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Using Leray-Schauder topological degree to solve a linear diffusion parabolic equation with periodic initial conditions

Raad Awad Hameed[a], Faez N. Ghaffoori[b], Hekmat Sh. Mustafa[c], Wafaa M. Taha[d], Maan A. Rasheed[b]

[a] *Department of Mathematics,College of Education for Pure Sciences, Tikrit University, Tikrit, Iraq*
[b] *Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq*
[c] *Department of Mathematics, College of Education, University of Al-Hamdaniya, Mosul, Iraq*
[d] *Department of Mathematics, College of Sciences, Kirkuk University, Kirkuk, Iraq*

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Abstract

Throughout this manuscript, we show time periodic solutions to a linear diffusion parabolic equation with Diriclet condition. Based on the topological degree theorem, we prove a time periodic solutions of the system such that we found the fixed point when the domain of the solution is sufficiently small.

Keywords: weakly nonlinear sources; Diriclet boundary conditions; Time-periodic solution; Topological degree theorem
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1. Introduction

In this manuscript, we consider a periodic solutions of a linear diffusion parabolic equation with Diriclet boundary conditions:

$$\frac{\partial v}{\partial t} - \Delta v + a(n) \cdot \nabla v - A(x,t,v) = h(x,t), \qquad (x,t) \in S_T, \tag{1.1}$$

$$v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \tag{1.2}$$

$$v(x,0) = v(x,T), \qquad x \in \Omega, \tag{1.3}$$

Email addresses: dr.raad.awad@tu.edu.iq (Raad Awad Hameed), fng-2022@uomustansiriyah.edu.iq (Faez N. Ghaffoori), hekmat78@uohamdaniya.edu.iq (Hekmat Sh. Mustafa), wafaamath2013@gmail.com (Wafaa M. Taha), maan.rasheed.edbs@uomustansiriyah.edu.iq (Maan A. Rasheed)

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The domain Ω is a bounded region in R^N,with appropriately smooth boundary $\partial\Omega$, $S_T = \Omega\times(0,T)$,Sometimes we use this model to describe some physical phenomena.Let $v(x,t)$ be a function respect to x and t, Δv represents the linear diffusion term and perturbations $(a(n)\cdot\nabla v)$ heavily depends on ∇v , the perturbation describes a convection effect with velocity field $a(n)$. In last decades,the existence of a periodic solution has been discussed by several authors [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Recently, Nakao studied the parabolic equation below:

$$v_t = \nabla^2(\beta(v)) + h(x,t) + A(x,t,v). \tag{1.4}$$

with boundary conditions,

$$v(x,t) = 0$$

where A, h both of them are periodic in the time with a period $T > 0$,. The author proved the existence solutions with periodic initial conditions by application the topological degree theorem.for more details about equation (1.4), see [10]. In our manuscript we made some important assumptions about the equation (1.1) and then we used the topological degree theorem to prove the existence solutions with periodic initial conditions of the problem (1.1)- (1.3).

2. Existence of Weak Solutions

Throughout this section we The most important results and hypotheses :

F1) $A(x,t,v)$ is Hölder continuous in $\mathbb{R}\times\mathbb{R}\times\overline{\Omega}$, here we can see a periodic in the time t with a period T and $A(x,t,v) \le a_0|v|^{\alpha+1}$ with $0 \le \alpha \le 1$ and nonnegative constants a_0.

F2) $h(x,t) \in C_T(\overline{S}_T) \cap L^\infty(0,T,W_0^{1,\infty}(\Omega))$ such that $h(x,t)$ nonnegative in $\Omega\times\mathbb{R}$, where $C_T(\overline{S}_T)$ denotes the set of functions which are continuous in $\overline{\Omega}\times\mathbb{R}$ and T-periodic with respect to t.

Because the equation (1.1) is degeneracy , the system (1.1)- (1.3) does not have generally a classical solution. So, we shall discuss rather the solutions of system (1.1)-(1.3) in a weak sense .

Definition 2.1. *Let v be a weak solution to problem (1.1)-(1.3), if $v \in L^2(0,T;H^1(T)) \cap C_T(\overline{S}_T)$ and satisfies*

$$\iint_{S_T}\left(-v\frac{\partial\vartheta}{\partial t} + \nabla v\nabla\vartheta - \beta(v)\cdot\nabla\vartheta - A(x,t,u)\vartheta - h(x,t)\vartheta\right)dxdt = 0, \tag{2.1}$$

for any $\vartheta \in C^1(\overline{S}_T)$ with the periodic initial value $\vartheta(x,0) = \vartheta(x,T)$. here $\beta(v) = \sum_{j=1}^N \beta_j(v)$ is a function of the form $\beta(v) = \int_0^v a_j(s)ds$, where $j \in N$

We denote $W^{p,m}(\Omega)$ and $L^p(\Omega)$ norms on Ω by $\|.\|_{p,m}$ and $\|.\|_p$, respectively. The first step, we prove the following a priori estimate which plays an important role in the proof of the main results of our manuscript.

Lemma 2.2. *Let $v(x,t)$ be a solution of*

$$\frac{\partial v}{\partial t} - \Delta v + a(n)\cdot\nabla v = \sigma A(x,t,v) + \sigma h(x,t), \qquad (x,t)\in S_T, \tag{2.2}$$

$$v(x,t) = 0, \qquad (x,t)\in\partial\Omega\times[0,T], \tag{2.3}$$

$$v(x,0) = v(x,T), \qquad x\in\Omega, \tag{2.4}$$

where $\sigma\in[0,1]$ and $R>0$ not dependent of σ such that

$$\|\nabla v(t)\|_{L^\infty}(S_T) \le R. \tag{2.5}$$

such that the measure of Ω is small enough.

Proof . After multiplying the equation (2.2))by $|v|^p v$ and then integrating The new equation over Ω. we obtain

$$\int_\Omega a(v)|v|^p v = \int_\Omega \sum_{j=1}^N a_j(v)|v|^p v\frac{\partial v}{\partial x_j}dx = \sum_{j=1}^N \int_\Omega \left(\int_0^v a_j(s)|s|^p s ds\right)_{x_j} dx$$
$$\le \sum_{j=1}^N \int_{\partial\Omega}\left(\int_0^v a_j(s)|s|^p s ds\right) cos(n,x_j)dx = 0,$$

we denote to outward normal vector by n

$$\frac{d}{dt}\int_\Omega |v(t)|^{p+2} - (p+2)\int_\Omega \Delta u|v(t)|^p v(t)dx \le (p+2)a_0\int_\Omega |v(t)|^{p+\alpha+1}dx + (p+2)\int_\Omega |v(t)|^p v(t)hdx, \tag{2.6}$$

we integration the second term of the left-hand side from (2.6), we have

$$-\int_\Omega \Delta v(t)|v(t)|^p v(t)dx = (p+1)\int_\Omega |v(t)|^p|\nabla v(t)|^2dx$$
$$=(p+1)\int_\Omega |v(t)|^p|\nabla v(t)|^2dx,$$

and

$$\int_\Omega |v(t)|^p v(t)hdx = \left(\int_\Omega |v(t)|^{p+2}dx\right)^{\frac{p+1}{p+2}}\left(\int_\Omega h^{p+2}dx\right)^{\frac{1}{p+2}} dx,$$

and hence, from (2.6), we get to

$$\frac{d}{dt}\|v(t)\|_{p+2}^{p+2} + D_1\|\nabla(|v(t)|^{\frac{p}{2}}v(t))\|_2^2 \le D_2(p+2)(\|v(t)\|_{p+\alpha+1}^{p+\alpha+1} + \|v(t)\|_{p+2}^{p+1}), \tag{2.7}$$

where for $j=1,2, D_j>0$ are a constant does not depend on $v(t)$ and p.

If $0\le\alpha\le 1$, we using two inequality together Young's and Hölder's, we obtain to

$$\begin{aligned}\int_\Omega |v(t)|^{p+\alpha+1}dx \le& |\Omega|^{\frac{1-\alpha}{p+2}}\left(\int_\Omega |v(t)|^{p+2}dx\right)^{\frac{p+\alpha+1}{p+2}} dx\\ \le& \max\{1,|\Omega|^{\frac{1}{2}}\}\|v(t)\|_{p+2}^{p+\alpha+1}\\ \le& \max\{1,|\Omega|^{\frac{1}{2}}\}\|v(t)\|_{p+2}^{(p+1)(1-\alpha)}\|v(t)\|_{p+2}^{(p+2)\alpha}\\ =&\|v(t)\|_{p+2}^{p+2}+\|v(t)\|_{p+2}^{p+1},\end{aligned} \tag{2.8}$$

Combined with (2.8), it yields

$$\frac{d}{dt}\|v(t)\|_{p+2}^{p+2}+D_1\|\nabla(|v(t)|^{\frac{p}{2}}v(t))\|_2^2 \le D_2(p+2)(\|v(t)\|_{p+2}^{p+2}+\|v(t)\|_{p+2}^{p+1}), \tag{2.9}$$

If $\alpha = 1$ directly from(2.8)we obtain eq. (2.9). set

$$v_k(t) = |v_t|^{\frac{p_k}{2}}v_t, \qquad p_k = \sum_{k=1}^{\infty}(2^k-2)$$

where $k \in N$, and then , $p_k = 2p_{k-1}+2$. From (2.9), we have

$$\frac{d}{dt}\|v(t)\|_2^2 + D_1\|\nabla(u_k(t))\|_2^2 \le D_2(p_k+2)\|v_k(t)\|_2^2 + D_2(p_k+2)\|v_k(t)\|_2^{\frac{2(p_k+1)}{p_k+2}},$$

Applying the Gagliardo-Nirenberg inequality, yields that

$$\|v_k(t)\|_2 \le D\|\nabla v_k(t)\|_2^\theta \|v_k(t)\|_1^{1-\theta}, \tag{2.10}$$

with

$$\theta = \frac{N}{N+2} \in (0,1).$$

From (2.10) with the fact that $\|v_k(t)\|_1 = \|v_{k-1}(t)\|_{\alpha_{k-1}}^{\alpha_{k-1}}$, we get the below inequality:

$$\begin{aligned}\frac{d}{dt}\|v_k(t)\|_\alpha^\alpha \le& - D_1\|v_k(t)\|_2^{\frac{2}{\theta}}\|v_k(t)\|_1^{\frac{2(\theta-1)}{\theta}} + D_2(p_k+1)\|v_k(t)\|_2^2 + D_2(p_k+2)\|v_k(t)\|_2^{\frac{2(p_k+1)}{p_k+2}}\\ \le& - D_1\|v_k(t)\|_2^{\frac{2}{\theta}}\|v_k(t)\|_2^{\frac{4(\theta-1)}{\theta}} + D_2(p_k+1)\|v_k(t)\|_2^2 + D_2(p_k+2)\|v_k(t)\|_2^{\frac{2(p_k+1)}{p_k+2}}.\end{aligned}$$

Let

$$\gamma_k = \max\{1, \sup_t \|v_k(t)\|_2\},$$

we have

$$\begin{aligned}\frac{d}{dt}\|v_k(t)\|_2^2 \le& \|v_k(t)\|_2^{\frac{2(p_k+1)}{p_k+2}}\{-D_1\|v_k(t)\|_2^{\frac{2}{\theta}-\frac{2(p_k+1)}{p_k+2}}\\ &\gamma_{k-1}^{\frac{4(\theta-1)}{\theta}} + D_2(p_k+2)\|v_k(t)\|_2^{\frac{2)}{p_k+2}} + D_2(p_k+2)\}.\end{aligned} \tag{2.11}$$

By young's inequality

$$cd \le \epsilon c^{p'} + \epsilon^{-\frac{q'}{p'}}d^{q'},$$

where $c>0, d>0, \epsilon>0, q'>1, p'>1$ and $\frac{1}{p'} = \frac{q'-1}{q'}$. we choose

$$c = \|v_k(t)\|_2^{\frac{2}{p_k+2}}, \qquad d = (p_k+2)^{p-1}, \qquad \epsilon = \frac{1}{2}\gamma_{k-1}^{\frac{4(\theta-1)}{\theta}},$$

$$p' = I_k = \left(\frac{p_k + 2 - \theta p_k - \theta}{\theta}\right) = \left(\frac{(p_k+m+1)(N+1)}{N)} - p_k - 1\right),$$

and then, we get

$$(p_k+2)\|v_k(t)\|_2^{\frac{2}{p_k+2}} \le \frac{1}{2}\|v_k(t)\|_2^{\frac{p}{\theta}-\frac{2(p_k+1)}{p_k+2}}\gamma_{k-1}^{\frac{4(\theta-1)}{\theta_k}} + D(p_k+1)^{\frac{l_k}{l_k-1}}\gamma_{k-1}^{\frac{4(1-\theta)}{\theta(l_k-1)}}. \tag{2.12}$$

Easily you can watch that

$$\lim_{k\to\infty} I_k = +\infty.$$

Denote

$$c_k = \frac{I_k}{I_k-1}, \qquad d_k = \frac{4(1-\theta)}{\theta(I_k-1)},$$

combining (2.11) and (2.12) we obtain

$$\frac{d}{dt}\|v_k(t)\|_2^2 \le \|v_k(t)\|_2^{\frac{2(p_k+1)}{p_k+2}}\{-\frac{D_1}{2}\|v_k(t)\|_2^{\frac{2}{\theta}-\frac{2(p_k+1)}{p_k+2}}\gamma_{k-1}^{\frac{4(\theta-1)}{\theta}} + D_2(p_k+2)^{c_k}\gamma_{k-1}^{d_k} + D_2(p_k+2)\}.$$

and then

$$(p_k+1)\frac{d}{dt}\|v_k(t)\|_2^{\frac{2}{p_k+2}} \le \frac{-D}{2}\|v_k(t)\|_2^{\frac{2}{\theta}-\frac{2(p_k+1)}{p_k+2}}\gamma_{k-1}^{\frac{4(\theta_k-1)}{\theta_k}} + D(p_k+2)^{c_k}\gamma_{k-1}^{d_k}. \tag{2.13}$$

Since $v_k(t)$ is periodic, there exists t' s.t. $\|v_k(t)\|_2$ takes the maximum value at this point. Therefore, the left hand side of (2.13) is vanished. It follows that

$$\|v_k(t)\|_2 \le \{D[(p_k+2) + (p_k+2)^{c_k} I_{k-1}^{d_k}]\gamma_{k-1}^{\frac{4(1-\theta)}{\theta}}\}^{\frac{1}{\Upsilon_k}},$$

where

$$\Upsilon_k = \frac{2(p_k+2) - 2\theta(p_k+1)}{\theta(p_k+2)} = \frac{2I_k}{p_k+2}.$$

We conclude from the above

$$\|v_k(t)\|_2 \le \{D(p_k+1)^{c_k}\gamma_{k-1}^{d_k+\frac{4(1-\theta)}{\theta}}\}^{\frac{1}{\Upsilon_k}} = \{D(p_k+1)^{c_k}\}^{\frac{p_k+2}{2I_k}}\gamma_{k-1}^{\frac{4(p_k+2)(1-\theta)}{2\theta(I_k-1)}}.$$

Since $\frac{p_k+2}{(I_k-1)\theta} = \frac{1}{1-\theta}$ and $\frac{p_k+2}{2I_k}$ are bounded, we get

$$\|v_k(t)\|_2 \le D2^{a'k}\gamma_{k-1}^2,$$

where the constant a' does not depend on k.

$$\ln\|v_k(t)\|_2 \le \ln\gamma_k \le \ln D + k\ln E + 2\ln\gamma_{k-1},$$

if $E = 2^{a'} > 1$. We get

$$\ln \|v_k(t)\|_2 \leq \ln D \sum_{j=0}^{k-2} 2^j + 2^{k-1} \ln \gamma_1 + \ln E(\sum_{i=0}^{k-2}(k-i)2^i)$$
$$\leq (2^{k-1}-1)\ln D + 2^{k-1}\ln \gamma_1 + f(k)\ln E,$$

where

$$f(k) = 2^k - 2^{k-1} - k - 2$$

And hence,

$$\|v_k(t)\|_{p_{k+2}} \leq \{D^{2^{k-1}} \gamma_1^{2^{k-1}} E^{f(k)}\}^{\frac{2}{p_k+2}},$$

Letting $k \to \infty$, we get

$$\|v_\tau(t)\|_\infty \leq D\gamma_1^2 \leq D(\max\{1, \sup_t \|v(t)\|_2\})^2. \tag{2.14}$$

From (2.14) and using the estimate $\|v(t)\|_2$, and then take value to $p = 0$. we obtain

$$\frac{d}{dt}\|v(t)\|_2^2 + D_1\|\nabla v(t)\|_2^2 \leq D_2\|v(t)\|_2^2 + D_2\|v(t)\|_2.$$

and thus by the Poincarë inequality, we obtain

$$D_p\|v(t)\|_2^2 \leq \|\nabla v(t)\|_2^2$$

when $|\Omega|$ is small then the measure of Ω very large and so, when D_p is a constant greater than zero which depends only on N

$$\frac{d}{dt}\|v(t)\|_2^2 + D_1 D_p\|v(t)\|_2^2 \leq D_2\|v(t)\|_2^2 + D_2\|v(t)\|_2.$$

and thus, when $|\Omega|$ is sufficiently small, we have $D_1 D_p > D_2$.Then by by using the Young's inequality, we get

$$\frac{d}{dt}\|v(t)\|_2^2 + D\|v(t)\|_2^2 \leq D.$$

for D are a constants independent of v. since v is periodic,it follows that

$$\|v(t)\|_2 \leq R, \tag{2.15}$$

here R is a constants does not depend on σ. Combining (2.15) with (2.14),we get (2.5). So, the Lemma is proved.

□

Theorem 2.3. *If assumptions(F1)and(F2)are satisfied, then problem (1.1)-(1.3) has at least one non-trivial non-negative periodic solution.*

Proof . ln this theorem, we can define a map by considering the system below:

$$\frac{\partial v}{\partial t} - \mathrm{div}(|\nabla v^m|^{p-2}\nabla v^m) + a(n)\cdot\nabla v = g(x,t), \qquad (x,t)\in S_T, \tag{2.16}$$
$$v(x,t) = 0, \qquad (x,t)\in\partial\Omega\times[0,T], \tag{2.17}$$
$$v(x,0) = v(x,T), \qquad x\in\Omega, \tag{2.18}$$

where $g(x,t)$ is a given function in $C_T(\overline{S}_T)$.and then by using an argument same way to (see [10]), that the problem (2.16) - (2.18)has a unique solution. and then, we can define a map $Q : [0,1]\times C_T(\overline{S}_T) \to C_T(\overline{S}_T)$.We define Q in the form $v = Qg$ we obtain to $v = Qg$ is compact and continuous map. by the same way in (see [9]), we can conclude that $\|v\|_{L^\infty(S_T)}$ is bounded where $g \in L^\infty(S_T)$ and $v, \nabla v \in C^\alpha(\overline{S}_T)$ respect to some $\alpha > 0$.Then (through the application the Arzela-Ascoli over a map Q) the compactness of the map is compact Q conclude from Hölder continuity of $v$and $\|v\|_{L^\infty(S_T)}$. from the Hölder continuity of ∇v we obtain to the continuity of the map Q.

Let $\phi(v) = A(x,t,v) + h(x,t)$, by using (F1)-(F2) and the above arguments, the map $Q(\sigma\phi)$ is a complete and continuous for $\sigma\in[0,1]$. We using Lemma 2.2 ,to get at least one fixed point v of the map $Q(\sigma\phi)$ satisfies

$$|v\|_\infty \le D$$

where $D > 0$ is not dependent of σ. By the topological degree theorem (see [11]), we infer that the problem (1.1) - (1.3), admits at least one periodic solution v. thus complete the proof.
□

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