligned}\Big\|K_1(x_1,x_2)-K_1(x_1',x_2')\Big\|_1+\Big\|K_2(x_1,x_2)-K_2(x_1',x_2')\Big\|_2\\ \le\ k_1\|x_1-x_1'\|_1+k_2\|x_2-x_2'\|_2\\ \le k\{\|x_1-x_1'\|_1+\|x_2-x_2'\|_2\},\end{aligned}\tag{3.16}$$

where $k_1=b_1+d_2, k_2=b_2+d_1$ and $k=\max\{k_1,k_2\}$.
Now, define the norm $\|.\|_\star$ on $X_1\times X_2$ by

$$\|(x_1,x_2)\|_\star=\|x_1\|_1+\|x_2\|_2,\quad \forall (x_1,x_2)\in X_1\times X_2.\tag{3.17}$$

Hence, it follows from (3.4), (3.16) and (3.17) that

$$\begin{aligned}\|V(x_1,x_2)-V(x_1',x_2')\|_\star &\le \|(K_1(x_1,x_2),K_2(x_1,x_2))-(K_1(x_1',x_2'),K_2(x_1',x_2'))\|_\star\\ &\le \|(K_1(x_1,x_2)-K_1(x_1',x_2'),K_2(x_1,x_2)-K_2(x_1',x_2'))\|_\star\\ &\le \|K_1(x_1,x_2)-K_1(x_1',x_2')\|_1+\|K_2(x_1,x_2)-K_2(x_1',x_2')\|_2\\ &\le\ k\{\|x_1-x_1'\|_1+\|x_2-x_2'\|_2\}.\end{aligned}\tag{3.18}$$

Since k =max $\{k_1,k_2\}<1$ by (3.3), it follows from (3.18) that V is a contraction mapping. Hence, by Banach contraction principle, it admits a unique fixed point $(x_1,x_2)\in X_1\times X_2$ such that $V(x_1,x_2)=(x_1,x_2)$, which implies that

$$g_1(x_1)=R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}\{(H_1(A_1,B_1)\circ g_1)(x_1)-\rho_1\{N_1(u_1,v_2)-Q_1(E_1(x_1),P_1(x_2))\}\},$$

$$g_2(x_2)=R^{\partial\phi_2(.,z_2),\eta_2}_{H_2(.,.),\rho_2}\{(H_2(A_2,B_2)\circ g_2)(x_2)-\rho_2\{N_2(u_2,v_1)-Q_2(E_2(x_2),P_2(x_1))\}\}.$$

It follows from Lemma 3.1, that (x_i,u_i,v_i,z_i) is a solution of SGNVLIP (2.7). This completes the proof.

When $X_i=L^p(R),\ 2\le p<\infty,\ i=1,2$, we have the following corollary:

Corollary 3.3. For $i\in\{1,2\}, j\in\{1,2\}\backslash\{i\}$, let $g_i:L^p\to L^p$ be q_i-strongly monotone and l_{g_i}-Lipschitz continuous, $\eta_i:L^p\times L^p\to L^p$ be τ_i-Lipschitz continuous. Let the mapping $H_i:L^p\times L^p\to L^p$ be $\alpha_i\beta_i$-symmetric η_i-monotone continuous w.r.t A_i and B_i. Let $\phi_i:L^p\times L^p\to R\cup\{+\infty\}$ be a proper, lower semicontinuous and η_i-subdifferential functional, $(H_i(A_i,B_i)\circ g_i)$ be l_{H_i}-Lipschitz continuous. Suppose that N_i be δ_i-strongly monotone w.r.t $(H_i(A_i,B_i)\circ g_i)$ in the first argument and ξ_i-strongly monotone w.r.t Q_i in the second argument and E_i,P_i be l_{E_i} and l_{P_i}-Lipschitz continuous, respectively. Let N_i be $l_{N_{i_1}},l_{N_{i_2}}$-Lipschitz continuous in the first and second arguments, respectively, Q_i be $l_{Q_{i_1}},l_{Q_{i_2}}$-Lipschitz continuous in the first and second arguments, respectively and $S_i,T_i,G_i:X_i\to C(X_i)$ be such that S_i is $l_{S_i}-\mathcal{D}$-Lipschitz continuous, T_i is $l_{T_i}-\mathcal{D}$-Lipschitz continuous, G_i is $l_{G_i}-\mathcal{D}$-Lipschitz continuous. In addition, if

$$\|R^{\partial\phi(.,z_i),\eta_i}_{H_i(.,.),\rho_i}(x_i)-R^{\partial\phi_i(.,z_i'),\eta_i}_{H_i(.,.),\rho_i}(x_i)\|\le r_i\|z_i-z_i'\|,\quad \forall x_i,z_i,z_i'\in L^p.$$

and

$$k_i=b_i+d_j<1,$$

where

$$b_i := \sqrt{1-2q_i+(p-1)l_{g_i}^2} + \frac{\tau_i}{\alpha_i-\beta_i}\Big(\sqrt{l_{H_i}^2-2\rho_i\delta_i+(p-1)\rho_i^2 l_{N_{i_1}}^2 l_{S_i}^2} + \rho_i l_{Q_{i_1}} l_{E_i}\Big) + r_i l_{G_i}$$

$$d_i := \frac{\tau_i}{\alpha_i-\beta_i}\rho_i\sqrt{l_{Q_{i_2}}^2 l_{P_i}^2 - 2\rho_i\xi_i + (p-1)\rho_i^2 l_{N_{i_2}}^2 l_{T_j}^2}$$

and $(p-1)$ is constant of smoothness of function space L^p, then SGNVLIP (2.7) has a solution.

4. Iterative Algorithm and Convergence Analysis

Lemma 3.1 is important from the numerical point of view. It allows us to suggest the following iterative algorithm for finding the approximate solution of SGNVLIP (2.7).

Iterative Algorithm 4.1. For each $i = 1, 2, j \in \{1, 2\} \setminus i$, given $(x_i^0, u_i^0, v_i^0, z_i^0)$, where $x_i^0 \in X_i, u_i^0 \in S_i(x_i^0), v_i^0 \in T_i(x_i^0), z_i^0 \in G_i(x_i^0)$ such that $S_i, T_i, G_i : X_i \to C(X_i)$, compute the sequences $\{x_i^n\}, \{u_i^n\}, \{v_i^n\}, \{z_i^n\}$ by the iterative schemes:

$$x_1^{n+1} = (1-a^n)x_1^n + a^n\Big\{x_1^n - g_1(x_1^n) + R_{H_1(.,.),\rho_1}^{\partial\phi_1^n(.,z_1^n),\eta_1}\{(H_1(A_1,B_1)\circ g_1)(x_1^n)$$
$$-\rho_1\{N_1(u_1^n, v_2^n) - Q_1(E_1(x_1^n), P_1(x_2^n))\}\}\Big\} + a^n e_1^n,$$
$$x_2^{n+1} = (1-a^n)x_2^n + a^n\Big\{x_2^n - g_2(x_2^n) + R_{H_2(.,.),\rho_2}^{\partial\phi_2^n(.,z_2^n),\eta_2}\{(H_2(A_2,B_2)\circ g_2)(x_2^n)$$
$$-\rho_2\{N_2(u_2^n, v_1^n) - Q_2(E_2(x_2^n), P_2(x_1^n))\}\}\Big\} + a^n e_2^n,$$
$$u_i^n \in S_i(x_i^n) : \|u_i^{n+1} - u_i^n\|_i \le \mathcal{D}(S_i(x_i^{n+1}), S_i(x_i^n))_i,$$
$$v_i^n \in T_i(x_i^n) : \|v_i^{n+1} - v_i^n\|_i \le \mathcal{D}(T_i(x_i^{n+1}), T_i(x_i^n))_i,$$
$$z_i^n \in G_i(x_i^n) : \|z_i^{n+1} - z_i^n\|_i \le \mathcal{D}(G_i(x_i^{n+1}), G_i(x_i^n))_i,$$

where $n = 0, 1, 2, \cdots$, $\rho_i > 0$ are constants and $\{e_1^n, e_2^n\}_{n\ge0}$ is sequence in $X_1 \times X_2$ introduced to take into account possible inexact computation which satisfies $\lim_{n\to\infty}\|e_1^n\| = \lim_{n\to\infty}\|e_2^n\| = 0$ and $\{a^n\}$ be a sequence of real numbers such that $a^n \in [0,1]$ and $\sum_{n=0}^{\infty} a^n = +\infty$.

Before proving the convergence criteria of the sequences generated by the above iterative algorithm 4.1, we consider the following condition:

Condition 4.2. Let for each $n \ge 0, \phi_i^n, \phi_i : X \times X \to R \cup \{\infty\}$ be a proper, lower semicontinuous and η_i-subdifferentiable functional. Then the sequence $\{\partial\phi_i^n\}$ approximates $\{\partial\phi_i\}$ in the following sense:

$$\lim_{n\to\infty} R_{H_i(.,.),\rho_i}^{\partial\phi_i^n(.,z_i^n),\eta_i}(x_i^n) = R_{H_i(.,.),\rho_i}^{\partial\phi_i(.,z_i),\eta_i}(x_i), \quad \forall x_i \in X_i.$$

Now, we prove the following theorem which ensures the convergence of the sequences generated by the Iterative Algorithm 4.1 for SGNVLIP (2.7).

Theorem 4.3. For $i \in \{1,2\}, j \in \{1,2\}\setminus\{i\}$, let X_i be a real 2-uniformly smooth Banach space, let $g_i : X_i \to X_i$ be q_i-strongly monotone and l_{g_i}-Lipschitz continuous, $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschitz continuous. Let the mapping $H_i : X_i \times X_i \to X_i$ be $\alpha_i\beta_i$-symmetric η_i-monotone continuous w.r.t A_i and B_i, $\phi_i^n, \phi_i : X_i \times X_i \to R \cup \{+\infty\}$ be a proper, lower semicontinuous and η_i-subdifferential functional and $(H_i(A_i, B_i) \circ g_i)$ be l_{H_i}-Lipschitz continuous. Let E_i, P_i be l_{E_i}

and l_{P_i}-Lipschitz continuous, respectively and N_i be $l_{N_{i_1}}, l_{N_{i_2}}$-Lipschitz continuous in the first and second arguments, respectively. Let Q_i be $l_{Q_{i_1}}, l_{Q_{i_2}}$-Lipschitz continuous in the first and second arguments, respectively and $S_i, T_i, G_i : X_i \to C(X_i)$ be such that S_i is l_{S_i}-$\mathcal{D}$-Lipschitz continuous, T_i is l_{T_i}-$\mathcal{D}$-Lipschitz continuous, G_i is l_{G_i}-$\mathcal{D}$-Lipschitz continuous. Suppose that H_i, g_i, Q_i, E_i, P_i be such that $\{(H_1(A_1, B_1) \circ g_1)(.) + \rho_1(Q_1(E_1(.), P_1(x_2^n)))\}$ is μ_1-cocoercive and $\{(H_2(A_2, B_2) \circ g_2)(.) + \rho_2(Q_2(E_2(.), P_2(x_1^n)))\}$ is μ_2-cocoercive. In addition, if

$$k'_i = b'_i + d'_j < 1, \tag{4.1}$$

where

$$b'_i := \sqrt{1 - 2q_i + cl_{g_i}^2} + \frac{\tau_i}{(\alpha_i - \beta_i)\mu_i} + \frac{\tau_i \rho_i}{(\alpha_i - \beta_i)} l_{N_{i_1}} l_{S_i}$$

$$d'_i := \frac{\tau_i \rho_i}{(\alpha_i - \beta_i)} \Big(l_{N_{i_2}} l_{T_j} + l_{Q_{i_2}} l_{P_i} \Big).$$

Then for each $i = 1, 2$, the sequences $\{x_i^n\}$, $\{u_i^n\}$, $\{v_i^n\}$, $\{z_i^n\}$ generated by Iterative Algorithm 4.1 converges strongly to x_i, u_i, v_i, z_i, respectively, where $(x_1, x_2, u_1, u_2, v_1, v_2, z_1, z_2)$ is a solution of SGNVLIP (2.7).

Proof. It follows from Theorem 3.2 that $(x_1, x_2, u_1, u_2, v_1, v_2, z_1, z_2)$ is a solution of SGNVLIP (2.7) and hence from Lemma 3.1, we have

$$\begin{aligned} x_1 &= (1 - a^n)x_1 + a^n \Big\{ x_1 - g_1(x_1) + R_{H_1(.,.),\rho_1}^{\partial\phi_1(.,z_1),\eta_1} \{ (H_1(A_1, B_1) \circ g_1)(x_1) \\ &\quad -\rho_1 \{ N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2)) \} \} \Big\}, \\ x_2 &= (1 - a^n)x_2 + a^n \Big\{ x_2 - g_2(x_2) + R_{H_2(.,.),\rho_2}^{\partial\phi_2(.,z_2),\eta_2} \{ (H_2(A_2, B_2) \circ g_2)(x_2) \\ &\quad -\rho_2 \{ N_2(u_2, v_1) - Q_2(E_2(x_2), P_2(x_1)) \} \} \Big\}. \end{aligned} \tag{4.2}$$

From Iterative Algorithm 4.1 and (4.2), we have

$$\begin{aligned} &\|x_1^{n+1} - x_1\|_1 \\ &= \Big\| (1 - a^n)x_1^n + a^n \Big\{ x_1^n - g_1(x_1^n) + R_{H_1(.,.),\rho_1}^{\partial\phi_1^n(.,z_1^n),\eta_1} \{ (H_1(A_1, B_1) \circ g_1)(x_1^n) \\ &\quad -\rho_1 \{ N_1(u_1^n, v_2^n) - Q_1(E_1(x_1^n), P_1(x_2^n)) \} \Big\} + a^n e_1^n \\ &\quad -\Big\{ (1 - a^n)x_1 + a^n \Big\{ x_1 - g_1(x_1) + R_{H_1(.,.),\rho_1}^{\partial\phi_1(.,z_1),\eta_1} \{ (H_1(A_1, B_1) \circ g_1)(x_1) \\ &\quad -\rho_1 \{ N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2)) \Big\} \Big\|_1 \\ &\le (1 - a^n)\|x_1^n - x_1\|_1 + a^n \|(x_1^n - x_1) - (g_1(x_1^n) - g_1(x_1))\|_1 \\ &\quad + a^n \Big\| R_{H_1(.,.),\rho_1}^{\partial\phi_1^n(.,z_1^n),\eta_1} \Big\{ (H_1(A_1, B_1) \circ g_1)(x_1^n) - \rho_1 \{ N_1(u_1^n, v_2^n) - Q_1(E_1(x_1^n), P_1(x_2^n)) \} \Big\} \\ &\quad - R_{H_1(.,.),\rho_1}^{\partial\phi_1^n(.,z_1^n),\eta_1} \Big\{ (H_1(A_1, B_1) \circ g_1)(x_1) - \rho_1 \{ N_1(u_1, v_2) - Q_1(E_1(x_1), P_1(x_2)) \} \Big\} \Big\|_1 \end{aligned}$$

$$+a^n\Big\|R^{\partial\phi_1^n(.,z_1^n),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1)-\rho_1\{N_1(u_1,v_2)-Q_1(E_1(x_1),P_1(x_2))\}\Big\}$$

$$-R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1)-\rho_1\{N_1(u_1,v_2)-Q_1(E_1(x_1),P_1(x_2))\}\Big\}\Big\|_1+a^n\|e_1^n\|_1. \quad (4.3)$$

Since g_i is q_i-strongly monotone and l_{g_i}-Lipschitz continuous, X_i is a real 2-uniformly smooth Banach space, by using Remark 2.6, we have

$$\begin{aligned}
&\|(x_1^n-x_1)-(g_1(x_1^n)-g_1(x_1))\|_1^2\\
&\quad\le ||x_1^n-x_1||_1^2-2\Big[g_1(x_1^n)-g_1(x_1),x_1^n-x_1\Big]_1\\
&\quad\quad +c||g_1(x_1^n)-g_1(x_1)||_1^2\\
&\quad\le ||x_1^n-x_1||_1^2-2q_1||x_1^n-x_1||_1^2+cl_{g_1}^2||x_1^n-x_1||_1^2\\
&\quad= (1-2q_1+cl_{g_1}^2)||x_1^n-x_1||_1^2
\end{aligned}$$

$$\Longrightarrow ||(x_1^n-x_1)-(g_1(x_1^n)-g_1(x_1))||_1\le\sqrt{1-2q_1+cl_{g_1}^2}\;||x_1^n-x_1||_1. \quad (4.4)$$

Now, from Theorem 2.16, it follows that

$$\begin{aligned}
&\Big\|R^{\partial\phi_1^n(.,z_1^n),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1^n)-\rho_1\{N_1(u_1^n,v_2^n)-Q_1(E_1(x_1^n),P_1(x_2^n))\}\Big\}\\
&\quad -R^{\partial\phi_1^n(.,z_1^n),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1)-\rho_1\{N_1(u_1,v_2)-Q_1(E_1(x_1),P_1(x_2))\}\Big\}\Big\|_1\\
&\quad\le \frac{\tau_1}{\alpha_1-\beta_1}\Big\|\Big\{(H_1(A_1,B_1)\circ g_1)(x_1^n)-\rho_1\{N_1(u_1^n,v_2^n)-Q_1(E_1(x_1^n),P_1(x_2^n))\}\Big\}\\
&\quad\quad -\Big\{(H_1(A_1,B_1)\circ g_1)(x_1)-\rho_1\{N_1(u_1,v_2)-Q_1(E_1(x_1),P_1(x_2))\}\Big\}\Big\|_1\\
&\quad\le \frac{\tau_1}{\alpha_1-\beta_1}\Big\|\Big\{(H_1(A_1,B_1)\circ g_1)(x_1^n)+\rho_1Q_1(E_1(x_1^n),P_1(x_2^n))\Big\}\\
&\quad\quad -\Big\{(H_1(A_1,B_1)\circ g_1)(x_1)+\rho_1Q_1(E_1(x_1),P_1(x_2^n))\Big\}\\
&\quad\quad -\rho_1\Big\{N_1(u_1^n,v_2^n)-N_1(u_1,v_2)\Big\}\\
&\quad\quad +\rho_1\Big\{Q_1(E_1(x_1),P_1(x_2^n))-Q_1(E_1(x_1),P_1(x_2))\Big\}\Big\|_1.
\end{aligned} \quad (4.5)$$

Since $\{(H_1(A_1,B_1)\circ g_1)(.)+\rho_1(Q_1(E_1(.),P_1(x_2^n)))\}$ is μ_1-cocoercive, we have

$$\begin{aligned}
&\Big\|\Big\{(H_1(A_1,B_1)\circ g_1)(x_1^n)+\rho_1Q_1(E_1(x_1^n),P_1(x_2^n))\Big\}\\
&\quad -\Big\{(H_1(A_1,B_1)\circ g_1)(x_1)+\rho_1Q_1(E_1(x_1),P_1(x_2^n))\Big\}\Big\|_1\;\|x_1^n-x_1\|_1
\end{aligned}$$

$$\geq \Big[\Big\{(H_1(A_1,B_1)\circ g_1)(x_1^n)+\rho_1 Q_1(E_1(x_1^n),P_1(x_2^n))\Big\}$$

$$-\Big\{(H_1(A_1,B_1)\circ g_1)(x_1)+\rho_1 Q_1(E_1(x_1),P_1(x_2^n))\Big\}, x_1^n - x_1\Big]_1$$

$$\geq \mu_1\Big\|\Big\{(H_1(A_1,B_1)\circ g_1)(x_1^n)+\rho_1 Q_1(E_1(x_1^n),P_1(x_2^n))\Big\}$$

$$-\Big\{(H_1(A_1,B_1)\circ g_1)(x_1)+\rho_1 Q_1(E_1(x_1),P_1(x_2^n))\Big\}\Big\|_1^2.$$

This implies

$$\Big\|\Big\{(H_1(A_1,B_1)\circ g_1)(x_1^n)+\rho_1 Q_1(E_1(x_1^n),P_1(x_2^n))\Big\}$$

$$-\Big\{(H_1(A_1,B_1)\circ g_1)(x_1)+\rho_1 Q_1(E_1(x_1),P_1(x_2^n))\Big\}\Big\|_1$$

$$\leq \frac{1}{\mu_1}\|x_1^n - x_1\|_1. \tag{4.6}$$

Also, since N_1 is $l_{N_{1_1}}, l_{N_{1_2}}$-Lipschitz continuous in the first and second arguments, respectively, we have

$$\Big\|N_1(u_1^n,v_2^n)-N_1(u_1,v_2)\Big\|_1$$

$$\leq l_{N_{1_1}}\|u_1^n-u_1\|_1 + l_{N_{1_2}}\|v_2^n - v_2\|_2$$

$$\leq l_{N_{1_1}}\mathcal{D}(S_1(x_1^n),S_1(x_1)) + l_{N_{1_2}}\mathcal{D}(T_2(x_2^n),T_2(x_2))$$

$$\leq l_{N_{1_1}} l_{S_1}\|x_1^n - x_1\|_1 + l_{N_{1_2}} l_{T_2}\|x_2^n - x_2\|_2. \tag{4.7}$$

Again, as P_1 is l_{P_1}-Lipschitz continuous and Q_1 is $l_{Q_{1_1}}, l_{Q_{1_2}}$-Lipschitz continuous in the first and second arguments, respectively, we have

$$\|Q_1(E_1(x_1),P_1(x_2^n)) - Q_1(E_1(x_1),P_1(x_2))\|_1$$

$$\leq l_{Q_{1_2}} l_{P_1}\|x_2^n - x_2\|_2. \tag{4.8}$$

Using (4.6),(4.7) and (4.8) in (4.5), we have

$$\Big\|R^{\partial\phi_1^n(.,z_1^n),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1^n)-\rho_1\{N_1(u_1^n,v_2^n)-Q_1(E_1(x_1^n),P_1(x_2^n))\}\Big\}$$

$$-R^{\partial\phi_1^n(.,z_1^n),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1)-\rho_1\{N_1(u_1,v_2)-Q_1(E_1(x_1),P_1(x_2))\}\Big\}\Big\|_1$$

$$\leq \frac{\tau_1}{\alpha_1-\beta_1}\Bigg\{\frac{1}{\mu_1}\|x_1^n-x_1\|_1 + \rho_1 l_{N_{1_1}} l_{S_1}\|x_1^n-x_1\|_1 + \rho_1 l_{N_{1_2}} l_{T_2}\|x_2^n-x_2\|_2$$

$$+\rho_1 l_{Q_{1_2}} l_{P_1}\|x_2^n - x_2\|_2\Bigg\}. \tag{4.9}$$

Combining (4.3),(4.4) and (4.9), we have

$$\|x_1^{n+1}-x_1\|_1 \leq (1-a^n)\|x_1^n - x_1\|_1 + a^n\sqrt{1-2q_1+cl_{g_1}^2}\;||x_1^n - x_1||_1$$

$$+a^n\frac{\tau_1}{\alpha_1-\beta_1}\Bigg\{\frac{1}{\mu_1}\|x_1^n-x_1\|_1 + \rho_1 l_{N_{1_1}} l_{S_1}\|x_1^n - x_1\|_1$$

$$+\rho_1 l_{N_{1_2}} l_{T_2}\|x_2^n-x_2\|_2 + \rho_1 l_{Q_{1_2}} l_{P_1}\|x_2^n - x_2\|_2\Bigg\} + a^n\Phi_1^n + a^n\|e_1^n\|_1,$$

where

$$\Phi_1^n := \Big\| R^{\partial\phi_1^n(.,z_1^n),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1) - \rho_1\{N_1(u_1,v_2) - Q_1(E_1(x_1),P_1(x_2))\}\Big\}$$
$$- R^{\partial\phi_1(.,z_1),\eta_1}_{H_1(.,.),\rho_1}\Big\{(H_1(A_1,B_1)\circ g_1)(x_1) - \rho_1\{N_1(u_1,v_2) - Q_1(E_1(x_1),P_1(x_2))\}\Big\}\Big\|_1,$$

and $\Phi_1^n \to 0$ as $n \to \infty$. Thus, we have

$$\begin{aligned}&\|x_1^{n+1} - x_1\|_1\\ &\le \Big\{(1-a^n) + a^n\Big(\sqrt{1-2q_1+cl_{g_1}^2} + \frac{\tau_1}{(\alpha_1-\beta_1)\mu_1} + \frac{\tau_1\rho_1}{(\alpha_1-\beta_1)} l_{N_{1_1}} l_{S_1}\Big)\Big\}\|x_1^n - x_1\|_1\\ &\quad + \frac{a^n\tau_1\rho_1}{(\alpha_1-\beta_1)}\Big(l_{N_{1_2}} l_{T_2} + l_{Q_{1_2}} l_{P_1}\Big)\|x_2^n - x_2\|_2 + a^n\Phi_1^n + a^n\|e_1^n\|_1. \qquad (4.10)\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&\|x_2^{n+1} - x_2\|_2\\ &\le \Big\{(1-a^n) + a^n\Big(\sqrt{1-2q_2+cl_{g_2}^2} + \frac{\tau_2}{(\alpha_2-\beta_2)\mu_2} + \frac{\tau_2\rho_2}{(\alpha_2-\beta_2)} l_{N_{2_1}} l_{S_2}\Big)\Big\}\|x_2^n - x_2\|_2\\ &\quad + \frac{a^n\tau_2\rho_2}{(\alpha_2-\beta_2)}\Big(l_{N_{2_2}} l_{T_1} + l_{Q_{2_2}} l_{P_2}\Big)\|x_1^n - x_1\|_1 + a^n\Phi_2^n + a^n\|e_2^n\|_2. \qquad (4.11)\end{aligned}$$

From (4.10) and (4.11), we have

$$\begin{aligned}&\|x_1^{n+1} - x_1\|_1 + \|x_2^{n+1} - x_2\|_2\\ &\le (1 - a^n(1-k_1'))\|x_1^n - x_1\|_1 + (1 - a^n(1-k_2'))\|x_2^n - x_2\|_2\\ &\quad + a^n(\Phi_1^n + \Phi_2^n) + a^n(\|e_1^n\|_1 + \|e_2^n\|_2)\\ &\le (1 - a^n(1 - \max\{k_1', k_2'\}))(\|x_1^n - x_1\|_1 + \|x_2^n - x_2\|_2)\\ &\quad + a^n(1 - \max\{k_1', k_2'\})\frac{(\Phi_1^n + \Phi_2^n + \|e_1^n\|_1 + \|e_2^n\|_2)}{(1 - \max\{k_1', k_2'\})}, \qquad (4.12)\end{aligned}$$

where $k_1' = b_1' + d_2'; \quad k_2' = b_2' + d_1'$

$$\text{if } \zeta^n = \|x_1^n - x_1\|_1 + \|x_2^n - x_2\|_2,\ \hbar^n = \frac{\Big\{(\Phi_1^n + \Phi_2^n + \|e_1^n\|_1 + \|e_2^n\|_2\Big\}}{(1 - \max\{k_1', k_2'\})} \text{ and}$$

$$\omega^n = a^n(1 - \max\{k_1', k_2'\}).$$

Then, we have

$$\zeta^{n+1} \le (1-\omega^n)\zeta^n + \omega^n\hbar^n.$$

Using Lemma 2.17, we have $\zeta^n \to 0$ as $n \to \infty$. This implies $x_1^n \to x_1, x_2^n \to x_2$ as $n \to \infty$. Since S_i is $l_{S_i} - \mathcal{D}$-Lipschitz continuous, it follows from iterative algorithm 4.1 that

$$\Big\|u_i^n - u_i\Big\|_i \le \mathcal{D}(S_i(x_i^n), S_i(x_i))_i \le l_{S_i}\Big\|x_i^n - x_i\Big\|_i.$$

This implies that

$$u_i^n \to u_i \text{ as } n \to \infty.$$

Further, we claim that $u_i \in S_i(x_i)$

$$\begin{aligned} d(u_i, S_i(x_i)) &\leq \left\| u_i - u_i^n \right\|_i + d(u_i^n, S_i(x_i))_i \\ &\leq \left\| u_i - u_i^n \right\|_i + \mathcal{D}(S_i(x_i^n), S_i(x_i))_i \\ &\leq \left\| u_i - u_i^n \right\|_i + l_{S_i} \left\| x_i^n - x_i \right\|_i \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $S_i(x_i)$ is compact, we have $u_i \in S_i(x_i)$.
Similarly, we can prove that $v_i \in T_i(x_i), z_i \in G_i(x_i)$.
Thus the approximate solution $(x_i^n, u_i^n, v_i^n, z_i^n)$ generated by iterative algorithm 4.1 converges strongly to (x_i, u_i, v_i, z_i) a solution of (2.7).

When $X_i = L^p(R),\ 2 \leq p < \infty,\ i = 1, 2$, we have the following corollary:

Corollary 4.4. For $i \in \{1,2\}, j \in \{1,2\}\backslash\{i\}$, let $g_i : L^p \to L^p$ be q_i-strongly monotone and l_{g_i}-Lipschitz continuous, $\eta_i : L^p \times L^p \to L^p$ be τ_i-Lipschitz continuous. Let the mapping $H_i : L^p \times L^p \to L^p$ be $\alpha_i\beta_i$-symmetric η_i-monotone continuous w.r.t A_i and $B_i, \phi_i^n, \phi_i : L^p \times L^p \to R \cup \{+\infty\}$ be a proper, lower semicontinuous and η_i-subdifferential functional and $(H_i(A_i, B_i) \circ g_i)$ be l_{H_i}-Lipschitz continuous. Let E_i, P_i be l_{E_i} and l_{P_i}-Lipschitz continuous, respectively and N_i be $l_{N_{i_1}}, l_{N_{i_2}}$-Lipschitz continuous in the first and second arguments, respectively. Let Q_i be $l_{Q_{i_1}}, l_{Q_{i_2}}$-Lipschitz continuous in the first and second arguments, respectively and $S_i, T_i, G_i : L^p \to C(L^p)$ be such that S_i is l_{S_i}-$\mathcal{D}$-Lipschitz continuous, T_i is l_{T_i}-$\mathcal{D}$-Lipschitz continuous, G_i is l_{G_i}-$\mathcal{D}$-Lipschitz continuous. Suppose that H_i, g_i, Q_i, E_i, P_i be such that $\{(H_1(A_1, B_1) \circ g_1)(.) + \rho_1(Q_1(E_1(.), P_1(x_2^n)))\}$ is μ_1-cocoercive and $\{(H_2(A_2, B_2) \circ g_2)(.) + \rho_2(Q_2(E_2(.), P_2(x_1^n)))\}$ is μ_2-cocoercive. In addition, if

$$k_i' = b_i' + d_j' < 1,$$

where

$$b_i' := \sqrt{1 - 2q_i + cl_{g_i}^2} + \frac{\tau_i}{(\alpha_i - \beta_i)\mu_i} + \frac{\tau_i \rho_i}{(\alpha_i - \beta_i)} l_{N_{i_1}} l_{S_i}$$

$$d_i' := \frac{\tau_i \rho_i}{(\alpha_i - \beta_i)} \Big(l_{N_{i_2}} l_{T_j} + l_{Q_{i_2}} l_{P_i} \Big).$$

Then for each $i = 1, 2$, the sequences $\{x_i^n\}$, $\{u_i^n\}$, $\{v_i^n\}$, $\{z_i^n\}$ generated by Iterative Algorithm 4.1 converges strongly to x_i, u_i, v_i, z_i, respectively, where $(x_1, x_2, u_1, u_2, v_1, v_2, z_1, z_2)$ is a solution of SGNVLIP (2.7).

5. Conclusion.

A system of generalized nonlinear variational-like inclusion problems involving $H(.,.)$-η-proximal mapping has been introduced in 2-uniformly smooth Banach spaces. Using $H(.,.)$-η-proximal mapping, an iterative algorithm has been constructed to solve the proposed system, and the convergence analysis of the algorithm has been investigated. Moreover the obtained results are generalized to solve the system of variational inclusions involving $H(.,.)$-η-proximal mappings. The obtained results generalize most of the results investigated in the literature, and offer a wide range of applications to future research on the sensitivity analysis, variational inclusion problems, variational inequality problems in Banach spaces. Researchers can use the proposed work in the future for research work. Proposed system of generalized nonlinear variational-like inclusion problems in 2-uniformly smooth finds and also will find greater applicability in various fields of real life in the future.

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