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Estimation of exponential Pareto parameters

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Abstract

In this paper, we simulated the methods of estimation of Pareto exponential distribution parameters, using various experiments including default parameter values (0.5, 1, 5), sample sizes (10,250,100), and Iterations (D = 1000). The Maximum likelihood method and the standard Bayes method were used with informative prior distribution (Gamma distribution) and the non-information prior distribution (Uniform distribution), and symmetric loss function (quadratic loss function), and asymmetric (General Entropy loss function). The Bayes method gave complex mathematical formulas that were solved by Lindley approximation. The results of the simulation experiments showed the advantage of the standard Bayes method in the information prior and the quadratic loss function.

Keywords: exponential Pareto distribution, maximum likelihood, standard Bays method, Lindley approximation.

1. Introduction and research aim

This distribution is attributed to the Italian economic world (Vilfredo Pareto), which established the basis of distribution in the subject of economics through the study of the distribution of income, and it is a very important distribution in the analysis of times of failure and models of stress and strength. Richard E.Quandt estimated the parameters of Pareto distribution using the method of moments, the method of least squares, and the method of Maximum likelihood; Hosain and William

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compared some methods for estimating Pareto type 1 parameters. This study aims at estimating exponential Pareto parameters using different methods, Maximum likelihood, Bayes with informative and non-informative priors, various simulations studies were used, including the sizes of samples and default parameter values, and the estimates were then compared with the mean squares error MSE.

2. Exponential Pareto distribution [5]

Gupta (1998) presents Pareto exponential or generalized distribution (EPD) as one of the survival time models. The probability density function is defined as follows:

$$f\left(z,\alpha,\beta\right)=\frac{1}{\left(1+z\right)^{\left(\alpha+1\right)}}\ \left[1-\left(1+z\right)^{-\alpha}\right]^{\beta-1},\qquad z,\alpha,\beta>0 \tag{2.1}$$

It has a cumulative function according to the following formula:

$$F\left(z,\alpha,\beta\right)=\left[1-\left(1+z\right)^{-\alpha}\right]^{\beta},\qquad z,\alpha,\beta>0, \tag{2.2}$$

(α) : represents the shape parameter, (β) : represents the parameter

3. Maximum Likelihood method [4]

(($z_1,\ z_2,\ldots,z_n$) is an independent random sample of variable observations followed by the Pareto exponential distribution with (α) parameter and (β) parameter. The possible function for random variable (Z) observations is:

$$L\left(z_1,\ z_2,\ldots,z_n|\alpha,\beta\right)=\alpha^n\beta^n\prod_{i=1}^{n}\left(1+z_i\right)^{-\left(\alpha+1\right)}\prod_{i=1}^{n}\left[1-\left(1+z_i\right)^{-\alpha}\right]^{\beta-1} \tag{3.1}$$

$$\ln\ L\left(\alpha,\beta;z_1,\ z_2,\ldots,z_n\right)=n\ln\left(\alpha\right)+\mathrm{nln}\left(\beta\right)+\left(\beta-1\right)\sum_{i=1}^{n}\ln\left\{1-\left(1+z_i\right)^{-\alpha}\right\}$$

$$-\left(\alpha-1\right)\sum_{i=1}^{n}\ln\left(1+z_i\right) \tag{3.2}$$

By deriving equation (3.2) for (α) and (β), we obtain:

$$\frac{\partial ln\ L\left(\alpha,\beta;z_1,\ z_2,\ldots,z_n\right)}{\partial\alpha}=\frac{n}{\alpha}+\left(\beta-1\right)\sum_{i=1}^{n}\frac{\left(1+z_i\right)^{-\alpha}\ ln(1+z_i)}{1-\left(1+z_i\right)^{-\alpha}}-\sum_{i=1}^{n}\ln\left(1+z_i\right) \tag{3.3}$$

$$\frac{\partial ln\ L\left(\alpha,\beta;z_1,\ z_2,\ldots,z_n\right)}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln\left\{1-\left(1+z_i\right)\right\}^{-\alpha} \tag{3.4}$$

To find solutions for equations (3.3) and (3.4), rely on the Fisher Information Matrix:

$$(\alpha,\beta)=\begin{bmatrix}\frac{\partial^2\ln\ L(\alpha,\beta;z_i)}{\partial\alpha^2} & \frac{\partial^2\ln\ L(\alpha,\beta;z_i)}{\partial\alpha\partial\beta}\\ \frac{\partial^2\ln\ L(\alpha,\beta;z_i)}{\partial\beta\partial\alpha} & \frac{\partial^2\ln\ L(\alpha,\beta;z_i)}{\partial\beta^2}\end{bmatrix}$$

Where:

$$\frac{\partial^2 \ln\ L(\alpha,\beta;z_i)}{\partial\alpha^2} = -\frac{n}{\alpha^2} - (\beta-1)\sum_{i=1}^{n}\frac{(1+z_i)^{-\alpha}\ \{ln(1+z_i)\}^2}{\left\{1-(1+z_i)^{-\alpha}\right\}^2}$$

$$\frac{\partial^2 \ln\ L(\alpha,\beta;z_i)}{\partial\alpha\partial\beta} = \frac{\partial^2 \ln\ L(\alpha,\beta;z_i)}{\partial\beta\partial\alpha} = \sum_{i=1}^{n}\frac{(1+z_i)^{-\alpha}\ ln(1+z_i)}{1-(1+z_i)},$$

$$\frac{\partial^2 \ln\ L(\alpha,\beta;z_i)}{\partial\beta^2} = -\frac{n}{\beta^2}$$

Standard Bayes Method (3.5) is a common method of estimation which finds the Posterior Probability Density Function, which represents all prior and current information about parameters to be estimated, and through (Bayes Inversion Formula), and that the reverse Bayes formula for Pareto's exponential distribution teachers is:

$$P(\alpha,\beta|z) = \frac{L(z|\alpha,\beta)\,\pi(\alpha)\,\pi(\beta)}{\int_{\forall\alpha}\int_{\forall\beta} L(z|\alpha,\beta)\,\pi(\alpha)\,\pi(\beta)d\alpha\,d\beta} \tag{3.5}$$

As:
$L(z\alpha,\beta)$: The joint probability functions of a sample of the size of (n) of the observations of the random variable Z.
(α): represents the prior probability density function of the parameter (α).
(β): represents the prior probability density function of the parameter (β).
$P(\alpha,\beta/z)$: represents the joint posterior function (α,β).

4. Non-Informative Prior Density Function

If there is not sufficient prior information about the parameter to be estimated or not yet available, the prior probability density selection is based on the Jeffery Formula based on the parameter field to be estimated. Since the parameter field is a positive $(0,\infty)$, then , the prior probability follows as log uniform distribution (α,β).

$$\pi(\alpha) \propto \frac{1}{\alpha},\quad \pi(\beta) \propto \frac{1}{\beta},\qquad \alpha,\beta>0 \tag{4.1}$$

Conjugate Prior

It probability functions has known parameters depend on the Likelihood function which it with parameter as a function of the parameter to be estimated. The joint prior conjugate Probability Density Function $(\alpha),(\beta)$ for exponential pareto distribution follows the gamma distribution of parameters $(a,b),(h,f)$.

$$\pi(\alpha) \propto\ \alpha^{a-1}\ e^{-\ b\alpha}\ ,\ \pi(\beta) \propto\ \beta^{h-1}\ e^{-f\beta}\ ,\qquad a\ ,b,h,f,\ \alpha,\beta>0 \tag{4.2}$$

Loss Functions

The loss function is a measure of the amount of loss resulting from the Bayes decision to estimate the parameter while the correct decision is the real value of the parameter. A Bayes estimator can be found by using loss functions, which is symbolized by the symbol $L(\widehat{\theta},\theta)$. The mathematical expectation of the loss function, called the Risk Function, is calculated as follows:

$$\text{Risk}\left(\widehat{\theta},\theta\right)=R\left(\widehat{\theta},\theta\right)=E\left\{L(\widehat{\theta},\theta)\right\}=\int_{\forall\theta}L\left(\widehat{\theta},\theta\right)P\left(\theta|x\right)d\theta \tag{4.3}$$

The estimate that makes the risk function at minimum (i.e., the least possible) is the standard bayes parameter of the parameter (θ).

There are many types of loss functions that can be divided according in to the symmetry criterion to two main types: Symmetric Loss Functions and Asymmetric Loss Functions. The research will use a Symmetrical Loss Function, namely the Quadratic Loss Function and Function Asymmetric loss is the general entropy loss function.

Lindley Approximation can then be used to find a solution for integration in Equation (9), i.e., to find a biz estimator for function G (α,β) as follows:

$$\widehat{G}=G\left(\widehat{\alpha},\ \widehat{\beta}\right)+0.5\left[\sum_{i}^{2}\sum_{j}^{2}X_{\text{ij}}Y_{\text{ij}}+l_{30}A_{12}+l_{03}A_{21}+l_{30}B_{12}+l_{30}B_{12}\right]+W_1C_{12}+W_2C_{21} \tag{4.4}$$

where

$$l_{ij}=\frac{\partial^{i+j}\ln L\left(\alpha,\beta\right)}{\partial\alpha^{i}\ \partial\beta^{j}},\qquad i,j=0,1,2,3,\quad i+j=3,$$

$$W_1=\frac{\partial^2\text{Ln }\ \{P\ \ (\alpha,\beta)\}}{\partial\alpha^2},\quad W_2=\frac{\partial^2\ln\{P\ \ (\alpha,\beta)\}}{\partial\beta^2},\quad X_{12}=\frac{\partial^2G\ \ (\alpha,\beta)}{\partial\alpha\partial\beta},$$

$$X_{21}=\frac{\partial^2G\ \ (\alpha,\beta)}{\partial\beta\partial\alpha},\quad X_{11}=\frac{\partial^2G\ \ (\alpha,\beta)}{\partial\alpha^2},$$

$$X_{22}=\frac{\partial^2G\left(\alpha,\beta\right)}{\partial\beta^2},\quad X_1=\frac{\partial G\ L\left(\alpha,\beta\right)}{\partial\alpha},\quad X_2=\frac{\partial G\ L\left(\alpha,\beta\right)}{\partial\beta},\quad A_{ij}=\left(X_i+Y_{ii}+X_j+Y_{ij}\right)Y_{ii},$$

$$B_{ij}=\left(X_i+Y_{ii}+X_j+Y_{ji}\right),\quad C_{ij}=3X_iY_{ii}Y_{ij}+X_j(Y_{ii}Y_{jj}+2{Y_{ij}}^2);\quad i\ ,\ j=1,2$$

$$Y_{11}=\frac{q}{pq-r^2},\quad Y_{22}=\frac{p}{pq-r^2},\quad Y_{12}=Y_{21}=\frac{r}{pq-r^2},\quad p=-\frac{\partial^2\ln L\left(\alpha,\beta\right)}{\partial\alpha^2},\quad q=-\frac{\partial^2\ln L\left(\alpha,\beta\right)}{\partial\beta^2},$$

$$r=-\frac{\partial^2\ln L\left(\alpha,\beta\right)}{\partial\alpha\partial\beta}=-\frac{\partial^2\ln L\left(\alpha,\beta\right)}{\partial\beta\partial\alpha}$$

Estimators of the Standard Bayes Method

The SB Method of Pareto exponential distribution parameters can be obtained by using the prior functions and the different loss functions by applying the Lindley approximation equations above as follows:

1. Determination of the Bayes Method using the informative prior distribution and the loss square function of the:

$$\begin{aligned}\widehat{\alpha}_{BS1}=\widehat{\alpha}^{-k}+0.5\ &\left\{\left(k(k+1)\widehat{\alpha}^{-(k+2)}\right)\ Y_{11}-\left(\frac{2n}{\widehat{\alpha}^3}\right)\right.\\&+\left(\widehat{\beta}-1\right)\sum_{i=1}^{n}\frac{\left(1+z_i\right)^{-\widehat{\alpha}}\left(1+\left(1+z_i\right)^{-\widehat{\alpha}}\right)\left(\text{Ln}\left(1+z_i\right)\right)^3}{\left(1-\left(1+z_i\right)\right)^3}\left(k\widehat{\alpha}^{-(k+1)}\right){Y_{11}}^2\\&\left.-\left(\frac{2n}{\widehat{\beta}^3}\right)\left(k\widehat{\alpha}^{-(k+1)}\right)Y_{21}Y_{22}+3Y_{11}Y_{12}\left(k\widehat{\alpha}^{-(k+1)}\right)\sum_{i=1}^{n}\frac{\left(1+z_i\right)^{-\widehat{\alpha}}\left(\text{Ln}\left(1+z_i\right)\right)^2}{\left(1-\left(1+z_i\right)\right)^2}\right\}\end{aligned} \tag{4.5}$$

$$\widehat{\beta}_{BS1} = \widehat{\beta}^{-k} + 0.5 \ \Bigg\{ \left(k(k+1)\widehat{\beta}^{-(k+2)} \right) \ Y_{22} - \left(\frac{2n}{\widehat{\alpha}^3} \right) + \left(\widehat{\beta} - 1 \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} \left(1 + (1+z_i)^{-\widehat{\alpha}} \right) (\mathrm{Ln}\,(1+z_i))^3}{(1-(1+z_i))^3} \left(k\widehat{\beta}^{-(k+1)} \right) Y_{21} Y_{22} - \left(\frac{2n}{\widehat{\beta}^3} \right) \left(k\widehat{\beta}^{-(k+1)} \right) {Y_{22}}^2 + \left(Y_{11}Y_{12} + 2{Y_{12}}^2 \right) \left(k\widehat{\beta}^{-(k+1)} \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} (\mathrm{Ln}\,(1+z_i))^2}{(1-(1+z_i))^2} \Bigg\} \tag{4.6}$$

2.

$$\widehat{\alpha}_{BE1} = \Bigg[\widehat{\alpha}^{-k} + 0.5 \ \Bigg\{ \left(k(k+1)\widehat{\alpha}^{-(k+2)} \right) \ Y_{11} - \left(\frac{2n}{\widehat{\alpha}^3} \right) + \left(\widehat{\beta} - 1 \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} \left(1 + (1+z_i)^{-\widehat{\alpha}} \right) (\mathrm{Ln}\,(1+z_i))^3}{(1-(1+z_i))^3} \left(k\widehat{\alpha}^{-(k+1)} \right) {Y_{11}}^2 - \left(\frac{2n}{\widehat{\beta}^3} \right) \left(k\widehat{\alpha}^{-(k+1)} \right) Y_{21} Y_{22} + 3Y_{11}Y_{12} \left(k\widehat{\alpha}^{-(k+1)} \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} (\mathrm{Ln}\,(1+z_i))^2}{(1-(1+z_i))^2} \Bigg\} \Bigg]^{-\frac{1}{k}} \tag{4.7}$$

$$\widehat{\beta}_{BE1} = \Bigg[\widehat{\beta}^{-k} + 0.5 \ \Bigg\{ \left(k(k+1)\widehat{\beta}^{-(k+2)} \right) \ Y_{22} - \left(\frac{2n}{\widehat{\alpha}^3} \right) + \left(\widehat{\beta} - 1 \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} \left(1 + (1+z_i)^{-\widehat{\alpha}} \right) (\mathrm{Ln}\,(1+z_i))^3}{(1-(1+z_i))^3} \left(k\widehat{\beta}^{-(k+1)} \right) Y_{21} Y_{22} - \left(\frac{2n}{\widehat{\beta}^3} \right) \left(k\widehat{\beta}^{-(k+1)} \right) {Y_{22}}^2 + \left(Y_{11}Y_{12} + 2{Y_{12}}^2 \right) \left(k\widehat{\beta}^{-(k+1)} \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} (\mathrm{Ln}\,(1+z_i))^2}{(1-(1+z_i))^2} \Bigg\} \Bigg]^{-\frac{1}{k}} \tag{4.8}$$

3. Bayes Method is estimated using the initial natural accompaniment function and quadratic loss function:

$$\widehat{\alpha}_{BS2} = \widehat{\alpha}^{-k} + 0.5 \ \Bigg\{ \left(k(k+1)\widehat{\alpha}^{-(k+2)} \right) \ Y_{11} - \left(\frac{2n}{\widehat{\alpha}^3} \right) + \left(\widehat{\beta} - 1 \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} \left(1 + (1+z_i)^{-\widehat{\alpha}} \right) (\mathrm{Ln}\,(1+z_i))^3}{(1-(1+z_i))^3} \left(k\widehat{\alpha}^{-(k+1)} \right) {Y_{11}}^2 - \left(\frac{2n}{\widehat{\beta}^3} \right) \left(k\widehat{\alpha}^{-(k+1)} \right) Y_{21} Y_{22} + 3Y_{11}Y_{12} \left(k\widehat{\alpha}^{-(k+1)} \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} (\mathrm{Ln}\,(1+z_i))^2}{(1-(1+z_i))^2} \Bigg\} - \left\{ \left(\frac{a-1}{\widehat{\alpha}} - b \right) \left(k\widehat{\alpha}^{-(k+1)} \right) Y_{11} - \left(\frac{f-1}{\widehat{\beta}} - h \right) \left(k\widehat{\alpha}^{-(k+1)} \right) Y_{12} \right\} \tag{4.9}$$

$$\widehat{\beta}_{BS2} = \widehat{\beta}^{-k} + 0.5\ \Bigg\{ \left((k(k+1)\widehat{\beta}^{-(k+2)} \right)\ Y_{22} - \left(\frac{2n}{\widehat{\alpha}^3} \right) + \left(\widehat{\beta} - 1 \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} \left(1 + (1+z_i)^{-\widehat{\alpha}} \right) (\mathrm{Ln}\,(1+z_i))^3}{(1-(1+z_i))^3} \left(k\widehat{\beta}^{-(k+1)} \right) Y_{21}Y_{22} - \left(\frac{2n}{\widehat{\beta}^3} \right) \left(k\widehat{\beta}^{-(k+1)} \right) {Y_{22}}^2 + \left(Y_{11}Y_{12} + 2{Y_{12}}^2 \right) \left(k\widehat{\beta}^{-(k+1)} \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} (\mathrm{Ln}\,(1+z_i))^2}{(1-(1+z_i))^2} \Bigg\} - \left\{ \left(\frac{a-1}{\widehat{\alpha}} - b \right) \left(k\widehat{\beta}^{-(k+1)} \right) Y_{21} - \left(\frac{f-1}{\widehat{\beta}} - h \right) \left(k\widehat{\beta}^{-(k+1)} \right) Y_{22} \right\} \tag{4.10}$$

4. Determination of the Bayes Method using the prior conjugate function and the general entropy loss function:

$$\widehat{\alpha}_{BE2} = \Bigg[\widehat{\alpha}^{-k} + 0.5\ \Bigg\{ \left(k(k+1)\widehat{\alpha}^{-(k+2)} \right)\ Y_{11} - \left(\frac{2n}{\widehat{\alpha}^3} \right) + \left(\widehat{\beta} - 1 \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} \left(1 + (1+z_i)^{-\widehat{\alpha}} \right) (\mathrm{Ln}\,(1+z_i))^3}{(1-(1+z_i))^3} \left(k\widehat{\alpha}^{-(k+1)} \right) {Y_{11}}^2 - \left(\frac{2n}{\widehat{\beta}^3} \right) \left(k\widehat{\alpha}^{-(k+1)} \right) Y_{21}Y_{22} + 3Y_{11}Y_{12} \left(k\widehat{\alpha}^{-(k+1)} \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} (\mathrm{Ln}\,(1+z_i))^2}{(1-(1+z_i))^2} \Bigg\} - \left\{ \left(\frac{a-1}{\widehat{\alpha}} - b \right) \left(k\widehat{\alpha}^{-(k+1)} \right) Y_{11} - \left(\frac{f-1}{\widehat{\beta}} - h \right) \left(k\widehat{\alpha}^{-(k+1)} \right) Y_{12} \right\} \Bigg]^{-\frac{1}{k}} \tag{4.11}$$

$$\widehat{\beta}_{BE2} = \Bigg[\widehat{\beta}^{-k} + 0.5\ \Bigg\{ \left((k(k+1)\widehat{\beta}^{-(k+2)} \right)\ Y_{22} - \left(\frac{2n}{\widehat{\alpha}^3} \right) + \left(\widehat{\beta} - 1 \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} \left(1 + (1+z_i)^{-\widehat{\alpha}} \right) (\mathrm{Ln}\,(1+z_i))^3}{(1-(1+z_i))^3} \left(k\widehat{\beta}^{-(k+1)} \right) Y_{21}Y_{22} - \left(\frac{2n}{\widehat{\beta}^3} \right) \left(k\widehat{\beta}^{-(k+1)} \right) {Y_{22}}^2 + \left(Y_{11}Y_{12} + 2{Y_{12}}^2 \right) \left(k\widehat{\beta}^{-(k+1)} \right) \sum_{i=1}^{n} \frac{(1+z_i)^{-\widehat{\alpha}} (\mathrm{Ln}\,(1+z_i))^2}{(1-(1+z_i))^2} \Bigg\} - \left\{ \left(\frac{a-1}{\widehat{\alpha}} - b \right) \left(k\widehat{\beta}^{-(k+1)} \right) Y_{21} - \left(\frac{f-1}{\widehat{\beta}} - h \right) \left(k\widehat{\beta}^{-(k+1)} \right) Y_{22} \right\} \Bigg]^{-\frac{1}{k}} \tag{4.12}$$

But the researchers faced the problem of finding the posterior distribution in equation (4.4) in an exact manner and not by relying on Lindley approximation as in equations (4.5). Solve the following integrals

$$P(\alpha,\beta|z) = \frac{\int_{\forall\alpha}\int_{\forall\beta}\alpha L(z|\alpha,\beta)\,\pi(\alpha)\,\pi(\beta)\,d\alpha\,d\beta}{\int_{\forall\alpha}\int_{\forall\beta}L(z|\alpha,\beta)\,\pi(\alpha)\,\pi(\beta)\,d\alpha\,d\beta}$$

$$P(\alpha,\beta|z) = \frac{\int_{\forall\alpha}\int_{\forall\beta}\beta L(z|\alpha,\beta)\,\pi(\alpha)\,\pi(\beta)\,d\alpha\,d\beta}{\int_{\forall\alpha}\int_{\forall\beta}L(z|\alpha,\beta)\,\pi(\alpha)\,\pi(\beta)\,d\alpha\,d\beta}$$

$$P(\alpha,\beta|z) = \frac{\int_{\forall\alpha}\int_{\forall\beta}\alpha^{-k} L(z|\alpha,\beta)\,\pi(\alpha)\,\pi(\beta)\,d\alpha\,d\beta}{\int_{\forall\alpha}\int_{\forall\beta}L(z|\alpha,\beta)\,\pi(\alpha)\,\pi(\beta)\,d\alpha\,d\beta}$$

$$P(\alpha,\beta|z) = \frac{\int_{\forall\alpha}\int_{\forall\beta}\beta^{-k} L(z|\alpha,\beta)\,\pi(\alpha)\,\pi(\beta)\,d\alpha\,d\beta}{\int_{\forall\alpha}\int_{\forall\beta}L(z|\alpha,\beta)\,\pi(\alpha)\,\pi(\beta)\ \,d\alpha\,d\beta}$$

Empirical side

The Monte Carlo Simulation Experiments were used in the estimation of Pareto's exponential distribution parameters based on the R 3.5.1 program and according to the following steps:

1. Selection of the default values of the distribution teachers and the parameters of the natural accompanying function as shown in Table (1)

Table 1: Simulation experiments

Experiment	α	$a = f$	$b = h$
1	*0.5*	*2*	*0.5*
2	*1*	*2*	*1*
3	*5*	*2*	*5*

2. Selection of the sample sizes for the four components 10,25,50,100 as well as the selection of k = 2 in the generalized loss function, and the frequency of the experiments was (D = 1000).
3. The generation of random variables (Z_i), which follows the exponential Pareto distribution according to the parameter values assumed in step 1, as well as the generation of the variables following the gamma distribution, which represents the natural conjugate function, using the Inverse transformation tethod.
4. Estimating the distribution parameters according to the methods, equations and formulas mentioned in the theoretical aspect as:

$$\widehat{\alpha} = \frac{1}{D}\sum_{d=1}^{D}\widehat{\alpha}_r \quad , \quad \widehat{\beta} = \frac{1}{D}\sum_{d=1}^{D}\widehat{\beta}_r$$

As well as estimating Mean Squares Error (MSE) for those estimations as follows:

$$\text{MSE}[\widehat{\alpha}] = \frac{1}{D}\sum_{d=1}^{D}[\alpha-\widehat{\alpha}_r]^2 \quad , \quad MSE[\widehat{\alpha}] = \frac{1}{D}\sum_{d=1}^{D}\left[\beta-\widehat{\beta}_r\right]^2$$

The results of the simulations were presented in Tables (2-5).

Table 2: α ***Estimates For simulation experiments***

Real	n	$\widehat{\alpha}_{MLE}$	$\widehat{\alpha}_{BS1}$	$\widehat{\alpha}_{BS2}$	$\widehat{\alpha}_{BE1}$	$\widehat{\alpha}_{BE2}$
	10	0.7028365	0.634151	0.6024439	0.642078	0.61342041
0.5	25	0.682457	0.601444	0.58941533	0.60249	0.62759396
	50	0.60072147	0.59051	0.58460463	0.590756	0.60281201
	100	0.52601245	0.516232	0.51365081	0.516285	0.52149964
	10	1.865472	1.527782	1.43347466	1.697536	1.5089207
1	25	1.322145	1.151111	1.12613036	1.199074	1.149113
	50	1.2750236	1.059726	1.04869212	1.081353	1.05928497
	100	1.1124562	1.075577	1.07009008	1.086442	1.07546741
	10	6.4014525	5.753635	5.39847273	6.392928	5.68260288
5	25	6.2994421	5.323461	5.207935	5.545272	5.31421925
	50	5.8721457	5.559006	5.50112362	5.672455	5.55669053
	100	5.455201	5.381375	5.3539215	5.435732	5.38082563

Table 3: β ***Estimates For simulation experiments***

Real	n	$\widehat{\beta}_{MLE}$	$\widehat{\beta}_{BS1}$	$\widehat{\beta}_{BS2}$	$\widehat{\beta}_{BE1}$	$\widehat{\beta}_{BE2}$
	10	0.593353	0.577156	0.566057	0.577928	0.572378
0.5	25	0.588215	0.57441004	0.553136	0.57589824	0.565261
	50	0.577003	0.56430575	0.51728	0.56752104	0.544008
	100	0.533256	0.51096	0.505951	0.511015	0.50851
	10	1.545462	1.32548057	1.215024	1.3417546	1.286526
	25	1.301458	1.09918087	1.05847	1.10116449	1.080809
1	50	1.101324	1.06407202	1.05364	1.06439238	1.059176
	100	1.085502	1.037352	1.017403	1.037726	1.027751
	10	5.469548	5.43682031	5.332266	5.44118851	5.388911
	25	5.366302	5.32377958	5.271586	5.32539848	5.299302
5	50	5.308851	5.10394781	5.095285	5.1091452	5.104814
	100	5.223254	5.09774085	5.020047	5.09969566	5.060849

Table 4: MSE for α ***Estimates For simulation experiments***

Real	n	$\widehat{\alpha}_{MLE}$	$\widehat{\alpha}_{BS1}$	$\widehat{\alpha}_{BS2}$	$\widehat{\alpha}_{BE1}$	$\widehat{\alpha}_{BE2}$
	10	0.051143	0.027997	0.02049475	0.030186	0.022864
0.5	25	0.043291	0.020291	0.0179951	0.020504	0.020291
	50	0.020145	0.018192	0.01715794	0.018237	0.018192
	100	0.010677	0.010263	0.01018634	0.010265	0.010263
	10	0.759042	0.288554	0.19790028	0.496556	0.269
1	25	0.113777	0.032835	0.02590887	0.04963	0.032235
	50	0.085638	0.013567	0.01237092	0.016618	0.013515
	100	0.022646	0.015712	0.01491262	0.017472	0.015695
	10	1.974069	0.577966	0.16878052	1.950249	0.475947
5	25	1.69855	0.114627	0.05323696	0.307322	0.108734
	50	0.770638	0.322488	0.26112489	0.462196	0.319904
	100	0.217208	0.155447	0.13526043	0.199862	0.155028

Table 5: MSE for β ***Estimates For simulation experiments***

Real	n	$\widehat{\beta}_{MLE}$	$\widehat{\beta}_{BS1}$	$\widehat{\beta}_{BS2}$	$\widehat{\beta}_{BE1}$	$\widehat{\beta}_{BE2}$
	10	0.018715	0.01595305	0.014364	0.01607277	0.015239
0.5	25	0.017782	0.01553685	0.012823	0.01576054	0.014259
	50	0.015929	0.01413523	0.010299	0.01455909	0.011937
	100	0.011106	0.01012012	0.010035	0.01012133	0.010072
	10	0.307529	0.1159376	0.056235	0.12679621	0.092097
1	25	0.100877	0.01983685	0.013419	0.02023425	0.01653
	50	0.020267	0.01410522	0.012877	0.01414638	0.013502
	100	0.017311	0.01139517	0.010303	0.01142325	0.01077
	10	0.230475	0.20081198	0.120401	0.2046473	0.161252
5	25	0.144177	0.11483322	0.083759	0.11588417	0.099581
	50	0.105389	0.02080515	0.019079	0.02191267	0.020805
	100	0.059842	0.01955327	0.010402	0.01993923	0.013703

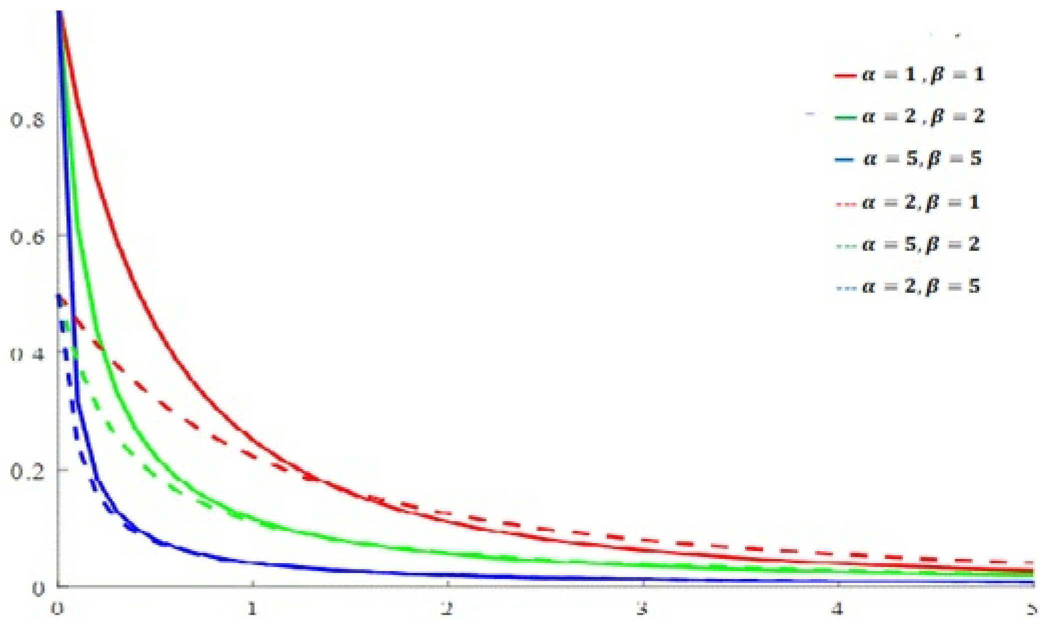

Figure 1: Probability density function under variety values of β , α

5. Analysis results and conclusions

In Tables (2-5), which represent the different simulation experiments for estimating Pareto exponential parameters, the following is illustrated:

1. All estimates under all estimation methods used in estimating the parameters α and β were identical to statistical theory. The larger the sample size, the estimated values of the function approached the default values, and the mean values of the MSE were estimated.
2. The Standard Bayes Method was superior to the Informative Prior Function, which is the prior natural conjugate function in the estimation of the two parameters (α, β) when it was in all experiments and for all sample sizes. The Bayes Method under non-informative prior came first with the second rank followed by the third maximum likelihood method.
3. The quadratic loss function with informative prior was the best estimate. The lowest estimation values were given to the real values; the (MSE) were among the other estimation methods. The general Entropy Loss Function was replaced by the Prior Informative Distribution. The Quadratic Loss Function in the first non-information distribution was ranked third, followed by the General Entropy Loss function by the prior distribution of non-information, and the method of the greatest potential was finally resolved. Note that all estimations for all estimation methods are approximated at sample size (n = 100).
4. The approximation of the Lindelly approach was well performed, given approximation values close to the assumed values, and the values of relatively small error squares.
5. Figure 1 shows that the distribution of the Pareto exponential is a skewedness distribution towards right and that the skewedness increases by increasing the value of the parameter (α).

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