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On the maximum number of limit cycles of a planar differential system

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Abstract

In this work, we are interested in the study of the limit cycles of a perturbed differential system in $\mathbb{R}^2$, given as follows

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(1 + \sin^m(\theta))\psi(x, y), \end{cases}$$

where ε is small enough, m is a non-negative integer, $\tan(\theta) = y/x$, and $\psi(x, y)$ is a real polynomial of degree $n \geq 1$. We use the averaging theory of first order to provide an upper bound for the maximum number of limit cycles. In the end, we present some numerical examples to illustrate the theoretical results.

Keywords: Periodic solution, averaging method, differential system.
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1. Introduction

One of the main problem in the qualitative of planar differential system is the determination of limit cycles. This notion was defined first by Poincaré [12] as a periodic orbit isolated in the set of all periodic orbits of a differential system. In contrast, a center is a critical point in the neighborhood of which all orbits are closed. Existence, number, stability and other properties of limit cycles were studied extensively by mathematicians and physicists and also by chemists, biologists, economists, etc (see [1, 7, 10]).

One of the main tools for studying the limit cycles is the averaging theory. This method appears early in the works of Lagrange and Laplace, and later in the first half of XXth century, in the works

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of Fatou, Bogoliubov and Krylov (see [11]). It is still a powerful tool in studying various types of dynamical systems (see [1, 3, 6, 9, 10, 13, 14] and the references therein).

In [5] T. Chen & J. Llibre considered a second-order differential equation

$$\ddot{x} + \varepsilon(1 + \cos^m(\theta))Q(x, y) + x = 0, \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, m is an arbitrary non-negative integer, $Q(x, y)$ is a polynomial of degree $n \geq 1$ and $\theta = \arctan(y/x)$. They determined an upper for the maximum number of limit cycles in equation (1.1) in the four cases where m and n are even and odd.

In our work, by using the averaging theory of first order, we study the maximum number of limit cycles of the following planar differential system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(1 + \sin^m(\theta))\psi(x, y). \end{cases} \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter, m is an arbitrary non-negative integer, $\psi(x, y)$ is a polynomial of degree $n \geq 1$ and $\tan(\theta) = y/x$.

2. Fundamental tools

In this section, we recall some preliminary notions and results, which will be useful in the sequel.

Theorem 2.1 (Averaging theory of first order). *Consider the differential system*

$$\dot{x}(t) = \varepsilon F(x, t) + \varepsilon^2 R(x, t, \varepsilon), \tag{2.1}$$

where $F : D \times \mathbb{R} \to \mathbb{R}^n$, $R : D \times \mathbb{R} \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the two following hypotheses hold.

1. *F and R are locally Lipschitz with respect to x.*
2. *We define $f : D \to \mathbb{R}^n$, called the averaged function associated to system (1.2), as*

$$f(z) = \frac{1}{T}\int_0^T F(z, s)ds.$$

For $z_* \in D$ with $f(z_*) = 0$, there exists a neighborhood V of z_* such that $f(z) \neq 0$ for all $z \in \bar{V} \setminus \{z_*\}$ and $d_B(f, V, z_*) \neq 0$.

Then, for $|\varepsilon| > 0$ small enough, there exists a T-periodic solution $\varphi(., \varepsilon)$ of system (2.1) such that $\varphi(0, \varepsilon) \to z_*$ as $\varepsilon \to 0$.

For a proof, see [4, 9].

Remark 2.2. *The expression $d_B(f, V, z_*) \neq 0$ means that the Brouwer degree of the function $f : V \to \mathbb{R}^n$ at the fixed point z_* is not zero. A sufficient condition for this inequality to be true is that the Jacobian of the function f at z_* is not zero.*

Next, in order to calculate the averaged function, we will use the following formulas (see [15]).

$$\int_0^{2\pi} \cos^p(\theta)\sin^{2q}(\theta)d\theta = \frac{(2q-1)!!}{(2q+p)(2q+p-2)...(p+2)}\int_0^{2\pi} \cos^p(\theta)d\theta, \tag{2.2}$$

$$p \in \mathbb{R} \setminus \{-2,-4,...\},\ q \in \mathbb{N},$$

$$\int_0^{2\pi} \cos^p(\theta)\sin^{2q+1}(\theta)d\theta = 0, \quad p \in \mathbb{R} \setminus \{-1,-3,...\},\ q \in \mathbb{N}. \tag{2.3}$$

$$\int_0^{2\pi} \cos^{2l}(\theta)d\theta = \frac{(2l-1)!!}{2^l l}2\pi, \quad l > 0, \tag{2.4}$$

$$\int_0^{2\pi} \cos^{2l+1}(\theta)d\theta = 0, \quad l \geq 0, \tag{2.5}$$

Theorem 2.3 (Descartes Theorem). *Let us consider the real polynomial*

$$p(r) = a_{i_1}r^{i_1} + a_{i_2}r^{i_2} + ... + a_{i_n}r^{i_n},$$

with $0 \leq i_1 < i_2 < ... < i_n$ *and* $a_{i_j} \neq 0$ *real constants for* $j \in 1,2,...,n$. *When* $a_{i_j}a_{i_{j+1}} < 0$, *we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of the signs is* m, *then* $p(r)$ *has at most* m *positive real zeros. In addition, it is always possible to choose the coefficients of* $p(r)$ *in such a way that* $p(r)$ *has exactly* $n-1$ *positive real zeros.*

For the proof, see [2].

3. Main result

The main contribution of this article is summed up in the following theorem.

Theorem 3.1. *Suppose that* $f(r)$ *the average function of first order is non-zero and* $\varepsilon > 0$ *is small enough. The maximum number of limit cycles bifurcating from the periodic solutions of the center is at most*

(a) $n-1$, *if* m *is odd and* n *is even.*

(b) n, *if* m *and* n *are odd.*

(c) $(n-2)/2$, *if* m *and* n *are even.*

(d) $(n-1)/2$, *if* m *is even and* n *is odd.*

Remark 3.2. *When* $m = 0$, *system (1.2) reads*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon P_n(x,y). \end{cases} \tag{3.1}$$

By Theorem 3.1, for system (3.1), the maximum number of limit cycles bifurcating from the periodic solutions of center is at most $(n-1)/2$ *or* $(n-2)/2$, *when* n *is odd or even, respectively.*

Proof .[Proof of Theorem 3.1] Suppose that the polynomial $\psi(x,y) = \sum_{i+j=0}^{n} a_{i_j}x^iy^j$. By performing the change of variable, $(x,y) \to (r,\theta)$, where $x = r\cos(\theta),\ y = r\sin(\theta)$, with $r > 0$, system (1.2) in polar coordinates (r,θ) writes

$$\begin{cases} \dot{r} = -\varepsilon \sum_{i+j=0}^{n} R_{ij}(\theta)r^{i+j}, \\ \dot{\theta} = -1 - \varepsilon \sum_{i+j=0}^{n} \Theta_{ij}(\theta)r^{i+j-1}, \end{cases}$$

where

$$\begin{cases} R_{ij}(\theta) = a_{ij}(\cos^i(\theta)\sin^{j+1}(\theta) + \cos^i(\theta)\sin^{m+j+1}(\theta), \\ \Theta_{ij}(\theta) = a_{ij}(\cos^{i+1}(\theta)\sin^j(\theta) + \cos^{i+1}(\theta)\sin^{m+j}(\theta). \end{cases}$$

Taking θ as the new independent variable, the previous differential system becomes

$$\frac{dr}{d\theta} = \varepsilon \sum_{i+j=0}^{n} R_{ij}(\theta) r^{i+j} + O(\varepsilon^2) = \varepsilon F(r,\theta) + O(\varepsilon^2). \tag{3.2}$$

Note that this differential equation is written in the normal form (2.1) which allows us to apply the averaging theory of first order. Therefore, from the previous section, we consider two four cases in order to study the averaged function associated to the differential equation (3.2).

Case (a) If m is odd and n is even, we have

$$\begin{aligned}
f_1(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{i+j=0}^{n} a_{ij} r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta) + \cos^i(\theta)\sin^{j+m+1}(\theta))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+j=0}^{n} a_{ij} r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta) \right.\\
&\quad \left. + \sum_{i+j=0}^{n} a_{ij} r^{i+j}\cos^i(\theta)\sin^{j+m+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+2p-1=0}^{n} a_{i,2p-1} r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta) \right.\\
&\quad \left. + \sum_{i+2p=0}^{n} a_{i,2p} r^{i+2p}\cos^i(\theta)\sin^{2p+m+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+2p=2}^{n+1} a_{i,2p-1} r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta) \right.\\
&\quad \left. + \sum_{i+2p=1}^{n+1} a_{i,2p} r^{i+2p}\cos^i(\theta)\sin^{2p+m+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{2l+2p=2}^{n+1} a_{2l,2p-1} r^{2l+2p-1}\cos^{2l}(\theta)\sin^{2p}(\theta) \right.\\
&\quad \left. + \sum_{2l+2p=1}^{n+1} a_{2l,2p} r^{2l+2p}\cos^{2l}(\theta)\sin^{2p+m+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{2l+2p=2}^{n} a_{2l,2p-1} r^{2l+2p-1}\cos^{2l}(\theta)\sin^{2p}(\theta) \right.\\
&\quad \left. + \sum_{2l+2p=2}^{n} a_{2l,2p} r^{2l+2p}\cos^{2l}(\theta)\sin^{2p+m+1}(\theta)\right] d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p-1}r^{2l+2p-1}\cos^{2l}(\theta)\sin^{2p}(\theta)\right.\\
&\quad \left.+\sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p}r^{2l+2p}\cos^{2l}(\theta)\sin^{2p+m+1}(\theta)\right]d\theta\\
&= \frac{1}{2\pi}\sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p-1}r^{2l+2p-1}\int_0^{2\pi}\cos^{2l}(\theta)\sin^{2p}(\theta)d\theta\\
&\quad +\frac{1}{2\pi}\sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p}r^{2l+2p}\int_0^{2\pi}\cos^{2l}(\theta)\sin^{2p+m+1}(\theta)d\theta.
\end{aligned}
$$

By using (2.2), (2.4) and (2.5), we get

$$
\begin{aligned}
f_1(r) &= \frac{1}{2\pi}\sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p-1}r^{2l+2p-1}\times\left[\frac{(2p-1)!!}{(2p+2l)(2p+2l-2)...(2l+2)}\frac{(2l-1)!!2\pi}{2^l l!}\right]\\
&\quad +\frac{1}{2\pi}\sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p}r^{2l+2p}\\
&\quad \times\left[\frac{(2p+m+1-1)!!}{(2p+2l+m+1)(2p+2l+m+1-2)...(2l+2)}\frac{(2l-1)!!2\pi}{2^l(l)!}\right]\\
&= \sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p-1}r^{2l+2p-1}\left[\frac{(2p-1)!!(2l-1)!!}{2(p+l)2(p+l-1)...2(l+1)2^l(l)!}\right]\\
&\quad +\sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p}r^{2l+2p}\left[\frac{(2p+m)!!(2l-1)!!}{(2p+2l+m+1)(2p+2l+m-1)...(2l+2)2^l(l)!}\right]\\
&= \sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p-1}r^{2l+2p-1}\left[\frac{(2p-1)!!(2l-1)!!}{2^{p+l}(l)!(p+l)(p+l-1)...(l+1)}\right]\\
&\quad +\sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p}r^{2l+2p}\left[\frac{(2l-1)!!(2p+m)!!}{2^l(l)!(2p+2l+m+1)(2p+2l+m-1)...(2l+2)}\right]\\
&= \sum_{k=1}^{n} A_k r^k. \qquad (3.3)
\end{aligned}
$$

Case (b) If m and n are odd, we have

$$
\begin{aligned}
f_2(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+j=0}^{n} a_{ij}r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta)+\cos^i(\theta)\sin^{j+m+1}(\theta))d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{i+j=0}^{n} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta) \\
&\quad + \sum_{i+j=0}^{n} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+m+1}(\theta)\Bigg]\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{i+2p-1=0}^{n} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{i+2p=0}^{n} a_{i,2p}r^{i+2p}\cos^i(\theta)\sin^{2p+m+1}(\theta)\Bigg]\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{i+2p=2}^{n+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{i+2p=1}^{n+1} a_{i,2p}r^{i+2p}\cos^i(\theta)\sin^{2p+m+1}(\theta)\Bigg]\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2l+2p=2}^{n+1} a_{2l,2p-1}r^{2l+2p-1}\cos^{2l}(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{2l+2p=2}^{n+1} a_{2l,2p}r^{2l+2p}\cos^{2l}(\theta)\sin^{2p+m+1}(\theta)\Bigg]\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p-1}r^{2l+2p-1}\cos^{2l}(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p}r^{2l+2p}\cos^{2l}(\theta)\sin^{2p+m+1}(\theta)\Bigg]\,d\theta \\
&= \frac{1}{2\pi}\sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p-1}r^{2l+2p-1}\int_0^{2\pi}\cos^{2l}(\theta)\sin^{2p}(\theta)d\theta \\
&\quad + \frac{1}{2\pi}\sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p}r^{2l+2p}\int_0^{2\pi}\cos^{2l}(\theta)\sin^{2p+m+1}(\theta)d\theta.
\end{aligned}
$$

Thus, by using (2.2), (2.4) and (2.5), we get

$$
\begin{aligned}
f_2(r) &= \frac{1}{2\pi}\sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p-1}r^{2l+2p-1}\times\left[\frac{(2p-1)!!}{(2p+2l)(2p+2l-2)...(2l+2)}\frac{(2l-1)!!2\pi}{2^l l!}\right] \\
&\quad + \frac{1}{2\pi}\sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p}r^{2l+2p} \\
&\quad \times\left[\frac{(m+2p+1-1)!!}{(2p+2l+m+1)(2p+2l+m+1-2)...(2l+2)}\frac{(2l-1)!!2\pi}{2^l(l)!}\right]
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p-1} r^{2l+2p-1} \left[\frac{(2p-1)!!(2l-1)!!}{2(p+l)2(p+l-1)...2(l+1)2^l(l)!}\right] \\
&\quad + \sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p} r^{2l+2p} \left[\frac{(2p+m)!!(2l-1)!!}{(2p+2l+m+1)(2p+2l+m-1)...(2l+2)2^l(l)!}\right] \\
&= \sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p-1} r^{2l+2p-1} \left[\frac{(2p-1)!!(2l-1)!!}{2^{p+l}(l)!(p+l)(p+l-1)...(l+1)}\right] \\
&\quad + \sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p} r^{2l+2p} \left[\frac{(2p+m)!!(2l-1)!!}{2^l(l)!(2p+2l+m+1)(2p+2l+m-1)...(2l+2)}\right] \\
&= \sum_{k=1}^{n+1} B_k r^k. \qquad (3.4)
\end{aligned}$$

Case (c) If m and n are even, we have

$$\begin{aligned}
f_3(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{i+j=0}^{n} a_{ij} r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta) + \cos^i(\theta)\sin^{j+m+1}(\theta))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+j=0}^{n} a_{ij} r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right. \\
&\quad \left. + \sum_{i+j=0}^{n} a_{ij} r^{i+j}\cos^i(\theta)\sin^{j+m+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+2p=1}^{n} a_{i,2p-1} r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{i+2p=1}^{n} a_{i,2p-1} r^{i+2p-1}\cos^i(\theta)\sin^{2p+m}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+2p=2}^{n+1} a_{i,2p-1} r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{i+2p=2}^{n+1} a_{i,2p-1} r^{i+2p-1}\cos^i(\theta)\sin^{2p+m}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{2l+2p=2}^{n} a_{2l,2p-1} r^{2l+2p-1}\cos^{2l}(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{2l+2p=2}^{n} a_{2l,2p-1} r^{2l+2p-1}\cos^{2l}(\theta)\sin^{2p+m}(\theta)\right] d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\sum_{2l+2p=2}^{n} a_{2l,2p-1}r^{2l+2p-1} \\
&\quad\times\left[\int_0^{2\pi}\cos^{2l}(\theta)\sin^{2p}(\theta)+\int_0^{2\pi}\cos^{2l}(\theta)\sin^{2p+m}(\theta)\right]d\theta \\
&= \frac{1}{2\pi}\sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p-1}r^{2l+2p-1} \\
&\quad\times\left[\int_0^{2\pi}\cos^{2l}(\theta)\sin^{2p}(\theta)+\int_0^{2\pi}\cos^{2l}(\theta)\sin^{2p+m}(\theta)\right]d\theta.
\end{aligned}
$$

Hence, by using (2.2), (2.4) and (2.5), we get

$$
\begin{aligned}
f_3(r) &= \frac{1}{2\pi}\sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p-1}r^{2l+2p-1}\left[\frac{(2p-1)!!}{(2p+2l)(2p+2l-2)...(2l+2)}\frac{(2l-1)!!2\pi}{2^l l!}\right. \\
&\quad\left.+\frac{(m+2p-1)!!}{(2p+2l+m)(2p+2l+m-2)...(2l+2)}\frac{(2l-1)!!2\pi}{2^l(l)!}\right] \\
&= \sum_{l+p=1}^{\frac{n}{2}} a_{2l,2p-1}r^{2l+2p-1}\left[\frac{(2p-1)!!(2l-1)!!}{2^{l+p}(l)!(p+l)(p+l-1)...(l+1)}\right. \\
&\quad\left.+\frac{(2p+m-1)!!(2l-1)!!}{2^l(l)!(2p+2l+m)(2p+2l+m-2)...(2l+2)}\right] \\
&= \sum_{k=1}^{\frac{n}{2}} C_k r^{2k-1}. \qquad (3.5)
\end{aligned}
$$

Case (d) If m is even and n is odd, we have

$$
\begin{aligned}
f_4(r) &= \frac{1}{2\pi}\int_0^{2\pi}F(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+j=0}^{n} a_{ij}r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta)+\cos^i(\theta)\sin^{j+m+1}(\theta))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+j=0}^{n} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right. \\
&\quad\left.+\sum_{i+j=0}^{n} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+m+1}(\theta)\right]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p=1}^{n} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad\left.+\sum_{i+2p=1}^{n} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p+m}(\theta)\right]d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p=2}^{n+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right.\\
&\quad\left.+\sum_{i+2p=2}^{n+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p+m}(\theta)\right]d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2l+2p=2}^{n+1} a_{2l,2p-1}r^{2l+2p-1}\cos^{2l}(\theta)\sin^{2p}(\theta)\right.\\
&\quad\left.+\sum_{2l+2p=2}^{n+1} a_{2l,2p-1}r^{2l+2p-1}\cos^{2l}(\theta)\sin^{2p+m}(\theta)\right]d\theta\\
&= \frac{1}{2\pi}\sum_{2l+2p=2}^{n+1} a_{2l,2p-1}r^{2l+2p-1}\\
&\quad\times\left[\int_0^{2\pi}\sin^{2p}(\theta)\cos^{2l}(\theta)+\int_0^{2\pi}\cos^{2l}(\theta)\sin^{2p+m}(\theta)\right]d\theta\\
&= \frac{1}{2\pi}\sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p-1}r^{2l+2p-1}\\
&\quad\times\left[\int_0^{2\pi}\cos^{2l}(\theta)\sin^{2p}(\theta)+\int_0^{2\pi}\cos^{2l}(\theta)\sin^{2p+m}(\theta)\right]d\theta.
\end{aligned}
$$

Thus, by using (2.2), (2.4) and (2.5), we get

$$
\begin{aligned}
f_4(r) = &\ \frac{1}{2\pi}\sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p-1}r^{2l+2p-1}\left[\frac{(2p-1)!!}{(2p+2l)(2p+2l-2)...(2l+2)}\frac{(2l-1)!!2\pi}{2^l l!}\right.\\
&\left.+\frac{(2p+m-1)!!}{(2p+2l+m)(2p+2l+m-2)...(2l+2)}\frac{(2l-1)!!2\pi}{2^l(l)!}\right]\\
= &\ \sum_{l+p=1}^{\frac{n+1}{2}} a_{2l,2p-1}r^{2l+2p-1}\left[\frac{(2p-1)!!(2l-1)!!}{2^{l+p}(l)!(p+l)(p+l-1)...(l+1)}\right.\\
&\left.+\frac{(2l-1)!!(2p+m-1)!!}{2^l(l)!(2p+2l+m)(2p+2l+m-2)...(2l+2)}\right]\\
= &\ \sum_{k=1}^{\frac{(n+1)}{2}} D_k r^{2k-1}. \qquad (3.6)
\end{aligned}
$$

According to the expressions of A_k, B_k, C_k and D_k in (3.3), (3.4), (3.5) and (3.6) respectively, we get

$$
\mathbf{J}_1 = \frac{\partial(A_1, A_2, ..., A_n)}{\partial(a_{0,1}, a_{0,2}, ..., a_{0,n})}
$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{(m+2)!!}{(m+3)!!} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{(n-1)!!}{2^{\frac{n}{2}}(\frac{n}{2})!!} & 0 \\ 0 & 0 & \cdots & 0 & \frac{(m+n)!!}{(m+n+1)!!} \end{bmatrix},$$

$$\mathbf{J}_2 = \frac{\partial(B_1, B_2, ..., B_n)}{\partial(a_{0,1}, a_{0,2}, ..., a_{0,n})}$$
$$= \begin{bmatrix} \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{(m+2)!!}{(m+3)!!} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{n!!}{2^{\frac{n+1}{2}}(\frac{n+1}{2})!!} & 0 \\ 0 & 0 & \cdots & 0 & \frac{(m+n+1)!!}{(m+n+2)!!} \end{bmatrix},$$

$$\mathbf{J}_3 = \frac{\partial(C_1, C_3, ..., C_{n-1})}{\partial(a_{0,1}, a_{0,3}, ..., a_{0,n-1})}$$
$$= \begin{bmatrix} \frac{1}{2} + \frac{(m+1)!!}{(m+2)!!} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{(n-1)!!}{2^{\frac{n}{2}}(\frac{n}{2})!!} + \frac{(m+n-1)!!}{(m+n)!!} \end{bmatrix},$$

$$\mathbf{J}_4 = \frac{\partial(D_1, D_3, ..., D_n)}{\partial(a_{0,1}, a_{0,3}, ..., a_{0,n})}$$
$$= \begin{bmatrix} \frac{1}{2} + \frac{(m+1)!!}{(m+2)!!} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{n!!}{2^{\frac{n+1}{2}}(\frac{n+1}{2})!!} + \frac{(m+n)!!}{(m+n+1)!!} \end{bmatrix},$$

which clearly shows that $\det(\mathbf{J}_1) \neq 0$, $\det(\mathbf{J}_2) \neq 0$, $\det(\mathbf{J}_3) \neq 0$ and $\det(\mathbf{J}_4) \neq 0$. Consequently, the families $\{A_k\}, \{B_k\}, \{C_k\}$ and $\{D_k\}$ are independent. Hence, from (3.3) and (3.4), the averaged function $f(r)$ is generated by a linear combination of functions in the set $\mathcal{S}_1 = \{r, r^2, ..., r^l\}$, with $l \in \{n, n+1\}$. Using the Descartes Theorem, it follows that $f(r)$ can have at most $n-1$ simple zeros if m is odd and n is even, or at most n when n and m are odd numbers. Therefore, by Theorem 2.1, for $\varepsilon > 0$ sufficiently small, the differential system (1.2) can have at most $n-1$ or n limit cycles. Similarly, from (3.5) and (3.6), $f(r)$ is generated by a linear combination of functions in $\mathcal{S}_2 = \{r, r^3, ..., r^{2l-1}\}$, with $l \in \{n/2, (n+1)/2\}$, and so, by using the Descartes Theorem and Theorem 2.1, for $\varepsilon > 0$ sufficiently small, the differential system (1.2) can have at most $(n-2)/2$ or $(n-1)/2$ limit cycles.

□

4. Applications and simulations

In this section, four numerical examples are presented in order to confirm our results.

Example 4.1. *Consider the differential system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(1 + \sin^3(\theta))(y - x^2). \end{cases} \tag{4.1}$$

In polar coordinates,

$$\begin{cases} x = r\cos(\theta), \\ y = r\sin(\theta), \end{cases}$$

with

$$\begin{cases} \dot{r} = \dfrac{x\dot{x} + y\dot{y}}{r}, \\ \dot{\theta} = \dfrac{x\dot{y} - y\dot{x}}{r^2}, \end{cases}$$

system (4.1) becomes

$$\begin{cases} \dot{r} = -r\varepsilon\sin(\theta)(1 + \sin^3(\theta))(\sin(\theta) - r\cos^2(\theta)) \\ \dot{\theta} = -1 - \varepsilon\cos(\theta)(1 + \sin^3(\theta))(\sin(\theta) - r\cos^2(\theta)). \end{cases}$$

Taking θ as an independent time variable,

$$\frac{dr}{d\theta} = \frac{dr}{dt}\frac{dt}{d\theta},$$

we get the following equation

$$\frac{dr}{d\theta} = r\varepsilon\sin(\theta)(1 + \sin^3(\theta))(\sin(\theta) - r\cos^2(\theta)).$$

So, we find the normal form of the averaging theory of first order

$$\frac{dr}{d\theta} = \varepsilon F(r, \theta) + O(\varepsilon^2),$$

where $F(r, \theta)$ is given by

$$F(r, \theta) = r\sin(\theta)(1 + \sin^3(\theta))(\sin(\theta) - r\cos^2(\theta)).$$

The averaged function is calculated as follows

$$\begin{aligned} f_1(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r, \theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} r\sin(\theta)(1 + \sin^3(\theta))(\sin(\theta) - r\cos^2(\theta))d\theta \\ &= \frac{1}{2\pi}\left[\int_0^{2\pi} r\sin^2(\theta)d\theta - \int_0^{2\pi} r^2\cos^2(\theta)\sin(\theta)d\theta \right. \\ &\left. + \int_0^{2\pi} r\sin^5(\theta)d\theta - \int_0^{2\pi} r^2\cos^2(\theta)\sin^4(\theta)d\theta\right] \end{aligned}$$

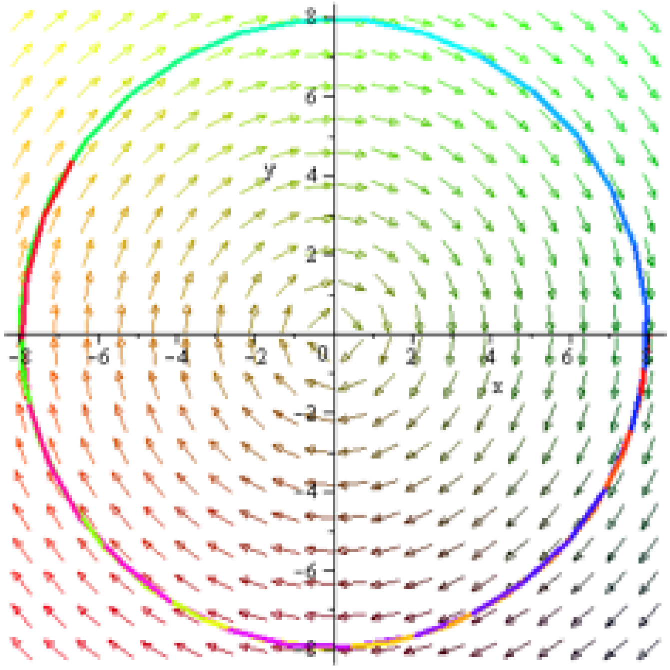

Figure 1: Limit cycle of amplitude 8 for system (4.1) when $\varepsilon = 0.001$.

$$= \frac{1}{2\pi}\left[r\pi - r^2\frac{\pi}{8}\right]$$
$$= \frac{1}{16}\left[8r - r^2\right],$$

which has only one positive zero, $r_* = 8$. Since

$$\frac{df_1(r)}{dr} = \frac{1}{16}[8 - 2r],$$

we obtain

$$\frac{df_1(8)}{dr} = \frac{-1}{2} \neq 0.$$

Since $m = 3$ and $n = 2$, from Theorem 3.1(a), system (4.1) can have at most one limit cycle, and this appears clearly in Fig. 1.

Example 4.2. *Consider the differential system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(1 + \sin^3(\theta))(y^3 - x^2). \end{cases} \tag{4.2}$$

In polar coordinates system (4.2) becomes

$$\begin{cases} \dot{r} = -r\varepsilon \sin(\theta)(1 + \sin^3(\theta))(r^2\sin^3(\theta) - r\cos^2(\theta)) \\ \dot{\theta} = -1 - \varepsilon\cos(\theta)(1 + \sin^3(\theta))(r^2\sin^3(\theta) - r\cos^2(\theta)). \end{cases}$$

Taking θ as an independent time variable, we get the following equation

$$\frac{dr}{d\theta} = r\varepsilon \sin(\theta)(1 + \sin^3(\theta))(r^2\sin^3(\theta) - r\cos^2(\theta)).$$

Hence, we find the normal form of the averaging theory of first order

$$\frac{dr}{d\theta} = \varepsilon F(r,\theta) + O(\varepsilon^2),$$

where $F(r,\theta)$ *is given by*

$$F(r,\theta) = r\sin(\theta)(1+\sin^3(\theta))(r^2\sin^3(\theta) - r\cos^2(\theta)).$$

The averaged function is calculated as follows

$$\begin{aligned} f_2(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} r\sin(\theta)(1+\sin^3(\theta))(r^2\sin^3(\theta) - r\cos^2(\theta))d\theta \\ &= \frac{1}{2\pi}\left[\int_0^{2\pi} r^3\sin^4(\theta)d\theta - \int_0^{2\pi} r^2\cos^2(\theta)\sin(\theta)d\theta \right. \\ &+ \left. \int_0^{2\pi} r^3\sin^7(\theta)d\theta - \int_0^{2\pi} r^2\cos^2(\theta)\sin^4(\theta)d\theta\right] \\ &= \frac{r}{2\pi}\left[\frac{3\pi}{4} - \frac{\pi}{8}r\right] \\ &= \frac{1}{16}r^2[6r-1], \end{aligned}$$

which has a unique positive zero, $r_* = \frac{1}{6}$. *Since*

$$\frac{df_2(r)}{dr} = \frac{1}{8}r[6r-1] + \frac{3}{8}r^2,$$

we get

$$\frac{df_2(\frac{1}{6})}{dr} = \frac{1}{96} \neq 0.$$

Since $m = 3$ *and* $n = 3$, *from Theorem 3.1(b), system (4.2) can have at most three limit cycles, see Fig. 2.*

Example 4.3. *Consider the differential system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(1+\sin^2(\theta))(y - x^2). \end{cases} \tag{4.3}$$

In polar coordinates, system (4.3) becomes

$$\begin{cases} \dot{r} = -r\varepsilon\sin(\theta)(1+\sin^2(\theta))(\sin(\theta) - r\cos^2(\theta)) \\ \dot{\theta} = -1 - \varepsilon\cos(\theta)(1+\sin^2(\theta))(\sin(\theta) - r\cos^2(\theta)). \end{cases}$$

Taking θ *as an independent time variable, we get the following equation*

$$\frac{dr}{d\theta} = r\varepsilon\sin(\theta)(1+\sin^2(\theta))(\sin(\theta) - r\cos^2(\theta)).$$

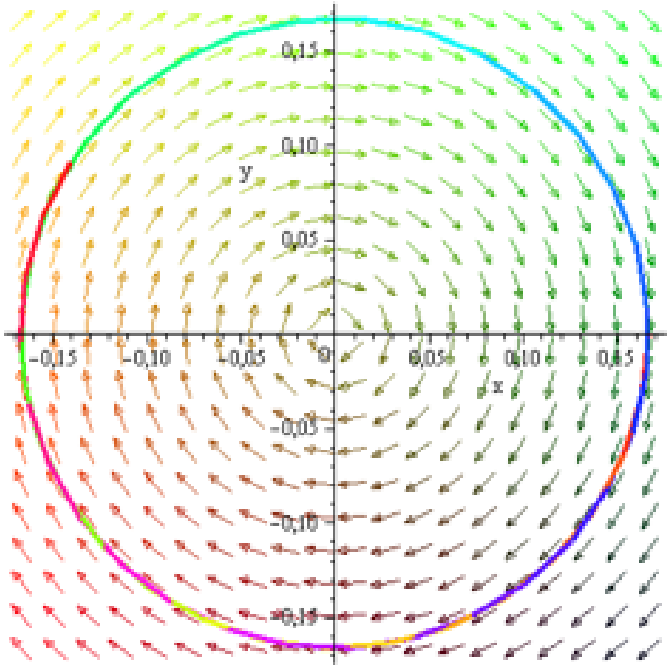

Figure 2: Limit cycle of amplitude 1/6 for system (4.2) when $\varepsilon = 0.001$.

So, we find the normal form of the averaging theory of first order

$$\frac{dr}{d\theta} = \varepsilon F(r,\theta) + O(\varepsilon^2),$$

where $F(r,\theta)$ *is given by*

$$F(r,\theta) = r\sin(\theta)(1+\sin^2(\theta))(\sin(\theta) - r\cos^2(\theta)),$$

The averaged function is calculated as follows

$$\begin{aligned} f_3(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} r\sin(\theta)(1+\sin^2(\theta))(\sin(\theta) - r\cos^2(\theta))d\theta \\ &= \frac{1}{2\pi}\left[\int_0^{2\pi} r\sin^2(\theta)d\theta - \int_0^{2\pi} r^2\cos^2(\theta)\sin(\theta)d\theta \right. \\ &\left. + \int_0^{2\pi} r\sin^4(\theta)d\theta - \int_0^{2\pi} r^2\cos^2(\theta)\sin^3(\theta)d\theta\right] \\ &= \frac{1}{2\pi}\left[r\frac{7\pi}{4}\right] \\ &= \frac{7}{8}r. \end{aligned}$$

$f_3(r)$ *does not have a positive zero, which implies that there is no limit cycle, which is confirmed by Theorem 3.1(c), since* $m = 2$ *and* $n = 2$ *(see Fig. 3).*

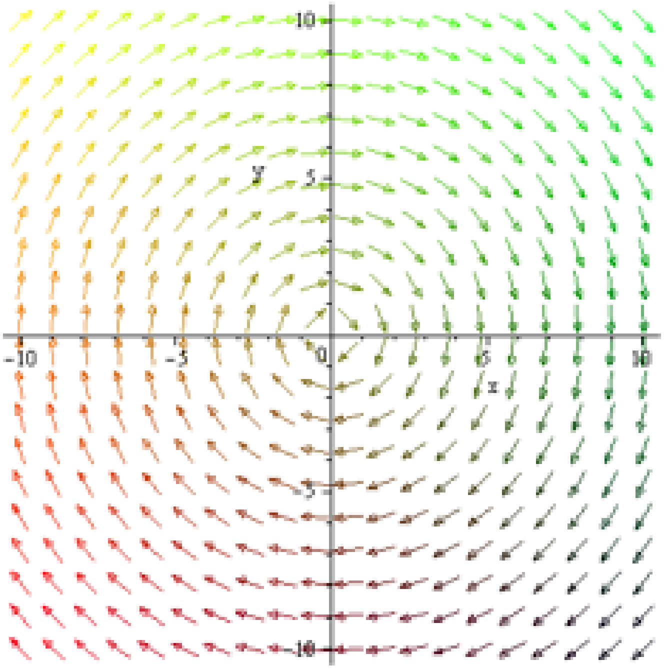

Figure 3: No limit cycle for system (4.3) when $\varepsilon = 0.001$.

Example 4.4. *Let the differential system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(1+\sin^2(\theta))(y^3 - x^2y^3). \end{cases} \tag{4.4}$$

In polar coordinates, system (4.4) becomes

$$\begin{cases} \dot{r} = -r\varepsilon\sin(\theta)(1+\sin^2(\theta))(r^2\sin^3(\theta) - r^4\cos^2(\theta)\sin^3(\theta)) \\ \dot{\theta} = -1 - \varepsilon\cos(\theta)(1+\sin^2(\theta))(r^2\sin^3(\theta) - r^4\cos^2(\theta)\sin^3(\theta)). \end{cases}$$

Taking θ as an independent time variable, we get the following equation

$$\frac{dr}{d\theta} = r\varepsilon\sin(\theta)(1+\sin^2(\theta))(r^2\sin^3(\theta) - r^4\cos^2(\theta)\sin^3(\theta)).$$

So, we find the normal form of the averaging theory of first order

$$\frac{dr}{d\theta} = \varepsilon F(r,\theta) + O(\varepsilon^2),$$

where $F(r,\theta)$ is given by

$$F(r,\theta) = r\sin(\theta)(1+\sin^2(\theta))(r^2\sin^3(\theta) - r^4\cos^2(\theta)\sin^3(\theta)),$$

The averaged function is calculated as follows

$$\begin{aligned} f_4(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} r\sin(\theta)(1+\sin^2(\theta))(r^2\sin^3(\theta) - r^4\cos^2(\theta)\sin^3(\theta))d\theta \end{aligned}$$

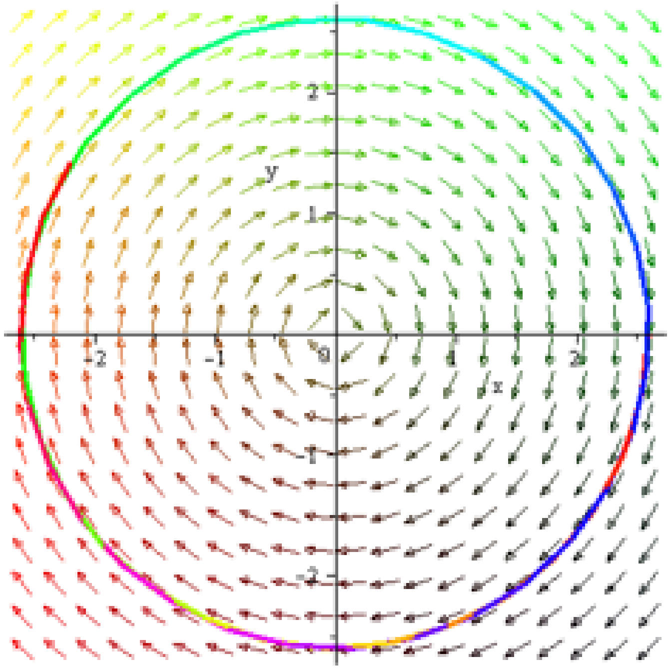

Figure 4: Limit cycle of amplitude $2\sqrt{22/13}$ for system (4.4) when $\varepsilon = 0.001$.

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} r^3\sin^4(\theta)d\theta - \int_0^{2\pi} r^5\cos^2(\theta)\sin^4(\theta)d\theta\right.$$
$$\left.+\int_0^{2\pi} r^3\sin^6(\theta)d\theta - \int_0^{2\pi} r^5\cos^2(\theta)\sin^6(\theta)d\theta\right]$$
$$= \frac{1}{2\pi}\left[\frac{3}{4}r^3 - \frac{1}{8}r^5 + \frac{5}{8}r^3 - \frac{5}{64}r^5\right],$$
$$= \frac{1}{2}\left[\frac{11}{8}r^3 - \frac{13}{64}r^5\right],$$

which has a unique positive zero, $r_* = 2\sqrt{22/13}$*. Since*

$$\frac{df_4(r)}{dr} = \frac{1}{16}\left[33r^2 - \frac{65}{8}r^4\right],$$

we obtain

$$\frac{df_4(r_*)}{dr} = -\frac{121}{13} \neq 0.$$

Since $m = 2$ *and* $n = 5$*, from Theorem 3.1(d), system (4.4) can have at most two limit cycles, see Fig. 4.*

References

[1] S. Badi and A. Makhlouf, *Limit cycles of the generalized Liénard differential equation via averaging theory*, Ann. Di. Eqs. 27(4) (2011) 472–479.

[2] I.S. Berezin and N.P. Zhidkv, *Computing Methods, vol. II*, Pergamon Press, Oxford, 1964.

[3] A. Buica, J. Giné and J. Llibre, *Bifurcation of limit cycles from a polynomial degenerate center*, Adv. Nonlinear Stud. 10 (2010) 597–609.

[4] A. Buica and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. 128 (2004) 7–22.

[5] T. Chen and J. Llibre, *Limit cycles of a second-order differential equation*, Appl. Math. Lett. 88 (2019) 111–117.

[6] A. Cima, J. Llibre, and M. A Teixeira, *Limit cycles of some polynomial differential system in dimension 2, 3 and 4 via averaging theory*, Appl. Anal. 87 (2008) 149–164.

[7] N. Hirano and S. Rybicki, *Existence of limit cycles for coupled Van der Pol equations*, J. Differ. Equ. 195(1) (2003) 194–209.

[8] J. Llibre and A.C. Mereu, *Limit cycles for generalized Kukles polynomial differential systems*, Nonlinear Anal. 74 (2011) 1261-–1271.

[9] J. Llibre, R. Moeckel and C. Simó, *Central Configurations, Periodic Orbits and Hamiltonian Systems*, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2015.

[10] N. Mellahi, A. Boulfoul and A. Makhlouf, *Maximum number of limit cycles for generalized Kukles polynomial differential systems*, Differ. Equ. Dyn. Syst. 27 (2019) 493-–514.

[11] L. Perko, *Differential Equations and Dynamical Systems*, Texts in Applied Mathematics, V. 7, 3rd edition.Springer, New York, 2000.

[12] H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle I, II*, J. Math. Pures Appl. 7 (1881) 375–422; 8 (1882) 251–296.

[13] E. Sáez and I. Szántó, *Bifurcations of limit cycles in Kukles systems of arbitrary degree with invariant ellipse*, Appl. Math. Lett. 25 (2012) 1695-–1700.

[14] H. Shi, Y. Bai and M. Han, *On the maximum number of limit cycles for a piecewise smooth differential system*, Bull. Sci. Math. 163 (2020) 102887.

[15] D. Zwillinger, *Table of Integrals*, Series and Products, 2014, ISBN: 978-0-12-384933-5 .