Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 2363-2378
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.5934

The bifurcation analysis of an ecological model involving SIS disease with a prey refuge

Azhar A. Majeed[a,*], Zina Kh. Alabacy[b]

[a]Department of mathematics, College of Science, University of Baghdad, Baghdad, Iraq
[b]Department of Control and Systems Engineering, University of Technology, Baghdad, Iraq

(Communicated by Madjid Eshaghi Gordji)

Abstract

This paper deals with an epidemiological system with stage-structured, harvesting and refuge for only prey, the disease of type (SIS) is just in the immature of the prey. The sufficient conditions guaranteeing the occurrence of local bifurcation and the Hopf bifurcation for the system are obtained. Further, the validity of our main results was demonstrated by numerical analysis.

Keywords: Epidemiological Model, Prey-Predator Model, Local bifurcation, Hopf bifurcation.

1. Introduction

The bifurcation means existence a change in the stabilization of an equilibrium point (EP) of the system at the value of a parameter. Most differential equations depend of parameters. The specific behavior of systems solution can be completely different because of the dependencies on the value of these parameters. Bifurcation theory studies the periodic orbits, the appearance and vanishing of equilibrium points (EPs), or more complicated features such as strange attractors. The methods and results of bifurcation theory are essential to understand the nonlinear dynamical systems.

The bifurcation is divided into two principal classes: local and global bifurcations. Local bifurcations (LB), which can be analyzed entirely through changes in the local stability properties of (EP), periodic orbit or other invariant sets as parameters cross through critical edges such as saddle node(SNB), transcritical (TB), pitchfork (PFB), period-doubling (flip), Hopf (HB) and Neimark (secondary Hopf) bifurcation. Global bifurcations occur when larger invariant sets, such as periodic orbits, crash with (EP) [5].

*Corresponding author
Email addresses: azharalthabeti2017@gmail.com (Azhar A. Majeed), zina.k.alabacy@uotechnology.edu.iq (Zina Kh. Alabacy)

Received: December 2021 *Accepted:* December 2021

In recent years, the bifurcation has been studied for its importance by many researchers such as Majeed and Alabacy [5] established the conditions of occurrence of (LB) for all the (EPs) and (HB) for the positive (EP) of a food chain prey-predator model with prey refuge and harvesting. Rihan et al. [16] studied the Allee effect on two prey and one predator system with time delays, also considered the bistability existence and (HB) for the interior (EP). Alidoust [2] studied the effects of scavengers, harvesting and fractional derivative on a prey-predator model and it's noticed the appearance of (HB) near the interior (EP).. Li et al. [4] established the conditions of occurrence of (LB) for all the (EPs) and (HB) for the positive (EP) of an SIS disease model with nonlinear contact rate, and from the results obtained is that the behavior of susceptible individuals may affect the spread of the disease. Zhang and Wan [20] calculated the occurrence of (HB) in a three-species ecological system with time delay and harvesting. Sen et al. [18] studied the Allee effect in prey-predator model with generalist whose reproduction follows a Beverton-Holt like function in the absence of prey and established the conditions of occurrence of (LB) and (HB) for all the (EPs). Mukherjee and Maji [15] established the conditions of occurrence of (LB) for all the (EPs) and (HB) for the positive (EP) of a prey-predator model with prey refuge. Ghosh et al. [3] established the conditions of occurrence of (LB) for all the (EPs) and (HB) for the positive (EP) of a prey-predator model and studied the extent of memory effect on the dynamic evolution. Saikh and Gazi [17] calculated the occurrence of (LB) for all the (EPs) and (HB) for the positive (EP) of an SIS epidemic model with with immigrants and treatment. Melese and Feyissa [14] established the conditions of occurrence of (LB) for all the (EPs) and (HB) for the positive (EP) of an eco-epidemiological model of a prey predator system where prey population is infected with a disease. And many other researchers have studied the (LB) like [6, 7, 8, 9, 19, 10, 11, 12, 13].

Finally, in this work, a set of basic outcomes and methods in the theory of (LB) around all (EPs) and a theory of (HB) around the positive (EP) for an epidemiological system [1] which consists on a single parameter. The system includes harvest and refuge for only prey, the disease of type (SIS) is just in the immature of the prey and the disease is spread by contact and by external source has been studied.

2. Model Formulation [1]

In this section, an epidemiological mathematical model has been suggested. The model includes of a stage-structured in "prey whose population density at time T is represented by" $U(T)$ and a predator is represented by $V(T)$. The following assumptions are assumed for this model:

1. The population density of the prey consists of stage structured, the immature represented by $U_1(T)$ and the mature which represented by $U_2(T)$, where $U(T) = U_1(T) + U_2(T)$.
2. An epidemic of type SIS disease in the immature prey's population which divides the population into two classes, namely S(T) that represents the susceptible immature prey's at time T and I(T) that represents the infected immature prey's at time T, where $U_1(T) = S(T) + I(T)$.
3. This disease is transmitted through contact between S and I and through an external source, it does not spread to the mature prey and predator. The proposed disease can be treated and does not give immunity to the immature.
4. The immature prey depends on the mature prey on their feeding.

The predator predate the immature (susceptible and infected) and the mature of prey by Lotka Voltera of functional response. Also, this model involving refuge and harvesting, and the parameters are described in Table 1.

Table 1: The model's parameters [1]

Parameters	***Symbolizing from a biological point of view***
$r > 0$	***The growth rate of immature's prey.***
$K > 0$	***The carrying capacity of the susceptible prey.***
β_i, ***i=1,2,3.***	***The maximum predation rate (MPR) of the predator over the susceptible, infected and the mature of prey respectively which are outside refuge.***
γ_j,***j=1,2.***	***The infection rate.***
$m_i, \ i = 1, 2, 3$	***The refuge rate of the susceptible, infected and the mature of prey respectively.***
γ_3	***The recovering rate.***
$\alpha > 0$	***The grown up rate of the immature into mature (in prey population).***
$\eta_i, \ \ i = 1, 2, 3$	***The conversion rate of food from susceptible, infected and the mature of prey respectively.***
d	***The natural death rate of the predator.***
$\theta_i, \ \ i = 1, 2, 3$	***The harvesting rate of the susceptible, infected and the mature of prey respectively.***

According to these assumptions, we propose the model by "first order non-linear differential equations".

$$\left.\begin{aligned}
\frac{\mathrm{dS}}{\mathrm{dT}} &= rU_2\left(1-\frac{U_2}{K}\right) - \alpha S - \theta_1 S - \beta_1\left(1-m_1\right)SV - \gamma_1 SI - \gamma_2 S + \gamma_3 I \\
\frac{\mathrm{dI}}{\mathrm{dT}} &= \gamma_1 SI + \gamma_2 S - \gamma_3 I - \beta_2\left(1-m_2\right)IV - \theta_2 I \\
\frac{dU_2}{\mathrm{dT}} &= \alpha S - \beta_3\left(1-m_3\right)U_2 V - \theta_3 U_2 \\
\frac{dV}{\mathrm{dT}} &= \eta_1\left(1-m_1\right)SV + \eta_2\left(1-m_2\right)IV + \eta_3\left(1-m_3\right)U_2 V - dV
\end{aligned}\right\} \tag{2.1}$$

Note that, the model has eighteen "parameters which make the analysis difficult, so to simplify it, we reduced the number of them by using dimensionless variables and parameters" as follows: $t = rT$, $h_1 = \frac{S}{K}$, $h_2 = \frac{I}{K}$, $h_3 = \frac{U_2}{K}$, $h_4 = \frac{V}{K}$, $p_1 = \frac{\alpha}{r}$, $p_2 = \frac{\gamma_2}{r}$, $p_3 = \frac{\theta_1}{r}$, $p_4 = \frac{\gamma_1 K}{r}$, $p_5 = \frac{\gamma_3}{r}$, $p_6 = \frac{\beta_1(1-m_1)K}{r}$, $p_7 = \frac{\beta_2(1-m_2)K}{r}$, $p_8 = \frac{\theta_2}{r}$, $p_9 = \frac{\beta_3(1-m_3)K}{r}$, $p_{10} = \frac{\theta_3}{r}$, $p_{11} = \frac{\eta_1(1-m_1)K}{r}$, $p_{12} = \frac{\eta_2(1-m_2)K}{r}$, $p_{13} = \frac{\eta_3(1-m_3)K}{r}$, $p_{14} = \frac{d}{r}$.
So the dimensional system (2.1) can be formulated as:

$$\left.\begin{aligned}
\frac{dh_1}{\mathrm{dt}} &= h_3\left(1-h_3\right) - \left(p_1+p_2+p_3\right)h_1 - p_4 h_1 h_2 + p_5 h_2 - p_6 h_1 h_4 = \widehat{f}_1\left(h_1,h_2,h_3,h_4\right) \\
\frac{dh_2}{\mathrm{dt}} &= p_2 h_1 + p_4 h_1 h_2 - p_5 h_2 - p_7 h_2 h_4 - p_8 h_2 = \widehat{f}_2\left(h_1,h_2,h_3,h_4\right) \\
\frac{dh_3}{\mathrm{dt}} &= p_1 h_1 - p_9 h_3 h_4 - p_{10} h_3 = \widehat{f}_3\left(h_1,h_2,h_3,h_4\right) \\
\frac{dh_4}{\mathrm{dt}} &= p_{11} h_1 h_4 + p_{12} h_2 h_4 + p_{13} h_3 h_4 - p_{14} h_4 = \widehat{f}_4\left(h_1,h_2,h_3,h_4\right)
\end{aligned}\right\} \tag{2.2}$$

With $h_1(0) \geq 0$, $h_2(0) \geq 0$, $h_3(0) \geq 0$ *and* $h_4(0) \geq 0$. It is noticed that the parameters' number have been reduced from eighteen in system (2.1) to fourteen in system (2.2). Clearly, the "interaction functions of system (2.2) are continuous and have continuous partial derivatives on the following positive four dimensional space".

3. The Local Bifurcation (LB)

In this section a study for dynamical behavior of system (2.2) under the impact of changing one parameter every time is achieved. The appearance of (LB) in the neighborhood of the (EPs) of system (2.2) is investigated. Recall that for occurring bifurcation, the existence of non- hyperbolic (EP) of system (2.2) is necessary condition but not sufficient. Therefore, Sotomayor's theory [5] has been applied in the following theorems.
Now, since the Jacobian matrix (JM) of system (2.2) which is given in [1]

$$J_i = \left[\widehat{f}_{\mathrm{ij}}\right]_{4\times 4}, \tag{2.3}$$

where $i, j = 1, 2, 3, 4$ and $\widehat{f}_{11} = -\left(p_1 + p_2 + p_3 + p_4 h_2 + p_6 h_4\right)$, $\widehat{f}_{12} = p_5 - p_4 h_1$, $\widehat{f}_{13} = 1 - 2h_3$, $\widehat{f}_{14} = -p_6 h_1$, $\widehat{f}_{21} = p_2 + p_4 h_2$, $\widehat{f}_{22} = p_4 h_1 - p_7 h_4 - (p_5 + p_8)$, $\widehat{f}_{23} = 0$, $\widehat{f}_{24} = -p_7 h_2$, $\widehat{f}_{31} = p_1$, $\widehat{f}_{32} = 0$, $\widehat{f}_{33} = -\left(p_9 h_4 + p_{10}\right)$, $\widehat{f}_{34} = -p_9 h_3$, $\widehat{f}_{41} = p_{11} h_4$, $\widehat{f}_{42} = p_{12} h_4$, $\widehat{f}_{43} = p_{13} h_4$, $\widehat{f}_{44} = p_{11} h_1 + p_{12} h_2 + p_{13} h_3 - p_{14}$
Clearly for any nonzero vector $\widehat{G} = \left(\widehat{G}_1, \widehat{G}_2, \widehat{G}_3, \widehat{G}_4\right)^T$ to prove that we have:

$$D^2 F\left(\widetilde{X}, \mu\right)\left(\widehat{G}, \widehat{G}\right) = \left[\widehat{g}_{\mathrm{ij}}\right]_{4\times 1}, \tag{2.4}$$

where:

$$\widehat{g}_{11} = -2\left[p_4 \widehat{G}_2 + p_6 \widehat{G}_4\right]\widehat{G}_1 - 2\widehat{G}_3^2 , \ \widehat{g}_{21} = 2\left[p_4 \widehat{G}_1 - p_7 \widehat{G}_4\right]\widehat{G}_2 , \ \widehat{g}_{31} = -2p_9 \widehat{G}_3 \widehat{G}_4 ,$$

$$\widehat{g}_{41} = 2\left[p_{11}\widehat{G}_1 + p_{12}\widehat{G}_2 + p_{13}\widehat{G}_3\right]\widehat{G}_4.$$

and

$$D^3 F\left(\widetilde{X}, \mu\right)\left(\widehat{G}, \widehat{G}, \widehat{G}\right) = [0]_{4\times 1}, \tag{2.5}$$

Where $\widetilde{X} = (h_1, h_2, h_3, h_4)$ and μ be any bifurcation parameter. Therefore system (2.2) has no pitch fork bifurcation (PFB) for all the (EPs).

In the next theorems the (LB) conditions near (EP) are determined.

Theorem 2.1. *System (2.2) at the (EP) $A_0(0,0,0,0)$ with the value of parameter $p_{10}^0 = p_{10} = \frac{p_1(p_5+p_8)}{p_2 p_8 + (p_1+p_3)(p_5+p_8)}$ has "transcritical bifurcation (TB) but, saddle–node bifurcation (SNB) can't occur at A_0.*

Proof . *The (JM) given in [1] of system (2.2) at the (EP) A_0 has an eigenvalue equal to zero (say $\lambda_{0h_3} = 0$) at $p_{10} = p_{10}^0$, and the (JM) of system (2.2) with $p_{10} = p_{10}^0$ becomes:*

$$J_0^0 = J\left(A_0, p_{10}^0\right) = \begin{bmatrix} -\left(p_1 + p_2 + p_3\right) & p_5 & 1 & 0 \\ p_2 & -\left(p_5 + p_8\right) & 0 & 0 \\ p_1 & 0 & -p_{10}^0 & 0 \\ 0 & 0 & 0 & -p_{14} \end{bmatrix}.$$

Now, let $\widehat{G}^{[0]} = \left(\widehat{G}_1^{[0]}, \widehat{G}_2^{[0]}, \widehat{G}_3^{[0]}, \widehat{G}_4^{[0]}\right)^T$ *be the eigenvector (EV) of* J_0^0 *for* $\lambda_{0h_3} = 0$.

Thus $(J_0^0 - \lambda_{0h_3} I)\,\widehat{G}^{[0]} = 0$, *that gives:* $\widehat{G}^{[0]} = \left(\alpha_1 \widehat{G}_2^{[0]}, \widehat{G}_2^{[0]}, \alpha_2 \widehat{G}_2^{[0]}, 0\right)^T$ *where* $\widehat{G}_2^{[0]} \neq 0$ *any real number and* $\alpha_1 = \frac{p_5+p_8}{p_2}$, $\alpha_2 = \frac{p_1(p_5+p_8)}{p_2 p_{10}^0}$.

Let $\breve{\psi}^{[0]} = \left(\breve{\psi}_1^{[0]}, \breve{\psi}_2^{[0]}, \breve{\psi}_3^{[0]}, \breve{\psi}_4^{[0]}\right)^T$ *be the (EV) of* $(J_0^0)^T$ *for* $\lambda_{0h_3} = 0$.

We get $\left((J_0^0)^T - \lambda_{0h_3} I\right)\breve{\psi}^{[0]} = 0$. *Now we can solve the previous equation for* $\breve{\psi}^{[0]}$ *we obtain,*

$\breve{\psi}^{[0]} = \left(\alpha_3 \breve{\psi}_2^{[0]}, \breve{\psi}_2^{[0]}, \alpha_4 \breve{\psi}_2^{[0]}, 0\right)^T$, $\breve{\psi}_2^{[0]} \neq 0$ *any real number and* $\alpha_3 = 1 + \frac{p_8}{p_5} > 1$, $\alpha_4 = \frac{p_5+p_8}{p_5 p_{10}^0}$.

Now, consider:

$\frac{\partial f}{\partial p_{10}} = f_{p_{10}}\left(\widetilde{X}, p_{10}\right) = \left(\frac{\partial \widehat{f}_1}{\partial p_{10}}, \frac{\partial \widehat{f}_2}{\partial p_{10}}, \frac{\partial \widehat{f}_3}{\partial p_{10}}, \frac{\partial \widehat{f}_4}{\partial p_{10}}\right)^T = (0, 0, -h_3, 0)^T$.

So, $f_{p_{10}}(A_0, p_{10}^0) = (0, 0, 0, 0)^T$ *and hence* $\left(\breve{\psi}^{[0]}\right)^T f_{p_{10}}(A_0, p_{10}^0) = 0$.

Therefore, by using Sotomayor's theorem the (SNB) condition can not satisfy at A_0. *But the first condition of (TB) is verified, as below,*

since $Df_{p_{10}}\left(\widetilde{X}, p_{10}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *where* $Df_{p_{10}}\left(\widetilde{X}, p_{10}\right)$ *noticed the derivative of* $f_{p_{10}}\left(\widetilde{X}, p_{10}\right)$ *for* $\widetilde{X} = (h_1, h_2, h_3, h_4)^T$.

Additional, it is observed that

$$Df_{p_{10}}(A_0, p_{10}^0)\,\widehat{G}^{[0]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \widehat{G}_2^{[0]} \\ \widehat{G}_2^{[0]} \\ \alpha_2 \widehat{G}_2^{[0]} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\alpha_2 \widehat{G}_2^{[0]} \\ 0 \end{bmatrix}, \text{ hence}$$

$$\left(\breve{\psi}^{[0]}\right)^T \left[Df_{p_{10}}(A_0, p_{10}^0)\,\widehat{G}^{[0]}\right] = \left(\alpha_3 \breve{\psi}_2^{[0]}, \breve{\psi}_2^{[0]}, \alpha_4 \breve{\psi}_2^{[0]}, 0\right)\left(0, 0, -\alpha_2 \widehat{G}_2^{[0]}, 0\right)^T = -\,\alpha_2 \alpha_4 \widehat{G}_2^{[0]} \breve{\psi}_2^{[0]} \neq 0.$$

Moreover, by substituting $\widehat{G}^{[0]}$ *in equation* (2.4) *we get:*

$$D^2 f(A_0, p_{10}^0)\left(\widehat{G}^{[0]}, \widehat{G}^{[0]}\right) = \begin{bmatrix} -2\left(\widehat{G}_2^{[0]}\right)^2 (p_4 \alpha_1 + \alpha_2^2) \\ 2 p_4 \alpha_1 \left(\widehat{G}_2^{[0]}\right)^2 \\ 0 \\ 0 \end{bmatrix},$$

Thus $\left(\breve{\psi}^{[0]}\right)^T \left[D^2 f(A_0, p_{10}^0)\left(\widehat{G}^{[0]}, \widehat{G}^{[0]}\right)\right] = 2\left(\widehat{G}_2^{[0]}\right)^2 \breve{\psi}_2^{[0]}[p_4 \alpha_1 (1 - \alpha_3) - \alpha_3 \alpha_2^2] \neq 0$.

Hence, by using Sotomayor's theorem system (2.2) *has (TB) at* A_0 *with the parameter* $p_{10}^0 = p_{10}$. □

Theorem 2.2. *Assume that the conditions* (4.9)*, (4.10) given in [1] and*

$$a_{14} a_{33} > a_{13} a_{34}, \tag{2.6}$$

$$a_{11} a_{33} a_{14} > a_{31}(a_{13} a_{34} - a_{14} a_{33}), \tag{2.7}$$

$$a_{34}\left(p_4 \widetilde{h}_1 - \frac{(p_5 + p_8)(p_1 + p_3) + p_2 p_8}{(p_1 + p_3)}\right) > -a_{11} a_{22} a_{14}, \tag{2.8}$$

are hold. Then system (2.2) *at the (EP)* $A_1 = \left(\widetilde{h}_1, 0, \widetilde{h}_3, 0\right)$ *with the parameter* $\widetilde{p}_{14} = p_{14} = p_{11}\widetilde{h}_1 + p_{13}\widetilde{h}_3$, *has (TB) but, (SNB) can't occur at* A_1 .

Proof . *According to the (JM) given in [1] for the system* (2.2) *at the (EP)* A_1 *has an eigenvalue equal to zero (say* $\lambda_{1h_4}=0$*) at* $\widetilde{p}_{14}=p_{14}=p_{11}\widetilde{h}_1+p_{13}\widetilde{h}_3$ *,obviously that* $\widetilde{p}_{14}>0$*, the (JM) of system* (2.2) *with* $\widetilde{p}_{14}=p_{14}$ *becomes:*

$$\widetilde{J}_1=J\left(A_1,\widetilde{p}_{14}\right)=\left[\widetilde{a}_{ij}\right]_{4\times 4},$$

where $\widetilde{a}_{ij}=a_{ij}, i,j=1,2,3,4$ *as given in [1] accept* $a_{44}=0$.

Now, let $\widehat{G}^{[1]}=\left(\widehat{G}_1^{[1]},\widehat{G}_2^{[1]},\widehat{G}_3^{[1]},\widehat{G}_4^{[1]}\right)^T$ *be the (EV) of* $\widetilde{J}_1$ *for* $\lambda_{1h_4}=0$.

Thus $\left(\widetilde{J}_1-\lambda_{1h_4}I\right)\widehat{G}^{[1]}=0$*, that gives:* $\widehat{G}^{[1]}=\left(\theta_1\widehat{G}_2^{[1]},\widehat{G}_2^{[1]},\theta_2\widehat{G}_2^{[1]},\theta_3\widehat{G}_2^{[1]}\right)^T$ *where* $\widehat{G}_2^{[1]}\neq 0$ *any real number and* $\theta_1=\frac{-a_{22}}{a_{21}}$, $\theta_2=\frac{a_{34}\left(p_4\widetilde{h}_1-\frac{(p_5+p_8)(p_1+p_3)+p_2p_8}{(p_1+p_3)}\right)+a_{11}a_{22}a_{14}}{a_{21}(a_{14}a_{33}-a_{13}a_{34})}$,

$\theta_3=\frac{a_{22}[a_{31}(a_{14}a_{33}-a_{13}a_{34})+a_{11}a_{33}a_{14}]+a_{33}a_{34}[\frac{(p_5+p_8)(p_1+p_3)+p_2p_8}{(p_1+p_3)}-p_4\widetilde{h}_1]}{a_{21}a_{34}(a_{14}a_{33}-a_{13}a_{34})}$.

Let $\breve{\psi}^{[1]}=\left(\breve{\psi}_1^{[1]},\breve{\psi}_2^{[1]},\breve{\psi}_3^{[1]},\breve{\psi}_4^{[1]}\right)^T$ *be the (EV) of* $\left(\widetilde{J}_1\right)^T$ *for* $\lambda_{1h_4}=0$.

We get $\left(\left(\widetilde{J}_1\right)^T-\lambda_{1h_4}I\right)\breve{\psi}^{[1]}=0$*. Now we can solve the previous equation for* $\breve{\psi}^{[1]}$ *we obtain,*

$\breve{\psi}^{[1]}=\left(0,0,0,\breve{\psi}_4^{[1]}\right)^T$, $\breve{\psi}_4^{[1]}\neq 0$ *any real number.*

Now, consider:

$\frac{\partial f}{\partial p_{14}}=f_{p_{14}}\left(\widetilde{X},p_{14}\right)=\left(\frac{\partial\widehat{f}_1}{\partial p_{14}},\frac{\partial\widehat{f}_2}{\partial p_{14}},\frac{\partial\widehat{f}_3}{\partial p_{14}},\frac{\partial\widehat{f}_4}{\partial p_{14}}\right)^T=(0,0,0,-h_4)^T$.

So, $f_{p_{14}}\left(A_1,\widetilde{p}_{14}\right)=(0,0,0,0)^T$ *and hence* $\left(\breve{\psi}^{[1]}\right)^T f_{p_{14}}\left(A_1,\widetilde{p}_{14}\right)=0$.

Therefore, by using Sotomayor's theorem the (SNB) condition can not satisfy at A_1*. But the first condition of (TB) is verified, as below,*

since $Df_{p_{14}}\left(\widetilde{X},p_{14}\right)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{bmatrix}$ *where* $Df_{p_{14}}\left(\widetilde{X},p_{14}\right)$ *noticed the derivative of* $f_{p_{14}}\left(\widetilde{X},p_{14}\right)$ *for* $\widetilde{X}=(h_1,h_2,h_3,h_4)^T$.

Additional, it is observed that

$$Df_{p_{14}}\left(A_1,\widetilde{p}_{14}\right)\widehat{G}^{[1]}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}\theta_1\widehat{G}_2^{[1]}\\\widehat{G}_2^{[1]}\\\theta_2\widehat{G}_2^{[1]}\\\theta_3\widehat{G}_2^{[1]}\end{bmatrix}=\begin{bmatrix}0\\0\\0\\-\theta_3\widehat{G}_2^{[1]}\end{bmatrix}, \text{ hence}$$

$\left(\breve{\psi}^{[1]}\right)^T\left[Df_{p_{14}}\left(A_1,\widetilde{p}_{14}\right)\widehat{G}^{[1]}\right]=\left(0,0,0,\breve{\psi}_4^{[1]}\right)\left(0,0,0,-\theta_3\widehat{G}_2^{[1]}\right)^T=-\theta_3\widehat{G}_2^{[1]}\breve{\psi}_4^{[1]}\neq 0$ *if the conditions (4.9), (4.10) given in [1] and* (2.6), (2.7) *hold.*

Moreover, by substituting $\widehat{G}^{[1]}$ *in equation* (2.4) *we get:*

$$D^2f\left(A_1,\widetilde{p}_{14}\right)\left(\widehat{G}^{[1]},\widehat{G}^{[1]}\right)=\begin{bmatrix}-2\left(\widehat{G}_2^{[1]}\right)^2\left[\theta_1(p_4+p_6\theta_3)-\theta_2^2\right]\\2\left(\widehat{G}_2^{[1]}\right)^2\left[p_4\theta_1-p_7\theta_3\right]\\-2p_9\theta_2\theta_3\left(\widehat{G}_2^{[1]}\right)^2\\2\theta_3\left(\widehat{G}_2^{[1]}\right)^2\left[p_{11}\theta_1+p_{12}+p_{13}\theta_2\right]\end{bmatrix},$$

Thus $\left(\breve{\psi}^{[1]}\right)^T\left[D^2f\left(A_1,\widetilde{p}_{14}\right)\left(\widehat{G}^{[1]},\widehat{G}^{[1]}\right)\right]=2\theta_3\left(\widehat{G}_2^{[1]}\right)^2\breve{\psi}_4^{[1]}[p_{11}\theta_1+p_{12}+p_{13}\theta_2]\neq 0$ *under conditions*

(4.9), (4.10) given in [1] and (2.6), (2.8).
Hence, by using Sotomayor's theorem system (2.2) *has (TB) at* A_1 *with the parameter* $\widetilde{p}_{14} = p_{14}$. □

Theorem 2.3. *Assume that the conditions (4.16), (4.17) given in [1] and*

$$n_{34}n_{21} > n_{24}n_{31}, \tag{2.9}$$

$$n_{13}n_{31}n_{22} > n_{33}\left(\frac{(p_5+p_8)(p_1+p_3)+p_8\left(p_2+p_4\overline{h}_2\right)}{(p_1+p_3)} - p_4\overline{h}_1\right), \tag{2.10}$$

$$n_{13}\left(n_{24}n_{31}-n_{21}n_{34}\right) > n_{33}\left(n_{24}n_{11}-n_{14}n_{21}\right), \tag{2.11}$$

$$n_{13}n_{22}n_{34} > n_{33}\left(n_{14}n_{22}-n_{12}n_{24}\right) \tag{2.12}$$

$$n_{12}\left(n_{21}n_{34}-n_{24}n_{31}\right) > n_{22}\left(n_{34}n_{11}-n_{31}n_{14}\right), \tag{2.13}$$

$$\sigma_4 > 1, \tag{2.14}$$

$$\sigma_2\sigma_4 > p_9\sigma_5, \tag{2.15}$$

$$p_{11}\sigma_1 + p_{12} < -p_{13}\sigma_2, \tag{2.16}$$

are hold. Then system (2.2) *at the (EPs)* $A_2\left(\overline{h}_1,\overline{h}_2,\overline{h}_3,0\right)$ *and* $A_3\left(\overline{h'}_1,\overline{h'}_2,\overline{h'}_3,0\right)$ *with the parameter* $\overline{p}_{14} = p_{14} = p_{11}\overline{h}_1 + p_{12}\overline{h}_2 + p_{13}\overline{h}_3$*, has (TB) but, (SNB) can't occur at* A_2 *and* A_3.

Proof . *According to the (JM) given in [1] for the system* (2.2) *at the (EP)* A_2 *that is the same for* A_3 *has an eigenvalue equal to zero (say* $\lambda_{2h_4} = 0$*) at* $\overline{p}_{14} = p_{14} = p_{11}\overline{h}_1 + p_{12}\overline{h}_2 + p_{13}\overline{h}_3$ *, the (JM) at* A_2 *is the same for* A_3 *with* $\overline{p}_{14} = p_{14}$ *becomes:*

$$\overline{J}_2 = J\left(A_2,\overline{p}_{14}\right) = \left[\overline{n}_{ij}\right]_{4\times 4},$$

where $\overline{n}_{ij} = n_{ij}, i,j = 1,2,3,4$ *as it given in* [20] *accept* $n_{44} = 0$.
Now, let $\widehat{G}^{[2]} = \left(\widehat{G}_1^{[2]},\widehat{G}_2^{[2]},\widehat{G}_3^{[2]},\widehat{G}_4^{[2]}\right)^T$ *be the (EV) of* $\overline{J}_2$ *for* $\lambda_{2h_4} = 0$.
Thus $\left(\overline{J}_2 - \lambda_{2h_4}I\right)\widehat{G}^{[2]} = 0$, *that gives:* $\widehat{G}^{[2]} = \left(\sigma_1\widehat{G}_2^{[2]},\widehat{G}_2^{[2]},\sigma_2\widehat{G}_2^{[2]},\sigma_3\widehat{G}_2^{[2]}\right)^T$ *where* $\widehat{G}_2^{[2]} \neq 0$ *any real number and* $\sigma_1 = \frac{n_{33}(n_{12}n_{24}-n_{14}n_{22})+n_{13}n_{22}n_{34}}{n_{33}(n_{14}n_{21}-n_{24}n_{11})+n_{13}(n_{24}n_{31}-n_{34}n_{21})}$, $\sigma_2 = \frac{n_{22}(n_{31}n_{14}-n_{34}n_{11})-n_{12}(n_{24}n_{31}-n_{21}n_{34})}{n_{33}(n_{14}n_{21}-n_{24}n_{11})+n_{13}(n_{24}n_{31}-n_{21}n_{34})}$,
$\sigma_3 = \frac{n_{33}\left(p_4\overline{h}_1-\frac{(p_5+p_8)(p_1+p_3)+p_8(p_2+p_4\overline{h}_2)}{(p_1+p_3)}\right)-n_{13}n_{31}n_{22}}{n_{33}(n_{14}n_{21}-n_{24}n_{11})+n_{13}(n_{24}n_{31}-n_{21}n_{34})}$.
Let $\breve{\psi}^{[2]} = \left(\breve{\psi}_1^{[2]},\breve{\psi}_2^{[2]},\breve{\psi}_3^{[2]},\breve{\psi}_4^{[2]}\right)^T$ *be the (EV) of* $\left(\overline{J}_2\right)^T$*for* $\lambda_{2h_4} = 0$.
We get $\left(\left(\overline{J}_2\right)^T - \lambda_{2h_4}I\right)\breve{\psi}^{[2]} = 0$. *Now we can solve the previous equation for* $\breve{\psi}^{[2]}$ *we obtain,*
$\breve{\psi}^{[2]} = \left(\sigma_4\breve{\psi}_2^{[2]},\breve{\psi}_2^{[2]},\sigma_5\breve{\psi}_2^{[2]},\breve{\psi}_4^{[2]}\right)^T$, $\breve{\psi}_2^{[2]} \neq 0$, $\breve{\psi}_4^{[2]} \neq 0$ *any real number and* $\sigma_4 = \frac{-n_{22}}{n_{12}}, \sigma_5 = \frac{n_{13}n_{22}}{n_{12}n_{33}}$.
Now, consider:
$\frac{\partial f}{\partial p_{14}} = f_{p_{14}}\left(\widetilde{X},p_{14}\right) = \left(\frac{\partial \widehat{f}_1}{\partial p_{14}},\frac{\partial \widehat{f}_2}{\partial p_{14}},\frac{\partial \widehat{f}_3}{\partial p_{14}},\frac{\partial \widehat{f}_4}{\partial p_{14}}\right)^T = (0,0,0,-h_4)^T$.
So, $f_{p_{14}}\left(A_2,\overline{p}_{14}\right) = (0,0,0,0)^T$ *and hence* $\left(\breve{\psi}^{[2]}\right)^T f_{p_{14}}\left(A_2,\overline{p}_{14}\right) = 0$.
Therefore, by using Sotomayor's theorem the (SNB) condition can not satisfy at A_2. *But the first condition of (TB) is verified, as below,*

since $Df_{p_{14}}\left(\widetilde{X},p_{14}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ *where* $Df_{p_{14}}\left(\widetilde{X},p_{14}\right)$ *noticed the derivative of* $f_{p_{14}}\left(\widetilde{X},p_{14}\right)$

for $\widetilde{X} = (h_1, h_2, h_3, h_4)^T$.
Additional, it is observed that

$$Df_{p_{14}}(A_2, \overline{p}_{14})\,\widehat{G}^{[2]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sigma_1 \widehat{G}_2^{[2]} \\ \widehat{G}_2^{[2]} \\ \sigma_2 \widehat{G}_2^{[2]} \\ \sigma_3 \widehat{G}_2^{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\sigma_3 \widehat{G}_2^{[2]} \end{bmatrix}, \text{ hence}$$

$$\left(\breve{\psi}^{[2]}\right)^T \left[Df_{p_{14}}(A_2, \overline{p}_{14})\,\widehat{G}^{[2]}\right] = \left(\sigma_4 \breve{\psi}_2^{[2]}, \breve{\psi}_2^{[2]}, \sigma_5 \breve{\psi}_2^{[2]}, \breve{\psi}_4^{[2]}\right)\left(0, 0, 0, -\sigma_3 \widehat{G}_2^{[2]}\right)^T = -\sigma_3 \widehat{G}_2^{[2]} \breve{\psi}_4^{[2]} \neq 0$$

under conditions (4.16) and (4.17) given in [1] with the conditions (2.9)-(2.11).
Moreover, by substituting $\widehat{G}^{[2]}$ *in equation* (2.4) *we get:*

$$D^2 f(A_2, \overline{p}_{14})\left(\widehat{G}^{[2]}, \widehat{G}^{[2]}\right) = \begin{bmatrix} -2\left(\widehat{G}_2^{[2]}\right)^2 [\sigma_1(p_4 + p_6\sigma_3) - \sigma_2^2] \\ 2\left(\widehat{G}_2^{[2]}\right)^2 [p_4\sigma_1 - p_7\sigma_3] \\ -2p_9\sigma_2\sigma_3\left(\widehat{G}_2^{[2]}\right)^2 \\ 2\sigma_3\left(\widehat{G}_2^{[2]}\right)^2 [p_{11}\sigma_1 + p_{12} + p_{13}\sigma_2] \end{bmatrix},$$

thus $\left(\breve{\psi}^{[2]}\right)^T \left[D^2 f(A_2, \overline{p}_{14})\left(\widehat{G}^{[2]}, \widehat{G}^{[2]}\right)\right] = 2\left(\widehat{G}_2^{[2]}\right)^2 (\breve{\psi}_2^{[2]} [\sigma_1 (p_4(1 - \sigma_4) - p_6\sigma_3\sigma_4)$
$+\sigma_3(\sigma_2 [p_9\sigma_5 - \sigma_2\sigma_4] - p_7)] + \sigma_3\breve{\psi}_4^{[2]} [p_{11}\sigma_1 + p_{12} + p_{13}\sigma_2]) \neq 0$ *under conditions (4.16) and (4.17) given in [1] with the conditions* (2.9)-(2.16).
Hence, by using Sotomayor's theorem system (2.2) *has (TB) at* A_2 *with the parameter* $\overline{p}_{14} = p_{14}$. □

Theorem 2.4. *Assume that the condition (4.24) given in [1] and the following conditions hold*

$$p_4\widehat{h}_1 > p_5 - p_8 - p_7\widehat{h}_4, \tag{2.17}$$
$$m_{14}(m_{41}m_{33} - m_{31}m_{43}) > m_{34}m_{13}m_{41}, \tag{2.18}$$
$$m_{14}m_{33} > m_{34}m_{13}, \tag{2.19}$$
$$\beta_5 \neq \beta_4, \tag{2.20}$$
$$\beta_5 < \beta_4, \tag{2.21}$$
$$m_{41}m_{22} > m_{42}\widehat{m}_{21}, \tag{2.22}$$
$$p_{12} + p_{13}\beta_2 < -p_{11}\beta_1, \tag{2.23}$$
$$\beta_6(p_{11}\beta_1 + p_{12} + p_{13}\beta_2) - p_6\beta_1\beta_4 - p_9\beta_2 > p_7\beta_5 \tag{2.24}$$
$$\beta_3[\beta_6(p_{11}\beta_1 + p_{12} + p_{13}\beta_2) - p_6\beta_1\beta_4 - p_7\beta_5 - p_9\beta_2] - \beta_2^2\beta_4 > p_4\beta_1(\beta_4 - \beta_5), \tag{2.25}$$

are hold. Then system (2.2) *at the (EP)* $A_4 = (\widehat{h}_1, 0, \widehat{h}_3, \widehat{h}_4)$ *with the parameter* $\widehat{p}_2 = p_2 = \frac{\widehat{L}_1}{\widehat{L}_2}$, *where*

$$\widehat{L}_1 = m_{22}([m_{14}(m_{41}m_{33} - m_{31}m_{43}) - m_{34}m_{13}m_{41}] - [p_1 + p_3 + p_6\widehat{h}_4]m_{34}m_{43}),$$
$$\widehat{L}_2 = m_{42}(m_{14}m_{33} - m_{13}m_{34}) - m_{34}m_{43}[p_8 + p_7\widehat{h}_4],$$

has (SNB) but, (TB) can't occur at A_4.
Proof . *According to the (JM) given in [1] for the system* (2.2) *at the (EP)* A_4 *has an eigenvalue equal to zero if and only if* $L_4 = 0$ *(say* $\lambda_{4h_3} = 0$*) at* $\widehat{p}_2 = p_2 = \frac{\widehat{L}_1}{\widehat{L}_2}$ *, obviously that* $\widehat{p}_2 > 0$ *under conditions* (2.17)-(2.19) *with* (4.24) *given in [1], the (JM) of system* (2.2) *with* $\widehat{p}_2 = p_2$ *becomes:*

$$\widehat{J}_4 = J(A_4, \widehat{p}_2) = [\widehat{m}_{ij}]_{4\times 4},$$

where $\widehat{m}_{ij} = m_{ij},\ i,j = 1,2,3,4$ *as it given in* [20] *accept* $\widehat{m}_{11} = -\left(p_1 + \widehat{p}_2 + p_3 + p_6\widehat{h}_4\right),\ \widehat{m}_{21} = \widehat{p}_2$.
Now, let $\widehat{G}^{[4]} = \left(\widehat{G}_1^{[4]}, \widehat{G}_2^{[4]}, \widehat{G}_3^{[4]}, \widehat{G}_4^{[4]}\right)^T$ *be the (EV) of* $\widehat{J}_4$ *for* $\lambda_{4h_3} = 0$.
Thus $\left(\widehat{J}_4 - \lambda_{4h_3} I\right)\widehat{G}^{[4]} = 0$, *that gives:* $\widehat{G}^{[4]} = \left(\beta_1\widehat{G}_2^{[4]}, \widehat{G}_2^{[4]}, \beta_2\widehat{G}_2^{[4]}, \beta_3\widehat{G}_2^{[4]}\right)^T$ *where* $\widehat{G}_2^{[4]} \neq 0$ *any real number and* $\beta_1 = \frac{-m_{22}}{\widehat{m}_{21}},\ \beta_2 = \frac{m_{41}m_{22} - m_{42}\widehat{m}_{21}}{\widehat{m}_{21}m_{43}},\ \beta_3 = \frac{m_{31}m_{22}m_{43} - m_{33}(m_{41}m_{22} - m_{42}\widehat{m}_{21})}{\widehat{m}_{21}m_{34}m_{43}}$.
Let $\breve{\psi}^{[4]} = \left(\breve{\psi}_1^{[4]}, \breve{\psi}_2^{[4]}, \breve{\psi}_3^{[4]}, \breve{\psi}_4^{[4]}\right)^T$ *be the (EV) of* $\left(\widehat{J}_4\right)^T$ *for* $\lambda_{4h_3} = 0$.
We get $\left(\left(\widehat{J}_4\right)^T - \lambda_{4h_3} I\right)\breve{\psi}^{[4]} = 0$. *Now we can solve the previous equation for* $\breve{\psi}^{[4]}$ *we obtain,*
$\breve{\psi}^{[4]} = \left(\beta_4\breve{\psi}_3^{[4]}, \beta_5\breve{\psi}_3^{[4]}, \breve{\psi}_3^{[4]}, \beta_6\breve{\psi}_3^{[4]}\right)^T$, $\breve{\psi}_3^{[4]} \neq 0$ *any real number and*
$\beta_4 = \frac{-m_{34}}{m_{14}}, \beta_5 = \frac{m_{12}m_{34}m_{43} + m_{42}(m_{14}m_{33} - m_{13}m_{34})}{m_{14}m_{43}m_{22}},\ \beta_6 = \frac{m_{13}m_{34} - m_{33}m_{14}}{m_{14}m_{43}}$.
Now, consider:
$\frac{\partial f}{\partial p_2} = f_{p_2}\left(\widetilde{X}, p_2\right) = \left(\frac{\partial f_1}{\partial p_2}, \frac{\partial f_2}{\partial p_2}, \frac{\partial f_3}{\partial p_2}, \frac{\partial f_4}{\partial p_2}\right)^T = (-h_1, h_1, 0, 0)^T$.
So, $f_{p_2}(A_4, \widehat{p}_2) = \left(-\widehat{h}_1, \widehat{h}_1, 0, 0\right)^T$ *and hence* $\left(\breve{\psi}^{[4]}\right)^T f_{p_2}(A_4, \widehat{p}_2) = \widehat{h}_1\breve{\psi}_3^{[4]}(\beta_5 - \beta_4) \neq 0$ *if the conditions (4.24) given in [1], with (4.15),(4.18) and* (2.15)-(2.18) *hold.*
Now substitute $\widehat{G}^{[4]}$ *in equation* (2.4) *we get*

$$D^2 f(A_4, \widehat{p}_2)\left(\widehat{G}^{[4]}, \widehat{G}^{[4]}\right) = \begin{bmatrix} -2\left(\widehat{G}_2^{[4]}\right)^2\left[\beta_1\left(p_4 + p_6\beta_3\right) + \beta_2^2\right] \\ 2\left(\widehat{G}_2^{[4]}\right)^2\left[p_4\beta_1 - p_7\beta_3\right] \\ -2p_9\beta_2\beta_3\left(\widehat{G}_2^{[4]}\right)^2 \\ 2\beta_3\left(\widehat{G}_2^{[4]}\right)^2\left[p_{11}\beta_1 + p_{12} + p_{13}\beta_2\right] \end{bmatrix},$$

thus $\left(\breve{\psi}^{[4]}\right)^T\left[D^2 f(A_4, \widehat{p}_2)\left(\widehat{G}^{[4]}, \widehat{G}^{[4]}\right)\right] =$
$2\left(\widehat{G}_2^{[4]}\right)^2\breve{\psi}_3^{[4]}(p_4\beta_1(\beta_5 - \beta_4) - \beta_2^2\beta_4 + \beta_3\left[\beta_6\left(p_{11}\beta_1 + p_{12} + p_{13}\beta_2\right) - p_6\beta_1\beta_4 - p_7\beta_5 - p_9\beta_2\right]) \neq 0$ *under conditions (*4.24*) given in [1], with* (2.17)-(2.25).
Therefore, by using Sotomayor's theorem the (SNB) conditions can satisfy at A_4. *But if the conditions* (2.20) *holds then there is no (SNB), to verify the conditions of (TB) is, as below,*

since $Df_{p_2}\left(\widetilde{X}, p_2\right) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *where* $Df_{p_2}\left(\widetilde{X}, p_2\right)$ *noticed the derivative of* $f_{p_2}\left(\widetilde{X}, p_2\right)$
for $\widetilde{X} = (h_1, h_2, h_3, h_4)^T$.
Additional, it is observed that

$$Df_{p_2}(A_4, \widehat{p}_2)\widehat{G}^{[4]} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \beta_1\widehat{G}_2^{[4]} \\ \widehat{G}_2^{[4]} \\ \beta_2\widehat{G}_2^{[4]} \\ \beta_3\widehat{G}_2^{[4]} \end{bmatrix} = \begin{bmatrix} -\beta_1\widehat{G}_2^{[4]} \\ \beta_1\widehat{G}_2^{[4]} \\ 0 \\ 0 \end{bmatrix},$$ *hence*

$\left(\breve{\psi}^{[4]}\right)^T\left[Df_{p_2}(A_4, \widehat{p}_2)\widehat{G}^{[4]}\right] = \beta_1\widehat{G}_2^{[4]}\breve{\psi}_3^{[4]}(\beta_5 - \beta_4) = 0$ *if the condition* (2.20) *hold.*
Hence, by using Sotomayor's theorem system (2.2) *can't has (TB) at* A_4 *with the parameter* $\widehat{p}_2 = p_2$.

□

Theorem 2.5. *Assume that the conditions*

$$r_{34}\left[\left(p_2+p_4h_2^*\right)\left(p_8+p_7h_4^*\right)-\left(p_1+p_3+p_6h_4^*\right)\left(p_4h_1^*-p_5-p_8-p_7h_4^*\right)\right]+r_{33}\left(r_{14}r_{21}-r_{11}r_{24}\right)$$
$$<r_{31}\left(r_{12}r_{24}-r_{14}r_{22}\right), \tag{2.26}$$
$$p_{11}^*\gamma_1+p_{13}<-p_{12}\gamma_2, \tag{2.27}$$
$$\gamma_6\left(p_{11}^*\gamma_1+p_{12}\gamma_2+p_{13}\right)-p_9\gamma_5> p_7\gamma_2, \tag{2.28}$$
$$1>p_6\gamma_1\gamma_3, \tag{2.29}$$

are hold. Then system (2.2) *at the (EP)* $A_5=(h_1^*,h_2^*,h_3^*,h_4^*)$ *with the parameter* $p_{11}^*=p_{11}=\frac{R_1}{R_2}$, *where*

$$R_1=r_{13}r_{34}r_{21}r_{42}-r_{43}[r_{34}[\left(p_2+p_4h_2^*\right)\left(p_8+p_7h_4^*\right)-\left(p_1+p_3+p_6h_4^*\right)\left(p_4h_1^*-p_5-p_8-p_7h_4^*\right)]$$
$$+r_{33}\left(r_{14}r_{21}-r_{11}r_{24}\right)-r_{13}\left(r_{12}r_{24}-r_{14}r_{22}\right)],$$
$$R_2=h_4^*[r_{13}\left(r_{34}r_{22}+r_{24}r_{42}\right)+r_{33}\left(r_{12}r_{24}-r_{14}r_{22}\right)]$$

has (SNB) but, (TB) can't occur at A_5 .

Proof . *According to the (JM) given in [1] for the system (2.2) at the (EP)* A_5 *has an eigenvalue equal to zero if and only if* $E_4=0$ *(say* $\lambda_{5h_2}=0$*) at* $p_{11}^*=p_{11}=\frac{R_1}{R_2}$ *, obviously that* $p_{11}^*>0$ *under conditions (4.32)-(4.34) given in [1] with* (2.26)*, the (JM) of system* (2.2) *with* $p_{11}^*=p_{11}$ *becomes:*

$$J_5^*=J\left(A_5,p_{11}^*\right)=\left[r_{ij}^*\right]_{4\times4},$$

where $r_{ij}^*=r_{ij},i,j=1,2,3,4$ *as it given in [1] accept* $r_{41}^*=p_{11}^*h_4^*$.
Now, let $\widehat{G}^{[5]}=\left(\widehat{G}_1^{[5]},\widehat{G}_2^{[5]},\widehat{G}_3^{[5]},\widehat{G}_4^{[5]}\right)^T$ *be the (EV) of* J_5^* *for* $\lambda_{5h_2}=0$.
Thus $\left(J_5^*-\lambda_{5h_2}I\right)\widehat{G}^{[5]}=0$*, that gives:* $\widehat{G}^{[5]}=\left(\gamma_1\widehat{G}_3^{[5]},\gamma_2\widehat{G}_3^{[5]},\widehat{G}_3^{[5]},\gamma_3\widehat{G}_3^{[5]}\right)^T$ *where* $\widehat{G}_3^{[5]}\neq0$ *any real number and* $\gamma_1=\frac{-r_{33}}{r_{31}}$, $\gamma_2=\frac{r_{41}^*r_{33}-r_{43}r_{31}}{r_{31}r_{42}}$, $\gamma_3=\frac{r_{21}r_{33}r_{42}-r_{22}(r_{41}^*r_{33}-r_{43}r_{31})}{r_{31}r_{24}r_{42}}$.
Let $\breve{\psi}^{[5]}=\left(\breve{\psi}_1^{[5]},\breve{\psi}_2^{[5]},\breve{\psi}_3^{[5]},\breve{\psi}_4^{[5]}\right)^T$ *be the (EV) of* $(J_5^*)^T$*for* $\lambda_{5h_2}=0$.
We get $\left((J_5^*)^T-\lambda_{5h_2}I\right)\breve{\psi}^{[5]}=0$*. Now we can solve the previous equation for* $\breve{\psi}^{[5]}$ *we obtain,*
$\breve{\psi}^{[5]}=\left(\gamma_4\breve{\psi}_2^{[5]},\breve{\psi}_2^{[5]},\gamma_5\breve{\psi}_2^{[5]},\gamma_6\breve{\psi}_2^{[5]}\right)^T$, $\breve{\psi}_2^{[5]}\neq0$ *any real number and* $\gamma_4=\frac{-r_{24}}{r_{14}}$,
$\gamma_5=\frac{r_{24}r_{42}r_{13}-r_{43}(r_{12}r_{24}-r_{14}r_{22})}{r_{14}r_{42}r_{33}}$, $\gamma_6=\frac{r_{12}r_{24}-r_{14}r_{22}}{r_{14}r_{42}}$.
Now, consider: $p_{11}^*=p_{11}$, $\frac{\partial f}{\partial p_{11}}=f_{p_{11}}\left(\widetilde{X},p_{11}\right)=\left(\frac{\partial\widehat{f}_1}{\partial p_{11}},\frac{\partial\widehat{f}_2}{\partial p_{11}},\frac{\partial\widehat{f}_3}{\partial p_{11}},\frac{\partial\widehat{f}_4}{\partial p_{11}}\right)^T=(0,0,0,h_1h_4)^T$.
So, $f_{p_{11}}\left(A_5,p_{11}^*\right)=(0,0,0,h_1^*h_4^*)^T$ *and hence* $\left(\breve{\psi}^{[5]}\right)^T f_{p_{11}}\left(A_5,p_{11}^*\right)=h_1^*h_4^*\gamma_6\breve{\psi}_2^{[5]}\neq0$ *under condition (4.32) given in [1].*
Now substitute $\widehat{G}^{[5]}$ *in equation* (2.4) *we get*

$$D^2f\left(A_5,p_{11}^*\right)\left(\widehat{G}^{[5]},\widehat{G}^{[5]}\right)=\begin{bmatrix}-2\left(\widehat{G}_3^{[5]}\right)^2\left[\gamma_1(p_4\gamma_2+p_6\gamma_3)-1\right]\\ 2\gamma_2\left(\widehat{G}_3^{[5]}\right)^2\left[p_4\gamma_1-p_7\gamma_3\right]\\ -2p_9\gamma_3\left(\widehat{G}_3^{[5]}\right)^2\\ 2\gamma_3\left(\widehat{G}_3^{[5]}\right)^2\left[p_{11}^*\gamma_1+p_{12}\gamma_2+p_{13}\right]\end{bmatrix},$$

thus $\left(\breve{\psi}^{[5]}\right)^T\left[D^2 f\left(A_5, p_{11}^*\right)\left(\widehat{G}^{[5]}, \widehat{G}^{[5]}\right)\right]=2\left(\widehat{G}_3^{[5]}\right)^2 \breve{\psi}_2^{[5]}\left[\gamma_3\left[\gamma_6\left(p_{11}^* \gamma_1+p_{12} \gamma_2+p_{13}\right)-p_7 \gamma_2-p_9 \gamma_5\right)\right]+$ $p_4 \gamma_1 \gamma_2\left(1-\gamma_4\right)+\gamma_4\left(1-p_6 \gamma_1 \gamma_3\right) \neq 0$ *under conditions (4.32) and (4.33) given in [1] with* (2.27)-(2.29). *Therefore, by using Sotomayor's theorem system* (2.2) *has (SNB) at* A_5 *but, (TB) can't occurs* A_5 *with the parameter* $p_{11}^*=p_{11}$. $\square$

3. Hopf Bifurcation Analysis (HB)

In this section, the occurrence of (HB) near the positive (EP) of system (2.2) has been investigated according to the Haque and Venturino methods [10] for $n=4$ is stated in terms of the properties of eigenvalues as shown below:

$$P_4(\lambda)=\lambda^4+W_1 \lambda^3+W_2 \lambda^2+W_3 \lambda+W_4=0,$$

where $W_1=-tr(J(x^*))$, $W_2=M_1(J(x^*))$, $W_3=-\ M_2(J(x^*))$ and $W_4=det\ (J(x^*))$ with $M_1(J(x^*))$ and $M_2(J(x^*))$ represent the sum of the principal minors of order two and of $J(x^*)$ respectively. Clearly, the first condition of (HB) satisfies if and only if
$W_i>0$; $i=1\ ,3,4$, $\Delta_1=W_1 W_2-W_3>0$, $W_1^3\ -4\ \Delta_1>0$, and $\Delta_2=W_3\left(W_1 W_2-W_3\right)-W_1^2 W_4=0$.
Consequently, $W_4=\frac{W_3\left(W_1 W_2-W_3\right)}{W_1^2}$.
So, the characteristic equation becomes:

$$P_4(\lambda)=\left(\lambda^2+\frac{W_3}{W_1}\right)\left(\lambda^2+\ W_1 \lambda+\frac{\Delta_1}{W_1}\right)=0. \tag{3.1}$$

Clearly, the roots of eq. (4.1) are $\lambda_{1,2}=\pm i \sqrt{\frac{W_3}{W_1}}$ and $\lambda_{3,4}=\frac{1}{2}\left(-W_1 \pm \sqrt{W_1^2-4 \frac{\Delta_1}{W_1}}\right)$.
Now, in order to verify the transversality condition of (HB), we substitute $\lambda(\mu)=\rho_1(\mu) \mp i \rho_2(\mu)$ into eq. (3.1), and then calculating its derivative with respect to the bifurcation parameter μ, $P_4^{\prime}(\lambda(\mu))=0$, comparing the two sides of this equation and then equating their real and imaginary parts, we have:

$$\left.\begin{array}{l}\breve{\varphi}(\mu) \rho_1^{\prime}(\mu)-\breve{\Phi}(\mu) \rho_2^{\prime}+\breve{\theta}(\mu)=0, \\ \breve{\Phi}(\mu) \rho_1^{\prime}(\mu)+\breve{\varphi}(\mu) \rho_2^{\prime}(\mu)+\breve{\Gamma}(\mu)=0.\end{array}\right\} \tag{3.2}$$

where:

$$\left.\begin{array}{l}\breve{\varphi}(\mu)=4(\rho_1(\mu))^3+3 W_1(\mu)(\rho_1(\mu))^2+W_3(\mu)+2 W_2(\mu) \rho_1(\mu)-12 \rho_1(\mu)(\rho_2(\mu))^2 \\ \qquad -3 W_1(\mu)(\rho_2(\mu))^2, \\ \breve{\theta}(\mu)=12(\rho_1(\mu))^2 \rho_2(\mu)+6 W_1(\mu) \rho_1(\mu) \rho_2(\mu)+2 W_2(\mu) \rho_2(\mu)-4(\rho_2(\mu))^3, \\ \breve{\Phi}(\mu)=(\rho_1(\mu))^3\ W_1^{\prime}(\mu)+W_3^{\prime}(\mu) \rho_1(\mu)+W_2^{\prime}(\mu)(\rho_1(\mu))^2+W_4^{\prime}(\mu) \\ \qquad -3 W_1^{\prime}(\mu) \rho_1(\mu)(\rho_2(\mu))^2-W_2^{\prime}(\mu)(\rho_2(\mu))^2, \\ \breve{\Gamma}(\mu)=3 W_1^{\prime}(\mu)(\rho_1(\mu))^2 \rho_2(\mu)+W_3^{\prime}(\mu) \rho_2(\mu)+2 W_2^{\prime}(\mu) \rho_1(\mu) \rho_2(\mu)-W_1^{\prime}(\mu)(\rho_2(\mu))^3\end{array}\right\} \tag{3.3}$$

Solving the linear system (3.2) by using Cramer's rule for the unknowns $\rho_1^{\prime}(\mu)$ and $\rho_2^{\prime}(\mu)$, gives that

$$\rho_1^{\prime}(\mu)=-\frac{\breve{\theta}(\mu)\ \ \breve{\varphi}(\mu)+\breve{\Gamma}(\mu)\ \ \breve{\Phi}(\mu)}{(\breve{\varphi}(\mu))^2+(\breve{\Phi}(\mu))^2} \text { and } \rho_2^{\prime}(\mu)=\frac{-\breve{\Gamma}(\mu)\ \ \breve{\varphi}(\mu)+\breve{\theta}(\mu)\ \ \breve{\Phi}(\mu)}{(\breve{\varphi}(\mu))^2+(\breve{\Phi}(\mu))^2}.$$

Hence, the second condition of the (HB) which is necessary and sufficient condition (transversality condition) $\frac{d}{d\mu}\mathrm{Re}\,(\lambda)\,|_{\mu=\bar{\mu}} = \rho_1'\,(\mu)\,|_{\mu=\bar{\mu}}$ not being zero if and only if:

$$\breve{\theta}\,(\mu)\ \ \breve{\varphi}\,(\mu) + \breve{\Gamma}\,(\mu)\ \ \breve{\Phi}\,(\mu) \neq 0. \tag{3.4}$$

Theorem 3.1. *Suppose that the following conditions hold:*

$$r_{34}E_1 > b_1, \tag{3.5}$$
$$r_{12}r_{21} > r_{11}\,(r_{22} + r_{33}) + r_{22}r_{33} - r_{13}r_{31} - r_{14}r_{41} - r_{24}r_{42}, \tag{3.6}$$
$$r_{43}^*\,[b_1 - r_{34}E_1] - E_1\,[r_{11}\,(r_{22} + r_{33}) + r_{22}r_{33} - r_{12}r_{21} - r_{13}r_{31} - r_{14}r_{41} - r_{24}r_{42}] > b_2, \tag{3.7}$$
$$p_{11} < p_{12}, \tag{3.8}$$
$$E_3 < \Delta_1 < \frac{E_1^3}{4}, \tag{3.9}$$
$$r_{34}E_3 > b_1E_1, \tag{3.10}$$

Where

$$\Delta_1 = E_1E_2 - E_3 = r_{43}^*\,[b_1 - r_{34}E_1] - E_1\,[r_{11}\,(r_{22} + r_{33}) + r_{22}r_{33} - r_{12}r_{21} - r_{13}r_{31} - r_{14}r_{41} - r_{24}r_{42}] - b_2,$$
$$b_1 = r_{34}\,[(p_1 + p_3 + p_6h_4^*\ )\,(p_4h_1^* - p_5 - p_8 - p_7h_4^*) - (p_2 + p_4h_2^*)\,(p_8 + p_7h_4^*)] + r_{31}(r_{12}r_{24} - r_{14}r_{22}),$$
$$b_2 = r_{33}\,[r_{42}(r_{14}r_{21} - r_{11}r_{24}) + r_{41}(r_{12}r_{24} - r_{14}r_{22})] + r_{13}[r_{34}r_{22}h_4^*\,(p_{11} - p_{12}) + r_{24}r_{42}r_{41}].$$

Then for the parameter value $p_{13} = p_{13}^*$, system (2.2) has a (HB) at A_5 .

Proof . *The characteristic equation of system (2.2) at A_5 mentioned in [1].*

$$\lambda^4 + E_1\lambda^3 + E_2\lambda^2 + E_3\lambda + E_4 = 0, \tag{3.11}$$

"we need to find the parameter" (p_{13}^*) "to verify the necessary and sufficient conditions for (HB)" to occurs at the positive (EP) that satisfy; $E_i\,(p_{13}^*) > 0,\quad i = 1, 3,\ 4$, $\Delta_1\,(p_{13}^*) = E_1E_2 - E_3 > 0$, $E_1^3 - 4\Delta_1 > 0$ and $\Delta_2\,(p_{13}^*) = (E_1E_2 - E_3)\,E_3 - E_1^2E_4 = 0$.
$E_1\,(p_{13}^*) > 0$, provided condition of locally (4.32) given in [1], $E_3\,(p_{13}^*) > 0$, provided conditions of locally (4.32)-(4.35) given in [1], $E_4\,(p_{13}^*) > 0$, provided conditions of locally (4.32)-(4.34) and (4.36) given in [1], $\Delta_1\,(p_{13}^*) > 0$, provided conditions of locally (4.32) and (4.34) given in [1] with (3.5)-(3.9)*, $E_1^3 - 4\Delta_1 > 0$ provided conditions of locally (4.32) and (4.35) given in [1], with* (3.5)-(3.9) *hold. On the other hand, It is observed that $\Delta_2 = 0$, gives:*

$$B_1p_{13}^{*2} + B_2p_{13}^* + B_3 = 0, \tag{3.12}$$

where $B_1 = h_4^{*2}b_1\,[b_1 + r_{34}E_1]$, $B_2 = -h_4^*b_1\,[E_1 - b_2]$,
$B_3 = -b_2(b_2 - E_1\,[r_{11}\,(r_{22} + r_{33}) + r_{22}r_{33} - r_{12}r_{21} - r_{13}r_{31} - r_{14}r_{41} - r_{24}r_{42}])$.
Now, $B_1 > 0$ provided conditions of locally (4.32) and (4.34) given in [1] hold and $B_3 < 0$ if the conditions of locally (4.32) and (4.33) given in [1] with (3.6) *and* (3.8) *are hold.*
By using "Descartes rule of sign", equation (3.12) *has a unique positive root*
$p_{13}^* = \frac{-1}{2B_1}(B_2 + \sqrt{B_2^2 - 4B_1B_3}\ \ )$.
Now, at ($p_{13} = p_{13}^*$), the characteristic equation (3.11) *can be rewritten as:*

$$P\,(\lambda) = \left(\lambda_5^2 + \frac{E_3}{E_1}\right)\left(\lambda_5^2 + E_1\lambda_5 + \frac{\Delta_1}{E_1}\right), \tag{3.13}$$

which have four roots: $\lambda_{5\ h_1,h_2} = \mp i\sqrt{\frac{E_3}{E_1}}$ *and* $\lambda_{5\ h_3,h_4} = \frac{1}{2}(-E_1 \mp \sqrt{E_1^2 - 4\frac{\Delta_1}{E_1}}\ )$.
Observe that at $(p_{13} = p_{13}^*)$, *there are two pure imaginary eigenvalues* $(\lambda_{5\ h_1}$ *and* $\lambda_{5\ h_2})$ *and two eigenvalues are real and negative* $(\lambda_{5\ h_3}$ *and* $\lambda_{5\ h_4})$.
Now for all values of p_{13} *in the neighborhood of* p_{13}^*, *roots are generally as follows*
$\lambda_{5\ h_1,h_2} = \tau_1 \pm i\tau_2$, $\lambda_{5\ h_3,h_4} = \frac{1}{2}(-E_1 \mp \sqrt{E_1^2 - 4\frac{\Delta_1}{E_1}}\ )$.
Clearly, $Re\left(\lambda_{5\ h_1,h_2}\,(p_{13})\right)|_{p_{13}=p_{13}^*} = \tau_1\left(p_{13}^*\right) = 0$ *that means the first condition of the necessary and sufficient condition for (HB) is satisfied at* $(p_{13} = p_{13}^*)$. *Now, to verify the transversally condition we must prove that:*
$\breve{\theta}\left(p_{13}^*\right)\breve{\varphi}\left(p_{13}^*\right) + \breve{\Gamma}\left(p_{13}^*\right)\breve{\Phi}\left(p_{13}^*\right) \neq 0,$
Where $\breve{\theta}$, $\breve{\varphi}$, $\breve{\Gamma}$ *and* $\breve{\Phi}$ *are given in* (3.3). *Note that for* $p_{13} = p_{13}^*$ *we have:*
$\tau_1 = 0$ *and* $\tau_2 = \sqrt{\frac{E_3}{E_1}}$, *substituting the value of* τ_2 *gives the following simplifications:*

$$\breve{\theta}\left(p_{13}^*\right) = \frac{h_4^*}{E_1\left(p_{13}^*\right)}\left[r_{34}E_3\left(p_{13}^*\right) - b_1E_1\left(p_{13}^*\right)\right], \quad \breve{\varphi}\left(p_{13}^*\right) = -2E_3\left(p_{13}^*\right),$$

$$\breve{\Gamma}\left(p_{13}^*\right) = h_4^*\sqrt{\frac{E_3\left(p_{13}^*\right)}{E_1\left(p_{13}^*\right)}}\left[r_{34}\left(r_{11}+r_{22}\right) - r_{14}r_{31}\right] \quad and \quad \breve{\Phi}\left(p_{13}^*\right) = 2\sqrt{\frac{E_3\left(p_{13}^*\right)}{E_1\left(p_{13}^*\right)}}\left(\frac{\Delta_1\left(p_{13}^*\right) - E_3\left(p_{13}^*\right)}{E_1\left(p_{13}^*\right)}\right),$$

Where

$$E_1'\left(p_{13}^*\right) = \left.\frac{\partial E_1}{\partial p_{13}}\right|_{p_{13}=p_{13}^*} = 0, \quad E_2'\left(p_{13}^*\right) = \left.\frac{\partial E_2}{\partial p_{13}}\right|_{p_{13}=p_{13}^*} = -h_4^* r_{34},$$

$$E_3'\left(p_{13}^*\right) = \left.\frac{\partial E_3}{\partial p_{13}}\right|_{p_{13}=p_{13}^*} = h_4^*\left[r_{34}\left(r_{11}+r_{22}\right) - r_{14}r_{31}\right], \quad E_4'\left(p_{13}^*\right) = \left.\frac{\partial E_4}{\partial p_{13}}\right|_{p_{13}=p_{13}^*} = -b_1h_4^*.$$

provided condition of locally (4.32)-(4.34) given in [1] with conditions (3.9) *and* (3.10) *are hold. Therefore system* (2.2) *at* A_5 *with the parameter* p_{13}^* *has a (HB).* □

4. Numerical Simulation of System (2.2)

In this section, "our obtained results in the previous sections are confirmed numerically by using the method of Runge Kutta beside the method of predictor corrector. Note that, in programming we used turbo C++ and for plotting Matlab, and then our obtained results have been discussed. System (2.2) is solved numerically for a set of parameters and sets of initial points". The purpose of studying numerical simulations is first to see the effectiveness of parameters and second confirm our obtained analytical results. It is noticed that, for the following set of hypothetical parameters that is assumed bellow which satisfies the stability conditions for the positive (EP), system (2.2) has a (GAS) positive (EP) .

$$\left.\begin{array}{l} p_1 = 0.4,\ p_2 = 0.3,\ p_3 = 0.01,\ p_4 = 0.5,\ p_5 = 0.9,\ p_6 = 0.2,\ p_7 = 0.3, \\ p_8 = 0.5,\ p_9 = 0.2,\ p_{10} = 0.06,\ p_{11} = 0.15,\ p_{12} = 0.2,\ p_{13} = 0.1,\ p_{14} = 0.11. \end{array}\right\} \tag{4.1}$$

The solution of system (2.2) "converges asymptotically to" $A_5\,(0.324, 0.063, 0.489, 1.025)$ beginning from different four initial points $(0.7, 0.1, 0.9, 0.5)$, $(0.2, 0.2, 0.7, 0.6)$, $(0.8, 0.4, 1, 0.7)$ and $(0.4, 0.3, 0.6, 0.4)$ and this confirms our analytical result that was obtained.

***Table 2: Numerical behavior for bifurcation of system* (2.2) *as varying some parameters and keeping the rest fixed as in eq.* (4.1).**

The range of the parameters	*Converge to*	*Bifurcation point*
$0.01 \leq p_1 < 0.8$	A_5	$p_1 = 0.8$
$0.8 \leq p_1 < 1$	A_2	
$0.0001 \leq p_2 < 0.001$	A_4	$p_2 = 0.001$
$0.001 \leq p_2 \leq 2$	A_5	
$0.001 \leq p_3 < 0.53$	A_5	
$0.53 \leq p_3 < 1$	A_2	$p_3 = 0.53$
$0.01 \leq p_4 \leq 1$	A_5	
$0.1 \leq p_5 < 1$	A_5	
$0.15 \leq p_6 \leq 2$	A_5	
$0.3 \leq p_7 \leq 1$	A_5	
$0.01 \leq p_8 < 1$	A_5	
$0.1 \leq p_9 \leq 1.5$	A_5	
$0.01 \leq p_{10} < 0.52$	A_5	
$0.52 \leq p_{10} < 0.89$	A_2	$p_{10} = 0.52$
$0.89 \leq p_{10} < 1$	A_0	$p_{10} = 0.89$
$0.1 \leq p_{11} < 0.19$	A_5	
$0.1 \leq p_{12} \leq 0.3$	A_5	
$0.1 \leq p_{13} \leq 0.2$	A_5	
$0.01 \leq p_{14} < 0.14$	A_5	
$0.14 \leq p_{14} < 1$	A_2	$p_{14} = 0.14$

To discuss the effectiveness of parameters on the behaviour of a dynamic system. By varying one parameter at each time for the data in (4.1) the system has been solved numerically and the observations are summarized in Table (2).

The parameter p_{10}. It is noticed that the solution converges to A_5 in the range $0.01 \leq p_{10} < 0.52$ as seen in Figure 1(a), for typical value $p_{10} = 0.3$, but if we increasing p_{10} in the range $0.52 \leq p_{10} < 0.89$ the solution converges to A_2 see Figure 1 (b), for typical value $p_{10} = 0.7$, but in the range $0.89 \leq p_{10} < 1$ the solution converges to A_0 ,see Figure 1 (c), for typical value $p_{10} = 0.99$

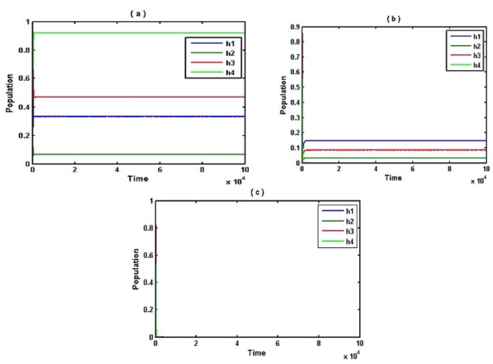

Figure 1: (a) Time series (TS) of the solution converges to $A_5 = (0.333, 0.066, 0.469, 0.920)$, for typical value $p_{10} = 0.1$, (b) (TS) of the solution converges to $A_2 = (0.148, 0.033, 0.085, 0)$, for typical value $p_{10} = 0.7$, (c) (TS) of the solution converges to $A_0 = (0, 0, 0, 0)$, for typical value $p_{10} = 0.99$.

Now, changing only the parameters p_2, p_8, p_{11} and p_{13} at the same time with the set of parameter given in eq. (4.1) it is noticed that for $0.0001 \leq p_2 < 0.002,\ \ 0.6 \leq p_8 < 0.99,\ 0.001 \leq p_{11} < 0.04$ *and* $0.01 \leq p_{13} < 0.105$ the solution converges to A_1 as seen in Figure 2, for typical value $p_2 = 0.01$, $p_8 = 0.7, p_{11} = 0.01$ *and* $p_{13} = 0.1$.

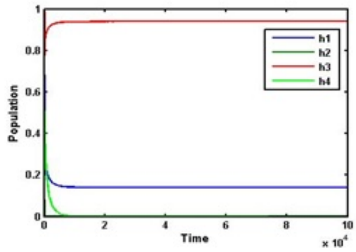

Figure 2: (TS) of the solution converges to $A_1 = (0.141, 0, 0.938\ , 0)$ for typical value $p_2 = 0.001, p_8 = 0.7, p_{11} = 0.01$ *and* $p_{13} = 0.1$.

5. Conclusion and Discussion

In this paper, the conditions of occurring the (LB) are established, it's noticed that near the trivial, axial and free predator (EPs) there is (TB), while at free disease and the positive (EPs) there is (SNB). On the other hand near all of these (EPs) there is no (PB). Further investigations for the (HB) near the positive (EP) are carried out. Finally, to illustrate the occurrence of (LB) of the system have been used the numerical simulations. Further, system (2.2) "has been solved numerically for different sets of initial points and one set of parameters starting with the hypothetical set of data given by eq. (4.1) and we obtained that:

1. For the set of parameters given that we have proposed in eq.(4.1) the system (2.2) has no periodic solution.
2. For the set of parameters given in eq. (4.1), the most effectiveness parameters on the stability of system (2.2) are $p_1,\ p_2, p_3, p_{10}\ , p_{11}, p_{13}$and p_{14}.
3. Varying only the parameters p_2, p_8, p_{11} and p_{13} at the same time with the rest of parameters as in eq. (4.1) it's noticed that for $0.0001 \leq p_2 < 0.002,\ \ 0.6 \leq p_8 < 0.99,\ \ 0.001 \leq p_{11} < 0.04$ and

$0.01 \leq p_{13} < 0.105$ "the solution converges to" A_1 .

References

[1] Z. Kh. Alabacy and A.A. Majeed, *The dynamics and analysis of an SIS disease of a stage-structured prey-predator model with a prey refuge*, Int. J. Nonlinear Anal. Appl. 13(1) (2022) 901–919.

[2] J. Alidoust, *Stability and bifurcation analysis of a fractional prey-predator scavenger mode*, Appl. Math. Model. 81 (2020) 342–355.

[3] U. Ghosh, S. Pal and M. Banerjee, *Memory effect on Bazykin's prey-predator model: Stability and bifurcation analysis*, Chaos Solitons Fractals 143 (2020) 110531.

[4] J. Li, Y. Zhaob and H. Zhub, *Bifurcation of an SIS model with nonlinear contact rate*, J. Math. Anal. Appl. 432(2) (2015) 1119–1138.

[5] A.A. Majeed and Z. Kh. Alabacy, *The persistence and bifurcation analysis of an ecological model with fear effect involving prey refuge and harvesting*, AIP Conf. Proc. (2021).

[6] A. A. Majeed and A.J. Kadhim, *The bifurcation analysis and persistence of the food chain ecological model with toxicant*, J. Phys. Conf. Ser. 1818 (2021) 012191.

[7] A.A. Majeed, *Refuge and age structures impact on the bifurcation analysis of an ecological model*, J. Southwest Jiaotong Univ. 55 (1) (2020).

[8] A.A. Majeed and M.H. Ismaeeb, *The bifurcation analysis of prey-predator model in the presence of stage structured with harvesting and toxicity*, J. Phys. Conf. Ser. 1362(1) (2019) 012155.

[9] A.A. Majeed and N.H. Ali, *The bifurcation of stage structured prey-predator food chain model with refuge*, Int. J. Modern Res. Engin. Technol. 1(8) (2016) 4–12.

[10] A.A. Majeed and H.G. Sufaeh, *The bifurcation of the dynamics of prey-predator model with harvesting involving diseases in both populations*, Sci. Int.(Lahore) 30(5) (2018) 793–805.

[11] A.A. Majeed and I.I. Shawka, *The bifurcation analysis of an eco-epidemiological model with two infectious diseases in prey*, Bull. Math. Stat. Res. 5(1) (2017).

[12] A.A. Majeed, *Local bifurcation and persistence of an ecological system consisting of a predator and stage structured prey*, Iraqi J. Sci. 54(3) (2013) 696–705.

[13] A. A. Majeed and M. A. Lafta, *The bifurcation analysis of food web prey-predator model with toxin*, J. Phys. IOP Pub. Conf. Ser. 1897(1) (2021) 012080.

[14] D. Melese and S. Feyissa, *Stability and bifurcation analysis of a diffusive modified Leslie-Gower prey-predator model with prey infection and Beddington DeAngelis functional response*, Heliyon 7(2) (2021) 06193.

[15] D. Mukherjee and C. Maji, *Bifurcation analysis of a Holling type II predator-prey model with refuge*, Chinese J. Phys. 65 (2020) 153–162.

[16] F.A. Rihan, H.J. Alsakaji and C. Rajivganthi, *Stability and hopf bifurcation of three-species prey-predator system with time delays and Allee Effect*, Complexity 2020 (2020) Article ID 7306412.

[17] A. Saikh and N.H. Gazi, *The effect of the force of infection and treatment on the disease dynamics of an SIS epidemic model with immigrants*, Results Control Optim. 2(100007) (2021).

[18] D. Sen, S. Ghorai, S. Sharma and M. Banerjee, *Allee effect in prey's growth reduces the dynamical complexity in prey-predator model with generalist predator*, Appl. Math. Model. 91 (2021) 768–790.

[19] D. Sen, S. Petrovskii, S. Ghorai and M. Banerjee, *Rich bifurcation structure of prey-predator model induced by the Allee effect in the growth of generalist predator*, Int. J. Bifurcat. Chaos 30(6) (2020) 2050084.

[20] Z. Zhang and A. Wan, *Bifurcation analysis of a three-species ecological system with time delay and harvesting*, Adv. Diff. Equ. 342 (2017) 2–12.