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Solving partial differential equations via a hybrid method between homotopy analytical method and Harris hawks optimization algorithm

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Abstract

In this article, a modern analytical technique called Homotopy Analysis Method (HAM) has been applied to several partial differential equations to obtain their sequential solution. The performance of the method was analysed on partial differential equations and found to be computationally efficient. The homotopy analysis method (HAM) has been found to be effective. The numerical results have been improved by proposing a new technique that combines the numerical method with the Harris Hawks Optimization algorithm. The current numerical results are compared to the exact solutions and it's found to be in very good results.

Keywords: homotopy analytical method, Harris hawks optimization algorithm, partial differential equations.

1. Introduction

After the role played by partial differential equations in the past few decades, it has become evident that partial differential equations have a fundamental role in many areas of mathematical and engineering sciences, especially biological mathematics, plasma, magnetic fluid dynamics and dynamic systems, and all this, for example. [3, 9, 8], after an in-depth study of the nonlinear part of partial differential equations, it became clear that it has many interesting advantages, including the phenomena hidden in the nonlinear part of partial differential equations [4, 7, 23]. And in view of the great importance of partial differential equations, attention has been directed to partial differential equations in order to be able to describe scientific phenomena in various sciences, including

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mathematics, physics, chemistry, and engineering and chaotic systems are especially those that have been linked with algorithms [24, 18, 6, 5, 10]. a partial differential equations became very effective in describing these scientific phenomena. In general, there is no numerical method that has an accurate solution for partial differential equations, but because they cannot be solved exactly, and therefore we have to use approximate and numerical methods [1, 2], The Homotopy method was proposed by Liao in his 1992 doctoral thesis and is a very effective method in nonlinear partial differential equations. In this method, an amazing convergence in the chain of solutions is obtained in most cases, and it is worth noting that in a few iterations (few limits) very precise solutions are obtained The HAM method is a successful and effective method for solving nonlinear problems and systems of equations as well as homogeneous and nonhomogeneous problems.

The principle idea of the HAM method is the presence of a parameter called (H), which is a very suitable method for controlling the convergence area and the convergence average to solve the series, and thus through (H) we were able to find clear analytical solutions to solve nonlinear problems.The HAM method is considered one of the best analytical methods for solving nonlinear problems due to its ability to deal with nonlinear partial differential equations [2, 19, 20, 21].

The "HHO" algorithm is distinguished by its flexibility, high performance, and wonderful results, which has earned it a growing interest among researchers. It is also considered free from gradation during many parts of the exploration and activities that change over time, and all of this is due to the feature of HHO algorithm of swarm-based optimization. The cooperative behaviour and hunting methods of Harris hawks in nature called "surprise" was the basis for designing the main logic of the proposed algorithm. Currently, there are many suggestions on how to improve HHO's functionality [6] first published in the Journal of Future Generation Computer Systems, (FGCS) in 2019 [17, 26, 12]. The proposed (HAM-HHO) method is based on finding the optimal parameters for partial differential

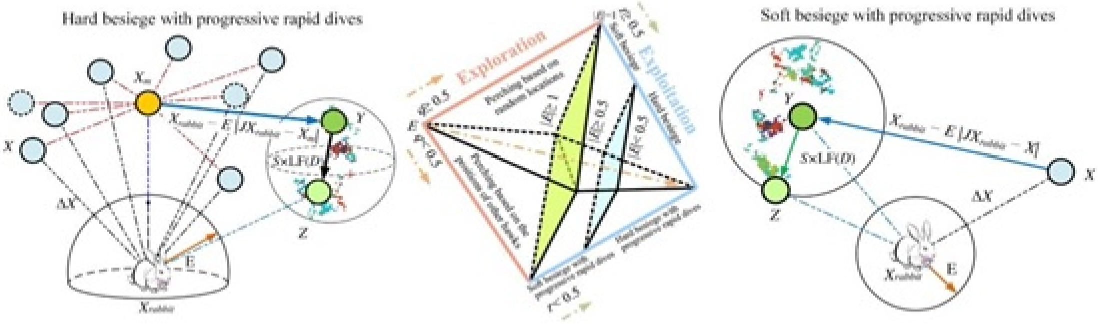

Figure 1: Explains how the Harris hawks optimization algorithm work

equations using Harris hawks optimization algorithm (HHO), When compared to well-established metaheuristic techniques, the HHO algorithm delivers highly promising and occasionally competitive results. with the (HAM) by formulating a fitness function from the HAM that is minimized by the (HHO) to attain optimal values for all parameters partial differential equations nonlinear.

In Section 1 of this paper, the introduction and related works are presented. In Section 2, the basic ideas of the numerical method are described, Section 3 introduces numerical examples and then compares the results with the proposed technique while Section 4 describes the concluding remarks.

2. Outline of the method

To apply the homotopy analytical method [5, 10] to partial differential equations to some problems, and to explain the basic idea, let us consider the next differential equation:

$$N[z(x,t)] = 0 \tag{2.1}$$

where M is represents a nonlinear operator, s and t indicate independent variables, $k(s,t)$ is an unknown function, respectively.
To treat the initial and boundary conditions, we ignore them in the same way, by generalizing the traditional homotopy method, Liao [1] constructs the so-called zero-order deformation equation

$$(1-p)\ \ \psi[\varnothing\,(s,t;p) - z_0(s,t)] = phM[\varnothing\,(s,t;p] \tag{2.2}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, Ψ is an auxiliary linear operator, $z_0(s,\ t)$ is an initial assessment of $k(s,t), \varnothing(s,t;p)$ is a unknown function, respectively. It is important to gain freedom by choose auxiliary, which gives a person high freedom, It is essential that one has high freedom of choosing of helpful items in HAM plainly, when $p = 0$ and $p = 1$, it holds.

$$\begin{aligned} \varnothing\,(s,t,0) &= \ z_0(s,\ t) \\ \varnothing\,(s,t,1) &= \ z(s,\ t) \end{aligned}$$

Straight. If the p-value changes from 0 to 1, the solution $\varnothing(x,t;p)$ varies from the initial guess $z_0\ (s,\ t)$ to the solution $z(s,t)$. Expanding $\varnothing(x,t;p)$ in the Taylor series with respect to p, one has

$$\varnothing\,(s,t;p) = z_0 + \sum_{m=1}^{+\infty} z_m\,(s,t)\,p^m \tag{2.3}$$

Where

$$Z_r = (s,t) = \frac{1}{r!}\frac{\partial^r p(s,t;y)}{\partial y^r}\bigg|_{y=0} \tag{2.4}$$

If we know the auxiliary linear factor, the initial guess, and the auxiliary parameter h is chosen correctly, and the string (2.3) converges at $p = 1$, one has

$$z\,(s,t) = z_0 + \sum_{m=1}^{+\infty} z_m\,(s,t)$$

According to Liao [5], one of the solutions of the original nonlinear equation must exist. Eq. (2.2) becomes Eq. (2.1) when $h = 1$.

$$(1-p)\ \ \psi[\varnothing\,(s,t;y) - z_0(s,t)] = phM[\varnothing\,(s,t;y]$$

Which is usually utilized in the homotopy perturbation approach, where the solution get immediately without the use of Taylor series [14, 15].
The compare between HAM and HPM, can be found in [22, 16]. According to (2.4), the commanding equation can be deduced from the zero-order deformation equation (2.2). Define the vector:

$$\vec{z}_n = \{z_o\,(s,t)\,,\ z_1\,(s,t)\,,\ \ldots,\ z_n\,(s,t)\}$$

Equation (2.2) z times with respect to the inclusion factor Y then set $Y = 0$ and divide them at the end by $z!$, we have the arrangement deformation equation zth.

$$\psi\left[V_r(s,t) - X_r\ V_{r-1}(s,t)\right] = h\ T_r\left(\vec{V}_{r-1}\right) \tag{2.5}$$

Where

$$T_r\left(\vec{V}_{r-1}\right) = \frac{1}{(Z-1)} \quad \left.\frac{\partial^{r-1} p(s,t;y)}{\partial y^{r-1}}\right|_{y=0} \tag{2.6}$$

And

$$X_r = \begin{cases} 0 & r \leq 1 \\ 1 & r > 1 \end{cases}$$

The linear boundary conditions that are taken from the original problem, which can be easily solved by symbolic mathematical programs such as Maple, then it must be indicated that $z_m(x,t)$ for $m \geq 1$ which is controlled due to the linear equation (2.5).

3. Numerical Examples

In this section, several partial differential equations were taken. The numerical results obtained through the proposed numerical and technical methods were constructive, observable, accurate and meaningful. Test problems discovered the strength and effectiveness of the proposed technique.

Example 3.1 ([13]). *The numerical solutions to the test problem 3.1, for the equation of* (3.1) *and constructed using HAM, are indicated in Table 1 for the values of* $h = -1$.
Demonstrate the efficiency and applicability of the proposed algorithm in comparison with the Homotopy (HAM) method. Indicated in Tables 2 for the various values of $h = -0.00000000000000001$.
Noting the comparison between the results of the HAM numerical method, the accurate solution, as well as the suggested algorithm with the precise solution, which confirms the strength of the proposed algorithm, and this confirms to one that it is more accurate and reliable.

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u = \left(\frac{\partial^2 u}{\partial t^2}\right)^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 2u^2 \tag{3.1}$$

$$Y_{Exact}\ (x,y) = \mathbf{e}^{(x+y)} \tag{3.2}$$

$$Y_0(x,y) = \left(1 + y + \frac{1}{2}y^2\right) e^x \tag{3.3}$$

$$
\begin{aligned}
Y_1(x,t) = u_1(x,y) &= \frac{1}{840}h\left(e^x\right)^2 y^7 + \frac{1}{120}h\left(e^x\right)^2 y^6 + \frac{1}{120}hy^5e^x + \frac{1}{30}h\left(e^x\right)^2 y^5 + \frac{1}{12}h\left(e^x\right)^2 y^4 \\
&- \frac{1}{6}hy^3e^x + \frac{1}{3}h\left(e^x\right)^2 y^3 \\
Y_2(x,y) &= -\frac{1}{554400}\left(e^x\right)^3 h^2y^{12} - \frac{1}{46200}\left(e^x\right)^3 h^2y^{11} - \frac{11}{75600}\left(e^x\right)^3 h^2y^{10} + \frac{23}{604800}y^{10}h^2\left(e^x\right)^2 \\
&- \frac{1}{1512}\left(e^x\right)^3 h^2y^9 + \frac{1}{4320}y^9h^2\left(e^x\right)^2 - \frac{31}{10080}y^8\left(e^x\right)^3 h^2 - \frac{1}{20160}y^8h^2\left(e^x\right)^2 + \frac{1}{40320}y^8h^2e^x \\
&- \frac{1}{180}y^7\left(e^x\right)^3 h^2 - \frac{1}{504}h^2\left(e^x\right)^2 y^7 - \frac{1}{5040}y^7h^2e^x + \frac{7}{360}h^2\left(e^x\right)^2 y^6 - \frac{1}{360}y^6h^2e^x - \frac{1}{10}h^2\left(e^x\right)^2 y^5 \\
&+ \frac{1}{60}h^2y^5e^x + \frac{1}{30}y^5\left(e^x\right)^3 h^2 - \frac{1}{3}h^2\left(e^x\right)^2 y^4 + \frac{1}{6}y^4\left(e^x\right)^3 h^2 + \frac{1}{24}h^2y^4e^x - \frac{1}{6}h^2y^3e^x + \frac{1}{3}h^2\left(e^x\right)^2 y^3 \\
&+ \frac{1}{840}h\left(e^x\right)^2 y^7 + \frac{1}{120}h\left(e^x\right)^2 y^6 + \frac{1}{120}hy^5e^x + \frac{1}{30}h\left(e^x\right)^2 y^5 + \frac{1}{12}h\left(e^x\right)^2 y^4 - \frac{1}{6}hy^3e^x + \frac{1}{3}h\left(e^x\right)y^2y^3 \\
Y_3(x,t) &= \frac{ht^2}{2} + h\sin(x)t^2 + \frac{1}{3}t^4h^2\sin(x) + \frac{t^4h^2}{12} + h^2t^2 + 2h^2\sin(x)t^2 + \frac{1}{90}t^6h^3\sin(x) + \frac{t^6h^3}{720} \\
&+ \frac{1}{3}t^4h^3\sin(x) + \frac{t^4h^3}{12} + \frac{h^3t^2}{2} + h^3\sin(x)t^2
\end{aligned}
$$

Table 1: $MSE\ OF\ HAM = 0.17722337784794133381 \times 10^{-5}$

$h=-1$			
0.	***1.1051709180756476248***	***- 0.00036755475079379726059***	***0.00036755475079379726059***
0.1	***1.2214027581601698339***	***- 0.00044643724640966827291***	***0.00044643724640966827291***
0.2	***1.3498588075760031040***	***- 0.0005419790385431744***	***0.0005419790385431744***
0.3	***1.4918246976412703178***	***- 0.0006576175814570732***	***0.0006576175814570732***
0.4	***1.6487212707001281468***	***- 0.00079747726287349887864***	***0.00079747726287349887864***
0.5	***1.8221188003905089749***	***- 0.00096650130822964643227***	***0.00096650130822964643227***
0.6	***2.0137527074704765216***	***- 0.0011706080227782996449***	***0.0011706080227782996449***
0.7	***2.2255409284924676046***	***- 0.0014168757273100677906***	***0.0014168757273100677906***
0.8	***2.4596031111569496638***	***- 0.0017137614097128614490***	***0.0017137614097128614490***
0.9	***2.7182818284590452354***	***- 0.0020713586828996934491***	***0.0020713586828996934491***
1.0	***3.0041660239464331121***	***- 0.0025017008206240074118***	***0.0025017008206240074118***

Table 2: $MSE\ of\ HAM - HHO = 0.96260042390254509955 \times 10^{-7}$

$h=-0.00000000000000001$			
	Exact	***HAM-HHO***	***Exact-(HAM_HHO)***
0.	***1.1051709180756476248***	***- 0.00017091807564832653381***	***0.00017091807564832653381***
0.1	***1.2214027581601698339***	***- 0.0001888936865801429***	***0.0001888936865801429***
0.2	***1.3498588075760031040***	***- 0.00020875980901666446662***	***0.00020875980901666446662***
0.3	***1.4918246976412703178***	***- 0.00023071526978848102012***	***0.00023071526978848102012***
0.4	***1.6487212707001281468***	***- 0.0002549798065264962***	***0.0002549798065264962***
0.5	***1.8221188003905089749***	***- 0.00028179626686999270145***	***0.00028179626686999270145***
0.6	***2.0137527074704765216***	***- 0.00031143303896743213220***	***0.00031143303896743213220***
0.7	***2.2255409284924676046***	***- 0.0003441867375952542***	***0.0003441867375952542***
0.8	***2.4596031111569496638***	***- 0.00038038517277825351582***	***0.00038038517277825351582***
0.9	***2.7182818284590452354***	***- 0.0004203906306224939***	***0.0004203906306224939***
1.0	***3.0041660239464331121***	***- 0.00046460349919642397856***	***0.00046460349919642397856***

Example 3.2. *Consider the equation* (3.2) *having the initial condition and exact solution [25]*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 0 \tag{3.4}$$

$$Y_0\,(x,t) = 1 + \sin\,(x) \tag{3.5}$$

$$Y_{Exact}\,(x,t) = \sin\,(x) + \cosh\,(t) \tag{3.6}$$

$$\begin{aligned}
Y_1(x,t) &= \frac{1}{2}h(1+2\sin(x))t^2 \\
Y_2(x,t) &= \frac{ht^2}{2} + h\sin(x)t^2 + \frac{1}{6}t^4h^2\sin(x) + \frac{t^4h^2}{24} + \frac{h^2t^2}{2} + h^2\sin(x)t^2 \\
Y_3(x,t) &= \frac{ht^2}{2} + h\sin(x)t^2 + \frac{1}{3}t^4h^2\sin(x) + \frac{t^4h^2}{12} + h^2t^2 + 2h^2\sin(x)t^2 + \frac{1}{90}t^6h^3\sin(x) \\
&\quad + \frac{t^6h^3}{720} + \frac{1}{3}t^4h^3\sin(x) + \frac{t^4h^3}{12} + \frac{h^3t^2}{2} + h^3\sin(x)t^2
\end{aligned}$$

Table 3: $MSE\ of\ HAM - HHO = 0.000218992414382082637$6

	$h = -2.0516$		
0.	1.0050041680558035990	- 0.01000000277802582122	0.01000000277802582122
0.1	1.1048375847026317513	- 0.01099667416347684057	0.010996674163476840572
0.2	1.2036734988508648145	- 0.01198338713790019119	0.011983387137900191188
0.3	1.3005243747171431741	- 0.01295028279141937977	0.012950282791419379771
0.4	1.3944225103644540907	- 0.01388770022227765206	0.013887700222277652058
0.5	1.4844297066600065993	- 0.01478627306537487677	0.014786273065374876775
0.6	1.5696466414508389562	- 0.01563702307789151748	0.015637023077891517481
0.7	1.6492218552934946527	- 0.01643144984692307288	0.016431449846923072880
0.8	1.7223602589553263606	- 0.017161615722279595567	0.0171616157222795955674
0.9	1.7883310776832869875	- 0.01782022512943919291	0.017820225129439192909
1.0	1.8464751528637001056	- 0.01840069745936893032	0.018400697459368930321

Table 4: MSE of HAM = 0.00069654518983683482837

	$h = -1$		
	Exact	HAM-HHO	Exact-(HAM_HHO)
0.	1.0050041680558035990	- 0.01583815304734099358	0.01583815304734099358
0.1	1.1048375847026317513	- 0.01800521079217687320	0.018005210792176873204
0.2	1.2036734988508648145	- 0.02015061601236040429	0.020150616012360404294
0.3	1.3005243747171431741	- 0.02225293252810819153	0.022252932528108191528
0.4	1.3944225103644540907	- 0.02429115468772832953	0.024291154687728329533
0.5	1.4844297066600065993	- 0.02624491724914856312	0.026244917249148563118
0.6	1.5696466414508389562	- 0.02809469886268321924	0.028094698862683219244
0.7	1.6492218552934946527	- 0.02982201712190636599	0.029822017121906365994
0.8	1.7223602589553263606	- 0.03140961323374734919	0.031409613233747349194
0.9	1.7883310776832869875	- 0.03284162446264615590	0.032841624462646155899
1.0	1.8464751528637001056	- 0.03410374262576360680	0.034103742625763606805

Example 3.3. *Consider the equation* (3.5) *having the initial condition*

$$u_t - 0.02u_{xx} + 0.1u_x = 0 \tag{3.7}$$

$$Y_0\,(x,t) = e^{(1.17712434446770x)} \tag{3.8}$$

and generates the exact solution given in [11] by

$$Y_{Exact}(x,t) = e^{(1.17712434446770x-0.09t)} \tag{3.9}$$

$$\begin{aligned}
Y_1(x,t) &= \frac{3615154728614}{40168385873489} h\mathbf{e}^{\left(\frac{2916231922x}{2477420449}\right)} t \\
Y_2(x,t) &= \frac{3615154728614}{40168385873489} h\mathbf{e}^{\left(\frac{2916231922x}{2477420449}\right)} t + \frac{3615154728614}{40168385873489} h^2 e^{\left(\frac{2916231922x}{2477420449}\right)} t \\
&+ \frac{9037838389}{2231565034321} h^2 e^{\left(\frac{2916231922x}{2477420449}\right)} t2 \\
Y_3(x,t) &= \frac{3615154728614}{40168385873489} he^{\left(\frac{2916231922x}{2477420449}\right)} t + \frac{7230367543678}{40168708575989} h^2 e^{\left(\frac{2916231922x}{2477420449}\right)} t \\
&+ \frac{18075676778}{2231565034321} h^2 e \left(\frac{2916231922x}{2477420449}\right) t^2 || + \frac{60252241}{495903218107} h^3 e^{\left(\frac{2916231922x}{2477420449}\right)} t^3 \\
&+ \frac{18075515426}{2231545114321} h^3 e^{\left(\frac{2916231922x}{2477420449}\right)} t^2 + \frac{3615154728614}{40168385873489} h^3 e^{\left(\frac{2916231922x}{2477420449}\right)} t
\end{aligned}$$

Table 5: MSE Of HAM = $0.31433112267182639614 \times 10^{-18}$

$h=-1$				
0.	0.99104037877288366216	$-0.27288363746 \times 10^{-9}$	0.27288363746	10^{-9}
0.1	1.11484171416739954 51	$-0.3069724188 \times 10^{-9}$	0.306972418801	10^{-9}
0.2	1.25410838374380131 18	$-0.3453195903 \times 10^{-9}$	***0.345319590282***	10^{-9}
0.3	***1.41077232596297250 22***	$-0.3884571126 \times 10^{-9}$	***0.388457112632***	10^{-9}
0.4	***1.58700681815198247 05***	$-0.4369833991 \times 10^{-9}$	***0.436983399080***	10^{-9}
0.5	***1.78525662469436839 77***	$-0.4915716171 \times 10^{-9}$	***0.491571617110***	10^{-9}
0.6	***2.00827191134972604 32***	$-0.5529790269 \times 10^{-9}$	***0.552979026860***	10^{-9}
0.7	***2.25914639616965344 25***	$-0.6220574856 \times 10^{-9}$	***0.622057485595***	10^{-9}
0.8	***2.54136026624810611 73***	$-0.6997652654 \times 10^{-9}$	***0.699765265419***	10^{-9}
0.9	***2.85882845565694571 46***	$-0.7871803458 \times 10^{-9}$	***0.787180345763***	10^{-9}
1.0	***3.21595495428902685 64***	$-0.88551536821 \times 0^{-9}$	***0.885515368196***	10^{-9}

Table 6: MSE of HAM-HHO = $0.16001525138627902803 10^{-22}$

$h=-0.99046$			
	Exact	***HAM-HHO***	***Exact-(HAM_HHO)***
0.	***0.99104037877288366216***	-0.19469934710^{-11}	0.19469934710^{-11}
0.1	***1.11484171416739954 51***	-0.2190213010^{-11}	0.219021300910^{-11}
0.2	***1.25410838374380131 18***	-0.2463815710^{-11}	0.246381565210^{-11}
0.3	***1.41077232596297250 22***	-0.2771597010^{-11}	0.277159699910^{-11}
0.4	***1.58700681815198247 05***	-0.3117826510^{-11}	0.311782650810^{-11}
0.5	***1.78525662469436839 77***	-0.3507307110^{-11}	0.350730711410^{-11}
0.6	***2.00827191134972604 32***	-0.3945441910^{-11}	0.394544191810^{-11}
0.7	***2.25914639616965344 25***	-0.4438308710^{-11}	0.443830873510^{-11}
0.8	***2.54136026624810611 73***	-0.4992745010^{-11}	0.499274496410^{-11}
0.9	***2.85882845565694571 46***	-0.5616441610^{-11}	0.561644155610^{-11}
1.0	***3.21595495428902685 64***	-0.6318050510^{-11}	0.631805054810^{-11}

4. Conclusion

The numerical method (HAM) was used successfully in order to obtain the analytical solutions of some nonlinear partial differential equations. The results were compared with the exact solution. On

the other hand, A new technique that combines the numerical method and the algorithm is presented (HHO).
The effectiveness of the proposed technique has been observed very effectively due to the rapid convergence of the solution according to the technique, as it can be applied to many nonlinear partial differential equations.

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